MPI Leipzig Summer School
in Algebraic Combinatorics

# The Koszul property in Algebraic Combinatorics

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Lectures

1: Motivation, definition of Koszul algebras
- the monomial case

2: Methods for proving Koszulity
and more examples

3: Bar complex, topology,
and inequalities

4: Group actions

# 3: Bar complex, topology, and inequalities

Recall $A$ is Koszul $\Longleftrightarrow$ $\mathrm{Tor}_i^A(\Bbbk,\Bbbk)_j = 0$ for $j \neq i$.

There is another way to compute $\mathrm{Tor}_*^A(\Bbbk,\Bbbk)$, via the Bar complex/resolution $\mathcal{B}_\bullet$ of $\Bbbk$:

$$\cdots \to \underset{\overset{\|}{F_3}}{A \otimes A_+ \otimes A_+ \otimes A_+} \xrightarrow{d} \underset{\overset{\|}{F_2}}{A \otimes A_+ \otimes A_+} \xrightarrow{d} \underset{\overset{\|}{F_1}}{A \otimes_{\Bbbk} A_+} \xrightarrow{d} \underset{\overset{\|}{F_0}}{A} \to \Bbbk \to 0$$

$$a \otimes a_1 \mapsto a a_1$$

$$a \otimes a_1 \otimes a_2 \mapsto a a_1 \otimes a_2 - a \otimes a_1 a_2$$

$$a \otimes a_1 \otimes a_2 \otimes a_3 \mapsto a a_1 \otimes a_2 \otimes a_3 - a \otimes a_1 a_2 \otimes a_3 + a \otimes a_1 \otimes a_2 a_3$$

It's more traditional to write a typical element

of $F_i = A \otimes \underbrace{A_+ \otimes A_+ \otimes \cdots \otimes A_+}^{i \text{ factors of } A_+}$

$$a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i$$

$$\updownarrow$$

as $a[a_1 | a_2 | \cdots | a_i]$

so the differential becomes the $A$-linear map

$$F_i \xrightarrow{d} F_{i-1}$$

$$\begin{aligned}
[a_1|a_2|a_3|\cdots|a_i] \longmapsto\ & a_1[a_2|a_3|\cdots|a_i] \\
& - [a_1a_2|a_3|\cdots|a_i] \\
& + [a_1|a_2a_3|\cdots|a_i] \\
& - \cdots \\
& + (-1)^{i-1}[a_1|a_2|\cdots|a_{i-2}|a_{i-1}a_i] \\
= \ & a_1[a_2|a_3|\cdots|a_i] \\
& + \sum_{j=1}^{i-1} (-1)^j [a_1|\cdots|a_j a_{j+1}|a_{j+2}|\cdots|a_i]
\end{aligned}$$

(Problem 9 shows why this complex $\mathcal{B}_0$ is exact)

Then $\mathrm{Tor}_i^A(\Bbbk,\Bbbk) = H_i(\Bbbk \otimes_A \mathcal{B}_\bullet)$

and note that since $\Bbbk = A/A_+$,

$$\Bbbk \otimes_A \mathcal{B}_\bullet$$

has $\mathbb{N}$-graded $\Bbbk$-linear differentials

$$\begin{array}{ccc} \Bbbk \otimes_A F_i & \xrightarrow{d} & \Bbbk \otimes_A F_{i-1} \\ \| & & \cap \\ \underbrace{A_+ \otimes A_+ \otimes \cdots \otimes A_+}_{i \text{ factors}} & & \underbrace{A_+ \otimes \cdots \otimes A_+}_{i-1 \text{ factors}} \end{array}$$

$$[a_1|a_2|\cdots|a_i] \overset{d}{\longmapsto} \sum_{j=1}^{i-1} (-1)^j [a_1|\cdots|a_{j-1}|a_j a_{j+1}|a_{j+2}|\cdots|a_i]$$

which looks very topological, and combinatorial.

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COROLLARY: $A$ Koszul $\Longleftrightarrow$

$$H_i(\Bbbk \otimes_A \mathcal{B}_\bullet)_j = 0 \text{ for } j \neq i.$$

It becomes topological for

affine semigroup / toric rings $\mathbb{k}[\Lambda]$

$\overset{\text{DEF'N}}{:=}$ $\mathbb{k}$-subalgebras of $\mathbb{k}[t_1^{\pm 1},\dots,t_d^{\pm 1}]$ generated by finitely many monomials

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EXAMPLES

- Hilbi rings $A_P = \mathbb{k}\left[t_0 \prod_{i\in I} t_i \ : \text{orders ideals } I \subseteq P\right]$

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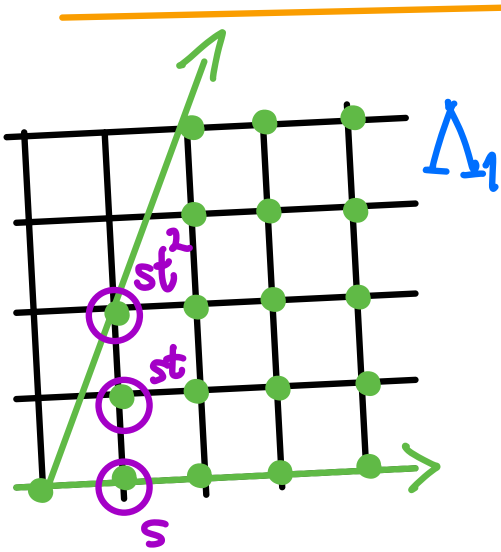
- 


$$A_1 = \mathbb{k}[\Lambda_1] = \mathbb{k}[s, st, st^2] \cong \mathbb{k}[x_1,x_2,x_3] \big/ (x_1x_3 - x_2^2)$$

