MPI Leipzig Summer School
in Algebraic Combinatorics

The Koszul property in Algebraic Combinatorics

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Lectures

1: Motivation, definition of Koszul algebras
 – the monomial case

2: Methods for proving Koszulity
 and more examples

3: Bar complex, topology,
 and inequalities

4: Group actions

Group actions on Koszul algebras

Many of our favorite Koszul algebras

$A = \Bbbk\langle x_1, \ldots, x_n \rangle / I$ with $I = (I_2)$

naturally have symmetries:

some subgroup $G < GL_n(\Bbbk)$

acts $\Bbbk$-linearly on $V = \mathrm{span}_{\Bbbk}\{x_1, \ldots, x_n\}$,

- preserving I_2 setwise,
- so preserving $I = (I_2)$, and
- acting via graded $\Bbbk$-algebra automorphisms on A.

Each graded component A_d in $A = \bigoplus_{d=0}^{\infty} A_d$ becomes a representation of G,

or a $\Bbbk G$-module.

How to keep track of them?

For simplicity, assume G is finite and $\Bbbk = \mathbb{C}$.

Then a $\Bbbk G$-module $\mathcal{U}$ (with $\dim_{\Bbbk} \mathcal{U}$ finite) is determined up to isomorphism by its

character
$$\begin{aligned} \chi_{\mathcal{U}} : G &\longrightarrow \Bbbk \\ g &\longmapsto \mathrm{Trace}(g|_{\mathcal{U}}) \end{aligned}$$

which is a class function $\chi_{\mathcal{U}}(hgh^{-1}) = \chi_{\mathcal{U}}(g)$.

$\chi_{\mathcal{U}}$ lives in a ring $\mathrm{Rep}(G)$ of all such class functions $f : G \to \Bbbk$ with pointwise $+$ and $\times$ corresponding to $\mathcal{U}_1 \oplus \mathcal{U}_2$ and $\mathcal{U}_1 \otimes \mathcal{U}_2$ for $\Bbbk G$-modules:

- $\chi_{\mathcal{U}_1 \oplus \mathcal{U}_2} = \chi_{\mathcal{U}_1} + \chi_{\mathcal{U}_2}$
- $\chi_{\mathcal{U}_1 \otimes \mathcal{U}_2} = \chi_{\mathcal{U}_1} \cdot \chi_{\mathcal{U}_2}$

EXAMPLE

$G = \mathfrak{S}_3$ has irreducible/simple $\Bbbk\mathfrak{S}_3$-modules $\chi_{(3)}$, $\chi_{(2,1)}$, $\chi_{(1,1,1)}$ indexed by partitions λ of 3.

Character table

conjugacy classes:

	e	(ij)	(ijk)
$\chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1,1,1)}$	1	-1	1

EXAMPLE (KRONECKER PRODUCT) CALCULATION:

$\chi_{(2,1)} \cdot \chi_{(2,1)}$ $= \chi_{(2,1)} \otimes \chi_{(2,1)}$ $= \chi_{(3)} + \chi_{(2,1)} + \chi_{(1,1,1)}$	4	0	1

Introduce for a Koszul algebra A with G-symmetry its ...

DEF'N: G-equivariant Hilbert series

$$\mathrm{Hilb}_G(A,t) := \sum_{d=0}^{\infty} \chi_{A_d} \cdot t^d \in \mathrm{Rep}(G)[[t]]$$

PROPOSITION: A Koszul $\Rightarrow$

$$\boxed{\mathrm{Hilb}_G(A^!,t) \cdot \mathrm{Hilb}_G(A^*,-t) = 1}$$

where $A^* :=$ graded dual $\bigoplus_{d=0} (A_d)^*$

with $U^* =$ contragredient $\Bbbk G$-module

where $g(\varphi)(u) = \varphi(g^{-1}u)$

and $\chi_{U^*}(g) = \chi_U(g^{-1})$.

In other words, $\forall d \geq 1$, one has

$$\boxed{\chi_{A^!_d} - \chi_{A^!_{d-1}} \cdot \chi_{A^*_1} + \chi_{A^!_{d-2}} \cdot \chi_{A^*_2} - \cdots \pm \chi_{A^*_d} = 0}$$

defining $\chi_{A^!_d}$ recursively in terms of $\chi_{A_1}, \ldots, \chi_{A_d}$.

