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Infinite graded free resolutions

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This paper is a survey on infinite graded free resolutions. We discuss their numerical invariants: Betti numbers, regularity, slope (rate), and rationality of Poincaré series. We also cover resolutions over complete intersections, Golod rings and Koszul rings.

1. Introduction

This paper is an expanded version of three talks given by I. Peeva during the Introductory Workshop in Commutative Algebra at MSRI in August 2013. It is a survey on infinite graded free resolutions, and includes many open problems and conjectures.

The idea of associating a free resolution to a finitely generated module was introduced in two famous papers by Hilbert [1890; 1893]. He proved Hilbert's Syzygy Theorem (Theorem 4.9), which says that the minimal free resolution of every finitely generated graded module over a polynomial ring is finite. Since then, there has been a lot of progress on the structure and properties of finite free resolutions. Much less is known about the properties of infinite free resolutions. Such resolutions occur abundantly since most minimal free resolutions over a graded nonlinear quotient ring of a polynomial ring are infinite. The challenges in studying them come from:

- The structure of infinite minimal free resolutions can be quite intricate.
- The methods and techniques for studying finite free resolutions usually do not work for infinite free resolutions. As noted by Avramov [1992]: "there seems to be a need for a whole new arsenal of tools."
- Computing examples with computer algebra systems is usually useless since we can only compute the beginning of a resolution and this is nonindicative for the structure of the entire resolution.

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Most importantly, there is a need for new insights and guiding conjectures. Coming up with reasonable conjectures is a major challenge on its own.

Because of space limitation, we have covered topics and results selectively. We survey open problems and results on Betti numbers (Section 4), resolutions over complete intersections (Section 5), rationality of Poincaré series and Golod rings (Section 6), regularity (Section 7), Koszul rings (Section 8), and slope (Section 9).

Some lectures and expository papers on infinite free resolutions are given in [Avramov 1992; 1998; Conca et al. 2013; Dao 2013; Fröberg 1999; Gulliksen and Levin 1969; Polishchuk and Positselski 2005; Peeva 2011; 2007].

2. Notation

Throughout, we use the following notation: $S = k[x_1, \dots, x_n]$ stands for a polynomial ring over a field k. We consider a quotient ring $R = S/I$, where I is an ideal in S. Furthermore, M stands for a finitely generated R-module. All modules are finitely generated unless otherwise stated. For simplicity, in the examples we may use x, y, z, etc. instead of $x_1, \dots, x_n$.

3. Free resolutions

Definition 3.1. *A left complex $\boldsymbol{G}$ of finitely generated free modules over R is a sequence of homomorphisms of finitely generated free R-modules*

$$\boldsymbol{G}: \quad \cdots \longrightarrow G_i \xrightarrow{d_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0,$$

such that $d_{i-1}d_i = 0$ for all i. The collection of maps $d = \{d_i\}$ is called the differential of $\boldsymbol{G}$. The complex is sometimes denoted $(\boldsymbol{G}, d)$. The i-th Betti number of $\boldsymbol{G}$ is the rank of the module G_i. The homology of $\boldsymbol{G}$ is $\mathrm{H}_i(\boldsymbol{G}) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1})$. The complex is exact at G_i, or at step i, if $\mathrm{H}_i(\boldsymbol{G}) = 0$. We say that $\boldsymbol{G}$ is acyclic if $\mathrm{H}_i(\boldsymbol{G}) = 0$ for all $i > 0$. A free resolution of a finitely generated R-module M is an acyclic left complex of finitely generated free R-modules

$$\boldsymbol{F}: \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0,$$

such that $M \cong F_0/\mathrm{Im}(d_1)$.

The idea to associate a resolution to a finitely generated R-module M was introduced in Hilbert's famous papers [1890; 1893]. The key insight is that *a* *free resolution is a description of the structure of M* since it has the form

$$\cdots \to F_2 \xrightarrow{\begin{pmatrix}\text{a generating}\\ \text{system of the}\\ \text{relations on the}\\ \text{relations in } d_1\end{pmatrix}} F_1 \xrightarrow{\begin{pmatrix}\text{a generating}\\ \text{system of the}\\ \text{relations on the}\\ \text{generators of } M\end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix}\text{a system of}\\ \text{generators}\\ \text{of } M\end{pmatrix}} M \to 0.$$

Therefore, the properties of M can be studied by understanding the properties and structure of a free resolution.