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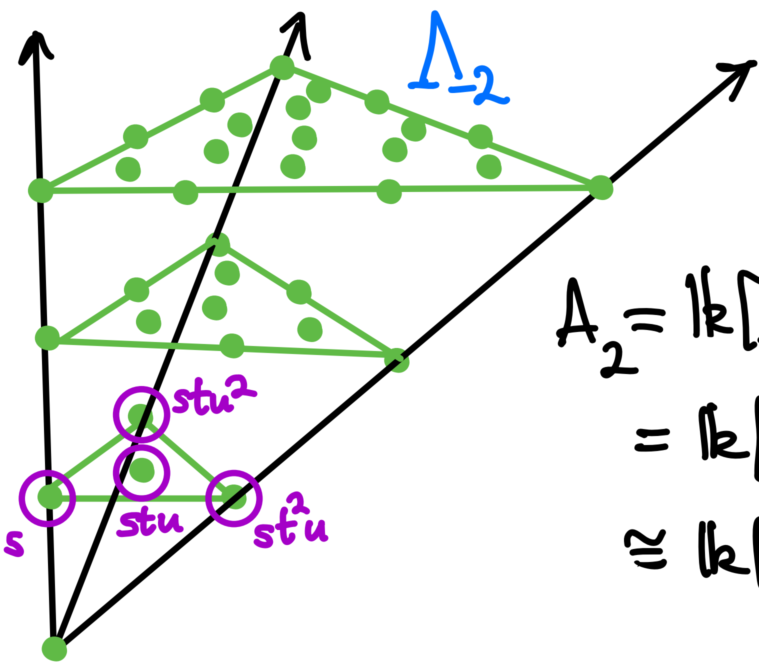
- 


$$A_2 = \mathbb{k}[\Lambda_2] = \mathbb{k}[s, st^2u, stu^2, stu] \cong \mathbb{k}[x_1,x_2,x_3,x_4] \big/ (x_1x_2x_3 - x_4^3)$$

These $A = \mathbb{k}[\underline{\Lambda}]$ naturally have a $\Lambda$-grading

$$= \bigoplus_{\lambda \in \Lambda} \mathbb{k} \cdot \underline{t}^{\lambda} \quad \text{with} \quad \underline{t}^{\lambda} \cdot \underline{t}^{\mu} = \underline{t}^{\lambda+\mu}$$

and same for A-resolutions of $\mathbb{k}$ and

$$\mathrm{Tor}_i^{\mathbb{k}[\Lambda]}(\mathbb{k},\mathbb{k}) = \bigoplus_{\lambda \in \underline{\Lambda}} \mathrm{Tor}_i^{\mathbb{k}[\Lambda]}(\mathbb{k},\mathbb{k})_{\lambda}$$

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THEOREM (Landal & Sletsjøe 1985)

reduced homology

$$\mathrm{Tor}_i^{\mathbb{k}[\underline{\Lambda}]}(\mathbb{k},\mathbb{k})_{\lambda} \cong \tilde{H}_{i-2}(\Delta_{\lambda} ; \mathbb{k})$$

where $\Delta_{\lambda}$ := order complex of the open interval $(0, \lambda)$ in the semigroup poset on $\underline{\Lambda}$:

$\lambda < \mu$ if $\lambda - \mu \in \underline{\Lambda}$

THEOREM (Landal & Sletsjøe 1985)

$$\mathrm{Tor}_i^{\Bbbk[\Lambda]}(\Bbbk,\Bbbk)_\lambda \cong \tilde{H}_{i-2}(\Delta_\lambda\,;\Bbbk)$$

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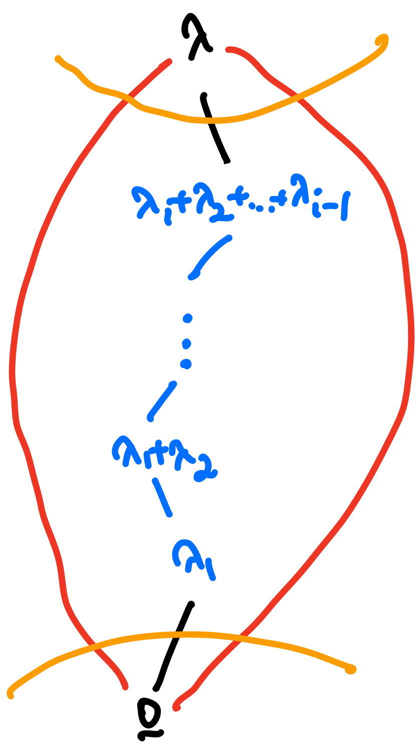
proof: $$\mathrm{Tor}_i^{\Bbbk[\Lambda]}(\Bbbk,\Bbbk) = H_i\left(\Bbbk \otimes_{\Bbbk[\Lambda]} \mathcal{B}_\bullet\right)$$

and

$$\left(\Bbbk \otimes_{\Bbbk[\Lambda]} \mathcal{B}_\bullet\right)_\lambda \cong \tilde{C}_{\bullet-2}(\Delta_\lambda)$$