PROPOSITION: A Koszul $\Rightarrow$

$$\boxed{\mathrm{Hilb}_G(A^!, t)\cdot \mathrm{Hilb}_G(A^*, -t) = 1}$$

proof: Recall Priddy's resolution of $\Bbbk$:

$$\cdots \to A\otimes(A^!_3)^* \to A\otimes(A^!_2)^* \to A\otimes(A^!_1)^* \to A \to \Bbbk \to 0$$

It turns out to be G-equivariant.

Its d^{th} graded component for $d\geq 1$ is

$$0 \to (A^!_d)^* \to A_1\otimes(A^!_{d-1})^* \to \cdots \to A_{d-1}\otimes(A^!_1)^* \to A_d \to 0$$

Exactness for this complex of $\Bbbk G$-modules gives

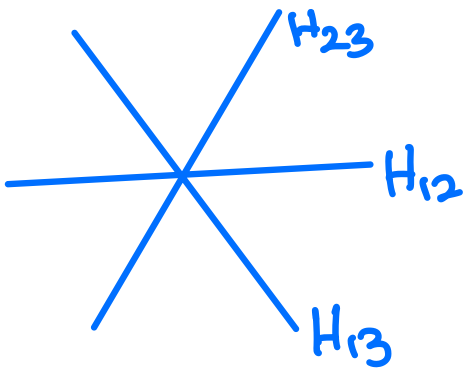
$$\chi^*_{A^!_d} - \chi_{A_1}\cdot\chi^*_{A^!_{d-1}} + \cdots \pm \chi_{A_{d-1}}\cdot\chi^*_{A^!_1} \mp \chi_{A_d} = 0$$

$$\Big\downarrow \; \chi_u \mapsto \chi_u^*$$

$$\chi_{A^!_d} - \chi^*_{A_1}\cdot\chi_{A^!_{d-1}} + \cdots \pm \chi^*_{A_{d-1}}\cdot\chi_{A^!_1} \mp \chi^*_{A_d} = 0$$

$\blacksquare$

EXAMPLE Reflection arrangement for $\mathfrak{S}_3$ is $\mathcal{H} = \{ \underset{x_1 = x_2}{H_{12}}, \underset{x_1 = x_3}{H_{13}}, \underset{x_2 = x_3}{H_{23}} \}$



and has symmetry of $\mathfrak{S}_3$ permuting indices:

$$\sigma(H_{ij}) = H_{\sigma(i)\sigma(j)}$$

$$OS(\mathcal{H}) = \Lambda_{\mathbb{k}}\langle x_{12}, x_{13}, x_{13} \rangle \Big/ \Big(\partial \underbrace{(x_{12} x_{13} x_{23})}_{x_{12}x_{13} - x_{12}x_{23} + x_{13}x_{23}} \Big)$$

$$VG(\mathcal{H}) = \mathbb{k}[x_{12}, x_{13}, x_{23}] \Big/ \Big(x_{ij}^2, \partial_{\pm} \underbrace{(x_{12} x_{13} x_{23})}_{x_{12}x_{13} - x_{12}x_{23} + x_{13}x_{23}} \Big)$$

$$\begin{array}{lccc} \text{degree:} & 0 & 1 & 2 \\ = \mathrm{span}_{\mathbb{k}} \Big\{ & 1, & \begin{matrix} x_{12} \\ x_{13} \\ x_{23} \end{matrix}, & \begin{matrix} x_{12}x_{13} \\ x_{12}x_{23} \end{matrix} \Big\} \end{array}$$

$$\mathrm{Hilb}_{\mathfrak{S}_3}(OS(\mathcal{H}), t) = \chi_{(3)} + (\chi_{(3)} + \chi_{(2,1)})\,t + \chi_{(2,1)} \cdot t^2$$

$$\mathrm{Hilb}_{\mathfrak{S}_3}(OS(\mathcal{H}), t) = \chi_{(3)} + (\chi_{(1,1,1)} + \chi_{(2,1)})\,t + \chi_{(2,1)} \cdot t^2$$