From a modern point of view, building a resolution amounts to repeatedly solving systems of polynomial equations. This is illustrated in Example 3.4 and implemented in Construction 3.3. It is based on the following observation.

Observation 3.2. If we are given a homomorphism $R^p \xrightarrow{A} R^q$, where A is the matrix of the map with respect to fixed bases, then describing the module $\mathrm{Ker}(A)$ is equivalent to solving the system of R-linear equations

$$A \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix} = 0$$

over R, where $Y_1, \dots, Y_p$ are variables that take values in R.

Construction 3.3. We will show that every finitely generated R-module M has a free resolution. By induction on homological degree we will define the differential so that its image is the kernel of the previous differential.

Step 0: Set $M_0 = M$. We choose generators $m_1, \dots, m_r$ of M_0 and set $F_0 = R^r$. Let $f_1, \dots, f_r$ be a basis of F_0, and define

$$\begin{aligned} d_0 :\ & F_0 \to M, \\ & f_j \mapsto m_j \quad \text{for } 1 \le j \le r. \end{aligned}$$

Step i: By induction, F_{i-1} and d_{i-1} are defined. Set $M_i = \mathrm{Ker}(d_{i-1})$. We choose generators $w_1, \dots, w_s$ of M_i and set $F_i = R^s$. Let $g_1, \dots, g_s$ be a basis of F_i and define

$$\begin{aligned} d_i :\ & F_i \to M_i \subset F_{i-1}, \\ & g_j \mapsto w_j \quad \text{for } 1 \le j \le s. \end{aligned}$$

By construction $\mathrm{Ker}(d_{i-1}) = \mathrm{Im}(d_i)$.

The process described above may never terminate; in this case, we have an infinite free resolution.

Example 3.4. Let $S = k[x, y, z]$ and $J = (x^2z, xyz, yz^6)$. We will construct a free resolution of S/J over S.

Step 0: Set $F_0 = S$ and let $d_0 : S \to S/J$.

Step 1: The elements x^2z, xyz, yz^6 are generators of $\mathrm{Ker}(d_0)$. Denote by f_1, f_2, f_3 a basis of $F_1 = S^3$. Defining d_1 by

$$f_1 \mapsto x^2z, \quad f_2 \mapsto xyz, \quad f_3 \mapsto yz^6,$$

we obtain the beginning of the resolution:

$$F_1 = S^3 \xrightarrow{\begin{pmatrix} x^2z & xyz & yz^6 \end{pmatrix}} F_0 = S \;\to\; S/J \to 0.$$

Step 2: Next, we need to find generators of $\mathrm{Ker}(d_1)$. Equivalently, we have to solve the equation $d_1(X_1 f_1 + X_2 f_2 + X_3 f_3) = 0$, or

$$X_1x^2z + X_2xyz + X_3yz^6 = 0,$$

where X_1, X_2, X_3 are indeterminates that take values in R. Computing with the computer algebra system Macaulay2, we find that $-yf_1 + xf_2$ and $-z^5 f_2 + xf_3$ are homogeneous generators of $\mathrm{Ker}(d_1)$. Denote by g_1, g_2 a basis of $F_2 = S^2$. Defining d_2 by

$$g_1 \mapsto -yf_1 + xf_2, \quad g_2 \mapsto -z^5 f_2 + xf_3,$$

we obtain the next step in the resolution:

$$F_2 = S^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -z^5 \\ 0 & x \end{pmatrix}} F_1 = S^3 \xrightarrow{\begin{pmatrix} x^2z & xyz & yz^6 \end{pmatrix}} F_0 = S.$$

Step 3: Next, we need to find generators of $\mathrm{Ker}(d_2)$. Equivalently, we have to solve $d(Y_1g_1 + Y_2g_2) = 0$; so we get the system of equations

$$\begin{aligned} -Y_1y &= 0, \\ Y_1x - Y_2z^5 &= 0, \\ Y_2x &= 0, \end{aligned}$$

where Y_1, Y_2 are indeterminates that take values in R. Clearly, the only solution is $Y_1 = Y_2 = 0$, so $\mathrm{Ker}(d_2) = 0$.