via $[\lambda_1|\lambda_2|\cdots|\lambda_i] \longmapsto$

with $\lambda_1+\lambda_2+\dots+\lambda_i=\lambda$

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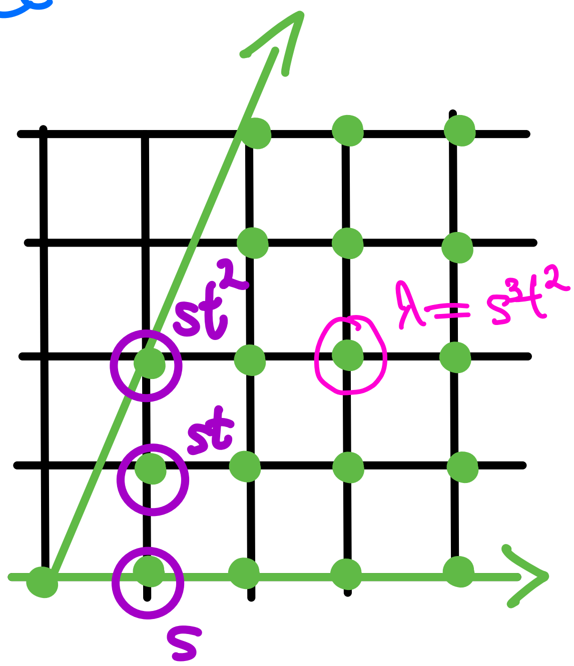
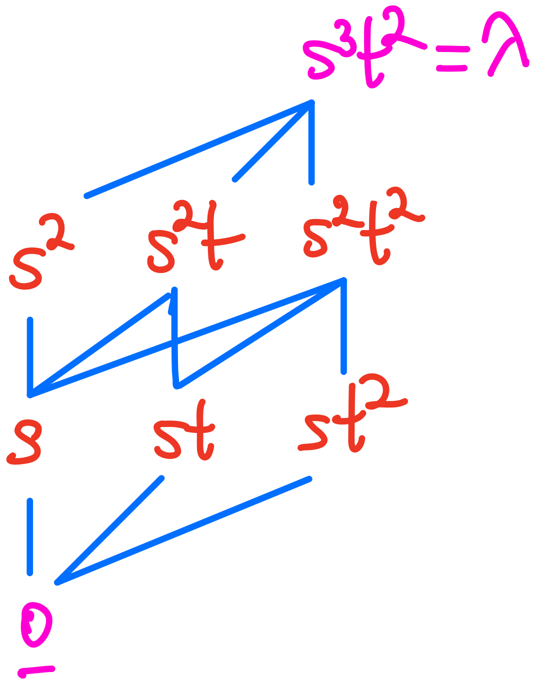
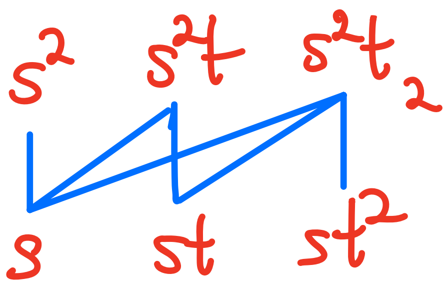
# EXAMPLE



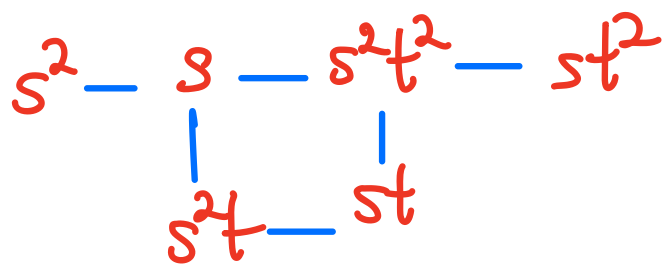

$$A_1 = \Bbbk[\Lambda_1] = \Bbbk[s, st, st^2] \cong \Bbbk[x_1, x_2, x_3] \big/ (x_1x_3 - x_2^2)$$




$\Delta_{s^3t^2} =$



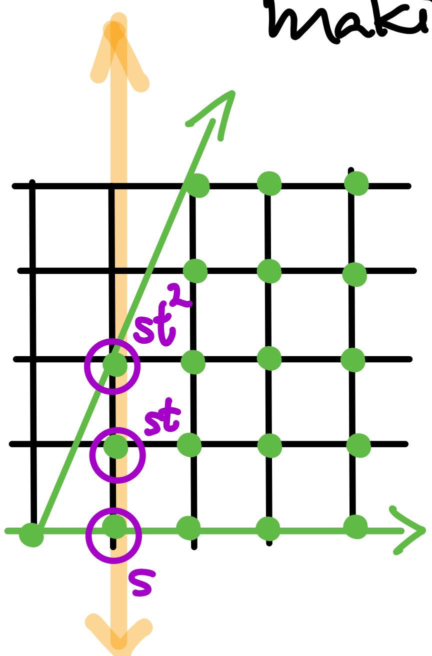

$\cong$

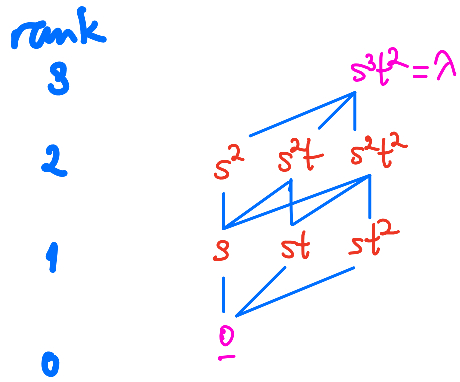



$$\Rightarrow \operatorname{Tor}_i^{\Bbbk[\Lambda_1]}(\Bbbk, \Bbbk)_{s^3t^2} = \tilde{H}_{i-2}(\Delta_{s^3t^2}) = \begin{cases} 0 & \text{if } i \neq 3 \\ \Bbbk^1 & \text{if } i = 3 \end{cases}$$

**PROPOSITION** $\Bbbk[\Lambda]$ is standard $\mathbb{N}$-graded

$\iff$ the generators for $\Lambda$ lie on some affine hyperplane in $\mathbb{Z}^d$, making $\Lambda$ a ranked poset.