Let's compute the first few $A^!_d$ for $A = OS(\mathcal{H})$, knowing $A = A_0 \oplus A_1 \oplus A_2$:

$$\begin{array}{ccccc} A_0 & \oplus & A_1 & \oplus & A_2 \\ \chi_{(3)} & & \chi_{(3)} + \chi_{(2,1)} & & \chi_{(2,1)} \end{array}$$

$$\chi_{A^!_0} = \chi^*_{A_0} = \chi^*_{(3)} = \chi_{(3)}$$

$$\chi_{A^!_1} = \chi^*_{A_1} = \chi^*_{(3)} + \chi^*_{(2,1)} = \chi_{(3)} + \chi_{(2,1)}$$

$$\begin{aligned}
\chi_{A^!_2} = \chi^*_{A_1} \cdot \chi_{A^!_1} - \chi^*_{A_2} &= (\chi_{(3)} + \chi_{(2,1)})(\chi_{(3)} + \chi_{(2,1)}) - \chi_{(2,1)} \\
&= \chi_{(3)} + 2\chi_{(2,1)} + \chi_{(2,1)} \cdot \chi_{(2,1)} - \chi_{(2,1)} \\
&= \chi_{(3)} + 2\chi_{(2,1)} + (\chi_{(2,1)} + \chi_{(3)} + \chi_{(1,1,1)}) - \chi_{(2,1)} \\
&= 2\chi_{(3)} + 2\chi_{(2,1)} + \chi_{(1,1,1)}
\end{aligned}$$

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similar calculation

$$\chi_{A^!_3} = 3\chi_{(3)} + 5\chi_{(2,1)} + 2\chi_{(1,1,1)}$$

$\vdots$

PROBLEM: (Thrall 1942)

For $A = \mathcal{VG}(\mathcal{H})$ or $OS(\mathcal{H})$

with $\mathcal{H}$ the $\mathfrak{S}_n$-reflection arrangement,

decompose each A_d explicitly into

irreducible $k\mathfrak{S}_n$-modules.

Generating function and plethysm

formulas are known, but

no explicit decomposition.

PROBLEM: (Almousa-R.-Sundaram 2024)

Do the same for each $A_d^!$

when $A = \mathcal{VG}(\mathcal{H})$ or $OS(\mathcal{H})$ as above.

The A_d and $A_d^!$ share some

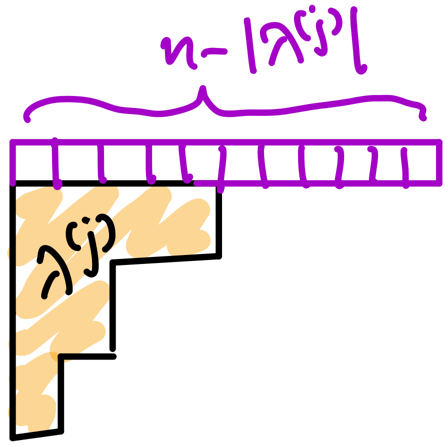
interesting features ...

Representation Stability

DEF'N: (Church & Farb 2013) A sequence of $\Bbbk \mathfrak{S}_n$-modules $\{V_n\}_{n=1,2,3,\dots}$ are called representation stable if $\exists$ some N and partitions $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(t)}$ and multiplicities $c_1, c_2, \dots, c_t$ such that

$\forall n \geq N$,

$$\chi_{V_n} = \sum_{j=1}^{t} c_j \chi$$



EXAMPLE (Church & Farb 2013) Fixing $d \geq 0$, and letting

$$A(n) = OS(\mathcal{H}) \text{ or } VG(\mathcal{H})$$

for $\mathcal{H}$ the $\mathfrak{S}_n$-reflection arrangement

$\{A(n)_d\}_{n=1,2,3,\ldots}$ is representation stable.

PROPOSITION (Almousa-R.-Sundaram 2024)

For any sequence of Koszul algebras $\{A(n)\}_{n=1,2,\ldots}$ having $\mathfrak{S}_n$-symmetry, if $\forall d \geq 0$, $\{A(n)_d\}_{n=1,2,\ldots}$ is representation stable then $\forall d \geq 0$, $\{A(n)^!_d\}_{n=1,2,\ldots}$ is also representation stable.