Thus, we obtain the free resolution

$$0 \to S^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -z^5 \\ 0 & x \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^2z & xyz & yz^6 \end{pmatrix}} S.$$

The next theorem shows that any two free resolutions of a finitely generated R-module are homotopy equivalent.

Theorem 3.5 (see [Peeva 2011, Theorem 6.8]). *For every two free resolutions $\boldsymbol{F}$ and $\boldsymbol{F}'$ of a finitely generated R-module M there exist homomorphisms of complexes $\varphi : \boldsymbol{F} \to \boldsymbol{F}'$ and $\psi : \boldsymbol{F}' \to \boldsymbol{F}$ inducing* $\mathrm{id} : M \to M$, *such that $\varphi\psi$ is homotopic to $\mathrm{id}_{\boldsymbol{F}'}$ and $\psi\varphi$ is homotopic to $\mathrm{id}_{\boldsymbol{F}}$.*

4. Minimality and Betti numbers

In the rest of the paper, we will use the *standard grading* of the polynomial ring $S = k[x_1, \dots, x_n]$. Set $\deg(x_i) = 1$ for each i. A monomial $x_1^{a_1} \dots x_n^{a_n}$ has *degree* $a_1 + \cdots + a_n$. For $i \in \mathbb{N}$, we denote by S_i the k-vector space spanned by all monomials of degree i. In particular, $S_0 = k$. A polynomial f is called *homogeneous* if $f \in S_i$ for some i, and in this case we say that f has *degree* i or that f is a *form* of degree i and write $\deg(f) = i$. By convention, 0 is a homogeneous element with arbitrary degree. Every polynomial $f \in S$ can be written uniquely as a finite sum of nonzero homogeneous elements, called the *homogeneous components of* f. This provides a direct sum decomposition $S = \bigoplus_{i \in \mathbb{N}} S_i$ of S as a k-vector space with $S_i S_j \subseteq S_{i+j}$. A proper ideal J in S is called *graded* if it has a system of homogeneous generators, or equivalently, $J = \bigoplus_{i \in \mathbb{N}} (S_i \cap J)$; the k-vector spaces $J_i = S_i \cap J$ are called the *graded components* of the ideal J.

If I is a graded ideal, then the quotient ring $R = S/I$ inherits the grading from S, so $R_i \cong S_i/I_i$ for all i. Furthermore, an R-module M is called *graded* if it has a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as a k-vector space and $R_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. The k-vector spaces M_i are called the *graded components* of M. An element $m \in M$ is called *homogeneous* if $m \in M_i$ for some i, and in this case we say that m has degree i and write $\deg(m) = i$. A homomorphism between graded R-modules $\varphi : M \to N$ is called *graded of degree* 0 if $\varphi(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$. It is easy to see that the kernel, cokernel, and image of a graded homomorphism are graded modules.

We use the following convention: For $p \in \mathbb{Z}$, the module M *shifted p degrees* is denoted by $M(-p)$ and is the graded R-module such that $M(-p)_i = M_{i-p}$ for all i. In particular, the generator $1 \in R(-p)$ has degree p since $R(-p)_p = R_0$.

Notation 4.1. In the rest of the paper, we assume that S is standard graded, I is a graded ideal in S, the quotient ring $R = S/I$ is graded, and M is a finitely generated graded R-module.

A complex $\boldsymbol{F}$ of finitely generated graded free modules

$$\boldsymbol{F} : \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots$$

is called *graded* if the modules F_i are graded and each d_i is a graded homomorphism of degree 0. In this case, the module $\boldsymbol{F}$ is actually bigraded since we have a homological degree and an internal degree, so we may write

$$F_i = \bigoplus_{j\in\mathbb{Z}} F_{i,j} \quad \text{for each } i.$$

An element in $F_{i,j}$ is said to have *homological degree* i and *internal degree* j.