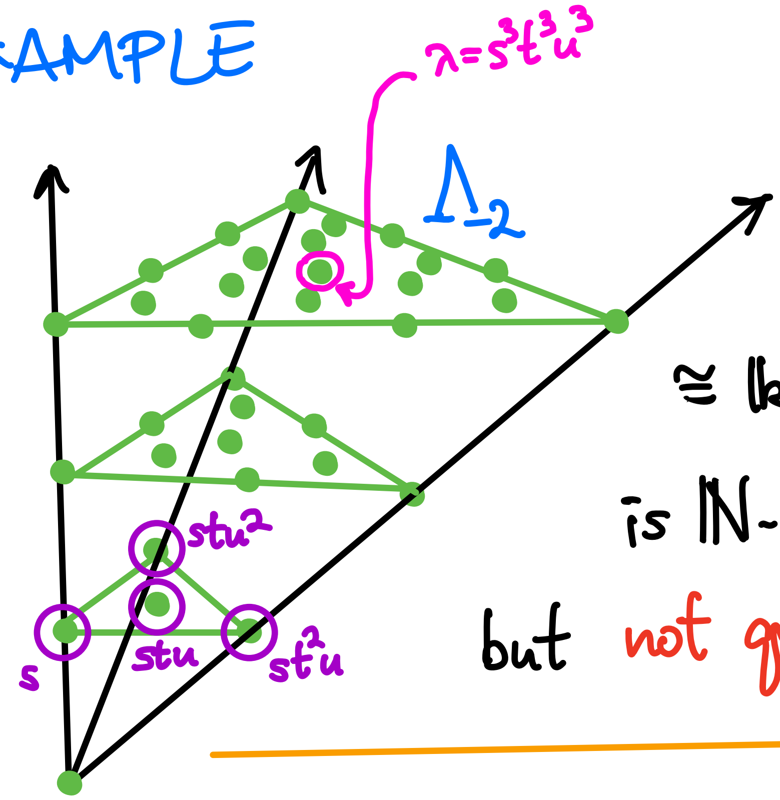

**COROLLARY** (Peeva–R.–Sturmfels 1998)

When $\Bbbk[\Lambda]$ is $\mathbb{N}$-graded,

$\Bbbk[\Lambda]$ is Koszul $\iff$ the poset $\Lambda$ is Cohen-Macaulay

i.e. all order complexes $\Delta_\lambda$ for open intervals $(\underline{0},\lambda)$ of rank $i$ have only top reduced homology $\tilde{H}_{i-2}(\Delta_\lambda)$

# EXAMPLE

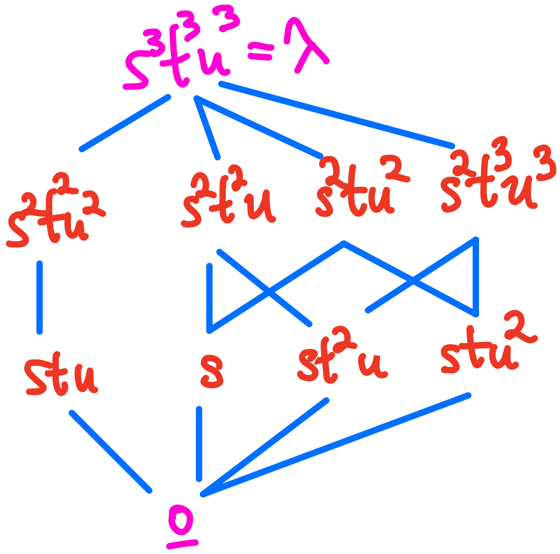



$$A_2 = \Bbbk[\Lambda_2]$$
$$= \Bbbk[s, st^2u, stu^2, stu]$$
$$\cong \Bbbk[x_1, x_2, x_3, x_4]/(x_1x_2x_3 - x_4^3)$$

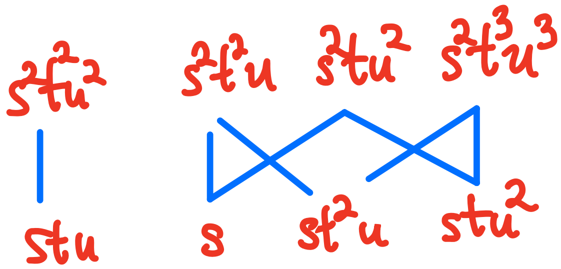
cubic

is $\mathbb{N}$-graded

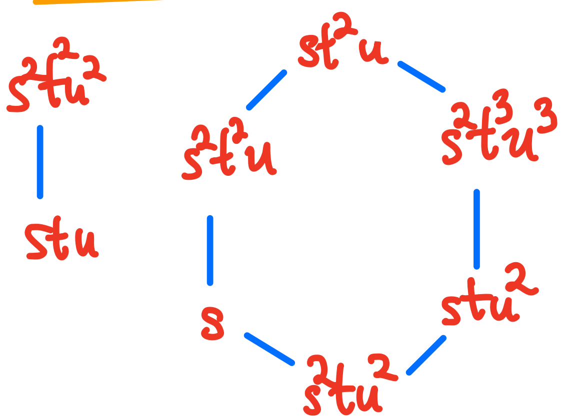
but not quadratic, so not Koszul




$\Delta_{s^3t^3u^3} =$




$\cong$




$$\operatorname{Tor}_i^{\Bbbk[\Lambda_2]}(\Bbbk,\Bbbk)_{s^3t^3u^3} = \tilde{H}_{i-2}(\Delta_{s^3t^3u^3}) = \begin{cases} 0 & \text{if } i \neq 2,3 \\ \Bbbk^1 & \text{if } i = 2 \leftarrow \text{NOT TOP} \\ \Bbbk^1 & \text{if } i = 3 \end{cases}$$

# Inequalities from Koszulity & bar complexes

When $A$ is Koszul,

$$H_i(\Bbbk \otimes_A \mathcal{B}_\bullet)_j = \mathrm{Tor}_i^A(\Bbbk,\Bbbk)_j = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{Tor}_j^A(\Bbbk,\Bbbk)_j & \text{if } i=j \end{cases}$$

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Note $\Bbbk \otimes_A \mathcal{B}_i = \underbrace{A_+ \otimes A_+ \otimes \cdots \otimes A_+}_{i \text{ factors}}$

with $A_+ = A_1 \oplus A_2 \oplus \cdots$

so $(\Bbbk \otimes_A \mathcal{B}_i)_j = 0$ unless $i \leq j$.