(Consequently, this holds for the $\mathfrak{S}_n$-reflection arrangements $\mathcal{H}$ and $A(n) := OS(\mathcal{H}), VG(\mathcal{H})$)

PROPOSITION

For any sequence of Koszul algebras $\{A(n)\}_{n=1,2,\ldots}$ having $\mathfrak{S}_n$-symmetry,
if $\forall d \geq 0$, $\{A(n)_d\}_{n=1,2,\ldots}$ is representation stable
then $\forall d \geq 0$, $\{A(n)^!_d\}_{n=1,2,\ldots}$ is also representation stable.

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proof: Induct on d.

$$\chi_{A(n)^!_d} = \sum_{i=1}^{d} \underbrace{\chi^*_{A(n)_i}}_{\text{rep. stable by Church \& Farb}} \cdot \underbrace{\chi_{A(n)^!_{d-i}}}_{\text{rep. stable by induction on } d}$$

Koszul recurrence for $A^!_d$

rep. stable by Murnaghan's Stability Theorem: 1938

stabilizes for large n

Branching rules

Both algebras
$$A(n) = OS(\mathcal{H}) \quad \text{or} \quad \mathcal{VG}(\mathcal{H})$$
for the $\mathfrak{S}_n$-reflection arrangements
have the same Hilbert series:

$$\mathrm{Hilb}(A(n), t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$
$$= \sum_{i=1}^{n} \underbrace{c(n, n-i)}_{\substack{\text{signless Stirling} \\ \text{numbers of the 1st kind}}} t^i$$

$c(n,k) = \#$ permutations in $\mathfrak{S}_n$ with k cycles

EXAMPLE

$$A(3) = \mathrm{span}_{\Bbbk}\left\{ 1, \quad \begin{array}{c} x_{12} \\ x_{13} \\ x_{23} \end{array}, \quad \begin{array}{c} x_{12}x_{13} \\ x_{12}x_{23} \end{array} \right\}$$

$(1)(2)(3)$	$(12)(3)$ $(13)(2)$ $(23)(1)$	(123) (132)
$c(3,3)=1$	$c(3,2)=3$	$c(3,1)=2$

Consequently,

$$\mathrm{Hilb}(A(n)^!, t) = \frac{1}{\mathrm{Hilb}(A(n), -t)}$$

$$= \frac{1}{(1-t)(1-2t)(1-3t)\cdots(1-(n-1)t)}$$

$$= \sum_{i=0}^{\infty} \underbrace{S(n-1+i, n-1)}_{\text{Stirling numbers of the 2nd kind}} t^i$$

$S(n,k) = \#$ of set partitions of $[n]$ into k blocks

Recall the Stirling number recurrences

$$c(n,k) = c(n-1, k-1) + (n-1) \cdot c(n-1, k)$$

$$S(n,k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

It turns out that they are hiding (Koszul dual) branching rules ...

Let $A(n) = OS(\mathcal{H})$ or $VG(\mathcal{H})$
for the $\mathfrak{S}_n$-reflection arrangement $\mathcal{H}$.

THEOREM: (Sundaram 1994, 2020)

The recurrence $c(n,k) = c(n-1,k-1) + (n-1) \cdot c(n-1,k)$
lifts to a branching rule for restricting from
$\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$-representations:

$$\chi_{A(n)_i}\Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \chi_{A(n-1)_i} + \chi_{def} \cdot \chi_{A(n-1)_{i-1}}$$

defining $\mathfrak{S}_{n-1}$-representation
via $(n-1)\times(n-1)$ permutation matrices

THEOREM: [Almousa-R.-Sundaram 2024]

The recurrence $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$
lifts to this Koszul dual branching rule:

$$\chi_{A(n)^!_i}\Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \chi_{A(n-1)^!_i} + \chi_{def} \cdot \left[\chi_{A(n)^!_{i-1}}\right]\Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$$

Interestingly, these two branching rules for $A(n)_i$ and $A(n)^!_i$ ...

- can be shown formally equivalent as a consequence of the inclusion of Koszul algebras

$$\begin{array}{ccc} A(n-1) & \subset & A(n) \\ \circlearrowleft & & \circlearrowleft \\ \mathfrak{S}_{n-1} & & \mathfrak{S}_n \end{array}$$

- both generalize to branching rules for $A = OS(\mathcal{H})$ or $VG(\mathcal{H})$ and their Koszul duals $A^!$ whenever $\mathcal{H}$ is supersolvable.

Thank you

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for your

attention!