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Fixing $j$, then this is exact:

$$0 \to \mathrm{Tor}_j^A(\Bbbk,\Bbbk)_j \to (\Bbbk \otimes_A \mathcal{B}_j)_j \to \cdots \to (\Bbbk \otimes_A \mathcal{B}_1)_j \to (\Bbbk \otimes \mathcal{B}_0)_j \to 0$$

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What exactly does this $j$ strand look like?

# EXAMPLE $j=4$

$$0 \to \mathrm{Tor}^A_4(\Bbbk,\Bbbk)_4 \to (A_+ \otimes A_+ \otimes A_+ \otimes A_+)_4 \to (A_+ \otimes A_+ \otimes A_+)_4$$

$$\to (A_+ \otimes A_+)_4 \to (A_+)_4 \to 0$$

or more concretely ...

$$0 \to \mathrm{Tor}^A_4(\Bbbk,\Bbbk)_4 \to A_1 \otimes A_1 \otimes A_1 \otimes A_1 \to \begin{matrix} A_2 \otimes A_1 \otimes A_1 \\ \oplus \\ A_1 \otimes A_2 \otimes A_1 \\ \oplus \\ A_1 \otimes A_1 \otimes A_2 \end{matrix}$$

$$\to \begin{matrix} A_3 \otimes A_1 \\ \oplus \\ A_2 \otimes A_2 \\ \oplus \\ A_1 \otimes A_3 \end{matrix} \to A_4 \to 0$$

Letting $a_i := \dim_{\Bbbk} A_i$, vanishing of Euler characteristic from exactness gives an inequality:

$$\underbrace{a_1a_1a_1a_1 - \begin{pmatrix} a_2a_1a_1 \\ + \\ a_1a_2a_1 \\ + \\ a_1a_1a_2 \end{pmatrix} + \begin{pmatrix} a_3a_1 \\ + \\ a_2a_2 \\ + \\ a_1a_3 \end{pmatrix} - a_4}_{\substack{\displaystyle\sum_{\substack{\text{compositions} \\ \underline{i}=(i_1,i_2,\ldots,i_\ell) \\ \text{of } 4}} (-1)^{4-\ell} a_{i_1}a_{i_2}\cdots a_{i_\ell}}} = \dim_{\Bbbk} \operatorname{Tor}_4^A(\Bbbk,\Bbbk)_4 \geq 0$$

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**COROLLARY** A Koszul $\Rightarrow$

$$\forall j \geq 0 \quad \sum_{\substack{\text{compositions} \\ \underline{i}=(i_1,i_2,\ldots,i_\ell) \\ \text{of } j}} (-1)^{j-\ell} a_{i_1}a_{i_2}\cdots a_{i_\ell} \geq 0$$

Using Backelin's criterion, Polishchuk & Positselski (2005) proved more generally:

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**THEOREM**: $A$ Koszul $\Rightarrow$ for all compositions $\underline{j} = (j_1, j_2, \ldots, j_m)$, the subcomplex of $\Bbbk \otimes_A B_\bullet$ that starts with

$$A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}$$

is exact except at the left end

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**EXAMPLE** For $A$ Koszul and any $\underline{j} = (b,c,d,e)$, this is exact:

$$0 \to \ker(d_4) \to A_b \otimes A_c \otimes A_d \otimes A_e \xrightarrow{d_4} \begin{array}{c} A_{b+c} \otimes A_d \otimes A_e \\ \oplus \\ A_b \otimes A_{c+d} \otimes A_e \\ \oplus \\ A_b \otimes A_c \otimes A_{d+e} \end{array}$$

$$\xrightarrow{d_3} \begin{array}{c} A_{b+c+d} \otimes A_e \\ \oplus \\ A_{b+c} \otimes A_{d+e} \\ \oplus \\ A_b \otimes A_{c+d+e} \end{array} \xrightarrow{d_2} A_{a+b+c+d} \to 0$$

(Euler characteristic)

**COROLLARY:** $A$ Koszul and $a_i := \dim_{\mathbb{k}} A_i$

$\Rightarrow$ for all compositions $\underline{j} = (j_1, j_2, \ldots, j_m)$

$$\sum_{\substack{\text{compositions} \\ \underline{i} = (i_1, i_2, \ldots, i_\ell) \\ \text{that coarsen } \underline{j}}} (-1)^{m-\ell} a_{i_1} a_{i_2} \cdots a_{i_\ell} \quad \left( = \dim_{\mathbb{k}} \ker(d_m) \right) \geq 0$$

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**EXAMPLE**

$$a_b a_c a_d a_e - \begin{pmatrix} a_{b+c} a_d a_e \\ + \\ a_b a_{c+d} a_e \\ + \\ a_b a_c a_{d+e} \end{pmatrix} + \begin{pmatrix} a_{b+c+d} a_e \\ + \\ a_{b+c} a_{d+e} \\ + \\ a_b a_{c+d+e} \end{pmatrix} - a_{b+c+d+e} \geq 0$$

These inequalities are among those that appear in theory of unimodality, log-concavity and totally positivity / Polya frequency sequences.

EXAMPLES (compare with Problem 3 for $\underline{j} = (1,1,\ldots,1)$)

$\underline{j} = (b,c)$: $a_b a_c - a_{b+c} = \det \begin{bmatrix} a_b & a_{b+c} \\ 1 & a_c \end{bmatrix} \geq 0$

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$\underline{j} = (b,c,d)$: $a_b a_c a_d - \begin{pmatrix} a_{b+c} a_d \\ + \\ a_b a_{c+d} \end{pmatrix} + a_{b+c+d}$

$$= \det \begin{bmatrix} a_b & a_{b+c} & a_{b+c+d} \\ 1 & a_c & a_{c+d} \\ 0 & 1 & a_d \end{bmatrix} \geq 0$$

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$\underline{j} = (b,c,d,e)$:

$$a_b a_c a_d a_e - \begin{pmatrix} a_{b+c} a_d a_e \\ + \\ a_b a_{c+d} a_e \\ + \\ a_b a_c a_{d+e} \end{pmatrix} + \begin{pmatrix} a_{b+c+d} a_e \\ + \\ a_{b+c} a_{d+e} \\ + \\ a_b a_{c+d+e} \end{pmatrix} - a_{b+c+d+e}$$

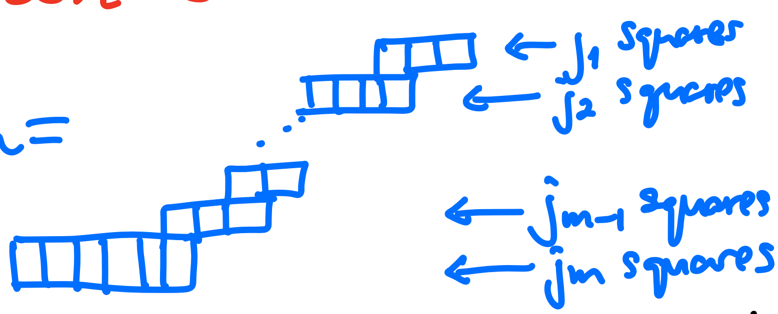
$$= \det \begin{bmatrix} a_b & a_{b+c} & a_{b+c+d} & a_{b+c+d+e} \\ 1 & a_c & a_{c+d} & a_{c+d+e} \\ 0 & 1 & a_d & a_{d+e} \\ 0 & 0 & 1 & a_e \end{bmatrix} \geq 0$$

These are all minor subdeterminants of the infinite Toeplitz matrix $T(\underline{a}) = T(1, a_1, a_2, a_3, \dots)$ for the Hilbert function $a_i = \dim_{\mathbb{k}} A_i$ of $A$.

$$T(\underline{a}) = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 & \cdots & \\ 0 & 1 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & 0 & 1 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & 0 & 1 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

In fact, the minors that are $\geq 0$ from Koszulity are exactly those corresponding to Jacobi-Trudi matrices for ribbon skew Schur functions $s_{\lambda/\mu}$ :

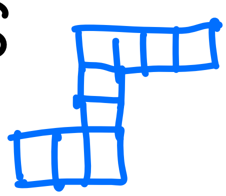
$\lambda/\mu =$




has

$$s_{\lambda/\mu} = \det \begin{bmatrix} h_{j_1} & h_{j_1+j_2} & h_{j_1+j_2+j_3} & & \\ 1 & h_{j_2} & h_{j_2+j_3} & & \\ 0 & 1 & \ddots & & \\ 0 & 0 & \ddots & \ddots & \\ 0 & 0 \cdots 0 & & 1 & h_{j_m} \end{bmatrix}$$

**EXAMPLE:** $\underline{j} = (j_1, j_2, j_3, j_4) = (4,1,1,3)$

$S$  $\begin{matrix}4\\1\\1\\3\end{matrix} = \det \begin{bmatrix} h_4 & h_{1+4} & h_{4+1+1} & h_{4+1+1+3} \\ & h_1 & h_{1+1} & h_{1+1+3} \\ & & h_1 & h_{1+3} \\ & & & h_3 \end{bmatrix}$

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**DEF'N:** $\underline{a} = (1, a_1, a_2, a_3, \_\_) \in \mathbb{R}$ is a Polya frequency sequence of order $r$ ($PF_r$ sequence) if all $r \times r$ minors of $T(\underline{a}) = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & \cdots \\ 0 & 1 & a_1 & a_2 & \cdots \\ 0 & 0 & 1 & a_1 & a_2 \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$ are $\geq 0$.

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$PF_1 \iff$ all $a_i \geq 0$

$PF_2 \iff$ all $\det \begin{bmatrix} a_i & a_{i+r} \\ a_{j-r} & a_j \end{bmatrix} \geq 0$

i.e. $a_i a_j \geq a_{i+r} a_{j-r}$

$\Downarrow$ $i=j$, $r=1$

$\boxed{a_i^2 \geq a_{i+1} a_{i-1}}$ log-concave

$\Downarrow$

$PF_2$ also implies no internal zeroes $\Longrightarrow$ unimodal

DEF'N: $\underline{a} = (1, a_1, a_2, \dots) \in \mathbb{R}$ is a

PF or $PF_\infty$ or Polya frequency sequence

if it is $PF_r$ $\forall r \geq 1$, so all minors of $T(\underline{a})$ are $\geq 0$.

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It generalizes polynomials with roots in $\mathbb{R}_{\leq 0}$:

THEOREM: (Aissen-Edrei-Schoenberg-Whitney 1951)

$\underline{a} = (1, a_1, a_2, \dots)$ is $PF_\infty$

$$\Leftrightarrow \quad A(t) := \sum_{i=0}^{\infty} a_i t^i = e^{\gamma t} \cdot \frac{\prod_{i=1}^{\infty} (1 + \alpha_i t)}{\prod_{j=1}^{\infty} (1 - \beta_j t)}$$

$$\text{with} \begin{cases} \gamma, \alpha_i, \beta_j \geq 0 \\ \sum_i \alpha_i < \infty \\ \sum_j \beta_j < \infty \end{cases}$$

COROLLARY $PF_\infty$ sequences $\underline{a}$ come in pairs $\underline{a} \leftrightsquigarrow \underline{a}^!$ defined by

$$A(t) := \sum_{i=0}^{\infty} a_i t^i \quad \text{and} \quad A^!(t) := \sum_{i=0}^{\infty} a_i^! t^i$$

satisfying $A^!(t) = \dfrac{1}{A(-t)}$

or $A^!(t) \cdot A(-t) = 1$.

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proof:

$$\underline{a} \text{ is } PF_\infty \Leftrightarrow A(t) = e^{\gamma t} \cdot \frac{\prod_{i=1}^{\infty} (1+\alpha_i t)}{\prod_{j=1}^{\infty} (1-\beta_j t)}$$

with $\begin{cases} \gamma, \alpha_i, \beta_j \geq 0 \\ \sum_i \alpha_i < \infty \\ \sum_j \beta_j < \infty \end{cases}$

$$\Leftrightarrow A^!(t) = e^{\gamma t} \cdot \frac{\prod_{j=1}^{\infty} (1+\beta_j t)}{\prod_{i=1}^{\infty} (1-\alpha_i t)} \qquad \blacksquare$$

## GENERAL QUESTIONS:

- Which Koszul algebras $A$ have their Hilbert functions $\underline{a} = (1, a_1, a_2, \ldots)$ where $a_i = \dim_{\mathbb{k}} A_i$ a $PF_\infty$ sequence?
  - ▷ We know some of their Toeplitz matrix minors are $\geq 0$.
  - ▷ We know they will come in Koszul dual pairs $A, A^!$.

- Is there some natural (homological?) strengthening of Koszulity for $A$ that implies $\underline{a}$ is $PF_\infty$?

# SOME MORE SPECIFIC QUESTIONS

- QUESTION: Which digraphs $D \subseteq [n] \times [n]$ have the Hilbert function $\underline{a}$ of the Koszul algebra

  Brenti (Thesis 1988)

  $$A_D = \Bbbk\langle x_1, \ldots, x_n \rangle / (x_i x_j : \underset{\text{not in } D}{i \to j})$$

  forming a $PF_\infty$ sequence.

  (He called them PF-digraphs.)

- CONJECTURE For simple matroids $M$ with maximal building set, the Chow rings $A_M$ have $PF_\infty$ Hilbert functions.

  Ferroni (& Schröter) 2022

  Koszul algebras (Mastroeni-McCullough) 2022

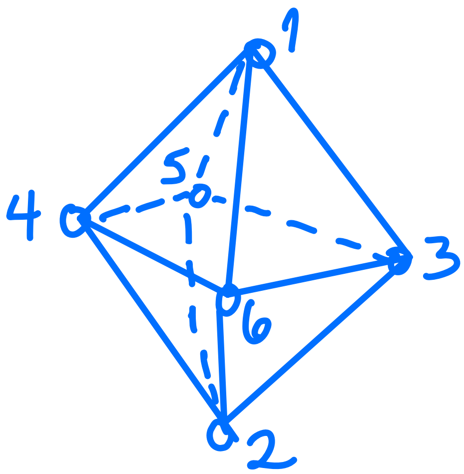
# Koszul Stanley-Reisner rings

**DEF'N:** For a simplicial complex $\Delta$ on vertex set $[n]$, the Stanley-Reisner ring is

$$\Bbbk[\Delta] := \Bbbk[x_1,\dots,x_n] \Big/ \left( \prod_{i\in G} x_i : G \text{ a non-face of } \Delta \right)$$

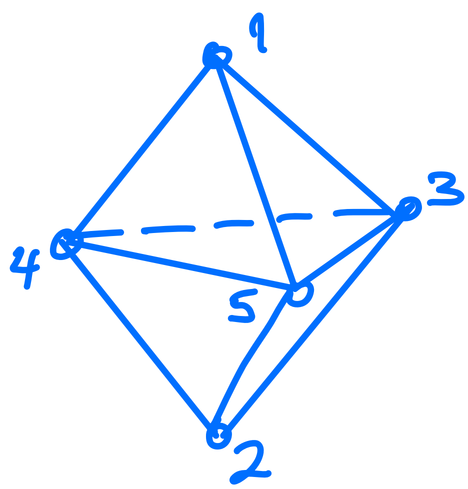
**EXAMPLES**

$\Delta_1 =$ boundary of octahedron




$$\Bbbk[\Delta_1] = \Bbbk[x_1,x_2,x_3,x_4,x_5,x_6] \Big/ (x_1x_2,\ x_3x_4,\ x_5x_6)$$

$\Delta_2 =$ boundary of bipyramid




$$\Bbbk[\Delta_2] = \Bbbk[x_1,x_2,x_3,x_4,x_5] \Big/ (x_1x_2,\ x_3x_4x_5)$$

Since one has

- Koszul $\Rightarrow$ quadratic
- quadratic monomial quotients $\Bbbk[x_1,\dots,x_n]/I$ are Koszul (by Fröberg 1975)

one concludes for Stanley-Reisner rings $\Bbbk[\Delta]$ that

$\Bbbk[\Delta]$ Koszul $\Leftrightarrow$
$\Bbbk[\Delta]$ quadratic $\Leftrightarrow$
$\Delta$ is a flag/clique complex

$\Delta$ is determined by its 1-skeleton:
vertices $F \subseteq [n]$
form a face of $\Delta$
$\Leftrightarrow$ pairwise they span
edges of $\Delta$

Questions on the Hilbert function/series for $A = \Bbbk[\Delta]$

$$\mathrm{Hilb}(\Bbbk[\Delta], t) = 1 + a_1 t + a_2 t^2 + \dots$$

are equivalent to questions about the

f-vector $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$

or h-vector $h(\Delta) = (h_0, h_1, h_2, \dots, h_d)$

where $f_i \overset{\text{DEF'N}}{:=}$ # of i-dimensional faces of $\Delta$:

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(EASY) PROPOSITION:

$$\mathrm{Hilb}(\Bbbk[\Delta], t) = \sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t}\right)^i$$

Take this as DEFINITION of $h(\Delta)$

$$= \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d}$$

COROLLARY:

Flag complexes $\Delta$ have $\Bbbk[\Delta]$ Koszul

$$\Downarrow$$

interesting inequalities

on $f(\Delta), h(\Delta)$

from the Toeplitz minors

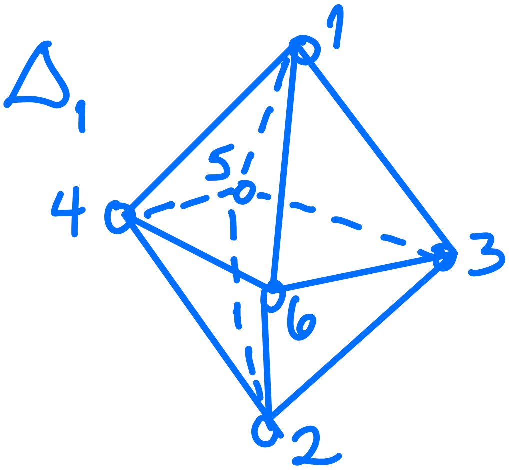
$$\det \begin{bmatrix} a_{j_1} & a_{j_1+j_2} & \cdots & \\ 1 & a_{j_2} & a_{j_2+j_3} & \\ 0 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots\ 0 & 1 & a_{j_m} \end{bmatrix} \geq 0$$

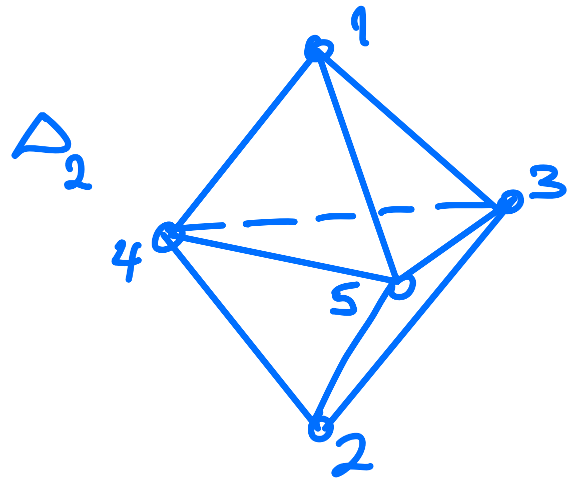
for $\mathrm{Hilb}(\Bbbk[\Delta], t) = 1 + a_1 t + a_2 t^2 + \cdots$

BETTER UNDERSTOOD:

Simplicial $(d-1)$-spheres $\Delta$,

e.g. boundaries of $d$-polytopes




$$f(\Delta_1) = (f_{-1}, f_0, f_1, f_2) = (1, 6, 12, 8)$$

$$f(\Delta_2) = (f_{-1}, f_0, f_1, f_2) = (1, 5, 9, 6)$$

$$h(\Delta_1) = (h_0, h_1, h_2, h_3) = (1, 3, 3, 1)$$

$$h(\Delta_2) = (h_0, h_1, h_2, h_3) = (1, 2, 2, 1)$$

$$\boxed{h_0 = 1\ , \ h_i = h_{d-i}}$$

Dehn-Sommerville equations

are known to give all equations satisfied by $f(\Delta)$ or $h(\Delta)$

In fact, all inequalities

satisfied by $f(\Delta)$ or $h(\Delta)$

for $(d-1)$-spheres are also known;

part of the celebrated

g-THEOREM of

P. McMullen 1971

Billera & Lee 1980

Stanley 1980

Adiprasito 2018

Adiprasito, Papadakis, Petrotou 2020

that completely characterizes

their $f(\Delta)$, $h(\Delta)$ (!)

We are much farther from characterizing $f(\Delta), h(\Delta)$ for flag simplicial spheres.

CONJECTURE: Charney-Davis 1995, Gal 2005

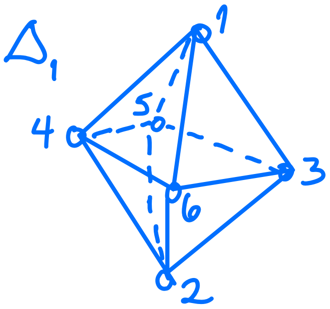
For $\Delta$ a flag simplicial $(d-1)$-sphere, the $\gamma$-vector

$$\gamma(\Delta) = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$$

has $\gamma_i \geq 0$ for all $i$,

where $h_0 + h_1 t + h_2 t^2 + \ldots + h_d t^d = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$

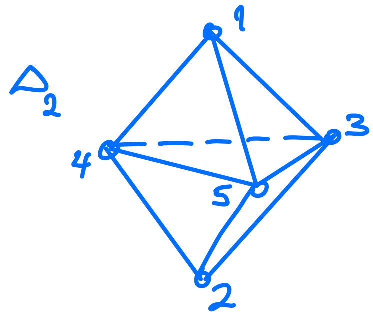
$$= \gamma_0 (1+t)^d + \gamma_1 t (1+t)^{d-2} + \gamma_2 t^2 (1+t)^{d-4} + \ldots$$




a flag sphere

$h(\Delta_1) = (h_0, h_1, h_2, h_3) = (1, 3, 3, 1)$

$\gamma(\Delta_1) = (\gamma_0, \gamma_1) = (1, 0)$




not a flag sphere

$h(\Delta_2) = (h_0, h_1, h_2, h_3) = (1, 2, 2, 1)$

$\gamma(\Delta_2) = (\gamma_0, \gamma_1) = (1, -1)$ $\gamma_1 < 0$

**PROPOSITION** Gal 2005 Whenever $\Bbbk[\Delta]$ has $PF_\infty$ Hilbert function, then $\gamma(\Delta)$ has $\gamma_i \geq 0$ $\forall i$.

(But some flag simplicial spheres fail to have $PF_\infty$ Hilbert function for $\Bbbk[\Delta]$)

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**QUESTIONS:**

(a) Which flag complexes $\Delta$ have $PF_\infty$ Hilbert functions for $\Bbbk[\Delta]$?

(b) Which flag spheres $\Delta$ have $PF_\infty$ Hilbert functions for $\Bbbk[\Delta]$?

(c) Does Koszulity of $\Bbbk[\Delta]$ have any role to play in proving Charney-Davis-Gal CONJECTURE $\gamma_i \geq 0$ for flag spheres?

[ WARNING: Koszul + Gorenstein rings can have $\gamma_i < 0$ (D'Alì - Venturello 2021) ]