Example 4.2. The graded version of the resolution of $S/(x^2z, xyz, yz^6)$ in Example 3.4 is

$$0 \longrightarrow S(-4)\oplus S(-8) \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -z^5 \\ 0 & x \end{pmatrix}} S(-3)^2 \oplus S(-7) \xrightarrow{\begin{pmatrix} x^2z & xyz & yz^6 \end{pmatrix}} S.$$

A graded free resolution can be constructed following Construction 3.3 by choosing a homogeneous set of generators of the kernel of the differential at each step.

The graded component in a fixed internal degree j of a graded complex is a subcomplex which consists of k-vector spaces; see [Peeva 2011, 3.7]. Thus, the grading yields the following useful criterion for exactness: a graded complex is exact if and only if each of its graded components is an exact sequence of k-vector spaces.

Another important advantage of having a grading is that Nakayama's Lemma holds, leading to the foundational Theorem 4.4. The proof of Nakayama's Lemma for local rings is longer (see [Matsumura 1989, Section 2]); for graded rings, the lemma follows immediately from the observation that a finitely generated R-module has a minimal generator of minimal degree, and we include the short proof.

Nakayama's Lemma 4.3. *If J is a proper graded ideal in R and M is a finitely generated graded R-module such that $M = JM$, then $M = 0$.*

Proof. Suppose that $M \neq 0$. We choose a finite minimal system of homogeneous generators of M. Let m be an element of minimal degree in that system. It follows that $M_j = 0$ for $j < \deg(m)$. Since J is a proper ideal, we conclude that every homogeneous element in JM has degree strictly greater than $\deg(m)$. This contradicts to $m \in M = JM$. ☐

Theorem 4.4 (see [Peeva 2011, Theorem 2.12]). *Let M be a finitely generated graded R-module. Consider the graded k-vector space $\overline{M} = M/(x_1, \dots, x_n)M$. Homogeneous elements $m_1, \dots, m_r \in M$ form a minimal system of homogeneous* *generators of M if and only if their images in $\overline{M}$ form a basis. Every minimal system of homogeneous generators of M has* $\dim_k(\overline{M})$ *elements.*

In particular, Theorem 4.4 shows that every minimal system of generators of M has the same number of elements.

Definition 4.5. *A graded free resolution of a finitely generated graded R-module M is minimal if*

$$d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i \quad \text{for all } i \geq 0.$$

This means that no invertible elements (*nonzero constants*) *appear in the differential matrices.*

The word "minimal" refers to the properties in the next two results. On the one hand, Theorem 4.6 shows that minimality means that at each step in Construction 3.3 we make an optimal choice, that is, we choose a minimal system of generators of the kernel in order to construct the next differential. On the other hand, Theorem 4.7 shows that minimality means that we have a smallest resolution which lies (as a direct summand) inside any other resolution of the module.

Theorem 4.6 (see [Peeva 2011, Theorem 3.4]). *The graded free resolution constructed in Construction* 3.3 *is minimal if and only if at each step we choose a minimal homogeneous system of generators of the kernel of the differential. In particular*, *every finitely generated graded R-module has a minimal graded free resolution.*

Theorem 4.7 (see [Peeva 2011, Theorem 3.5]). *Let M be a finitely generated graded R-module*, *and $\boldsymbol{F}$ be a minimal graded free resolution of M. If $\boldsymbol{G}$ is any graded free resolution of M*, *we have a direct sum of complexes $\boldsymbol{G} \cong \boldsymbol{F} \oplus \boldsymbol{P}$ for some complex $\boldsymbol{P}$*, *which is a direct sum of short trivial complexes*

$$0 \longrightarrow R(-p) \xrightarrow{1} R(-p) \longrightarrow 0$$

possibly placed in different homological degrees.

The minimal graded free resolution of M is unique up to an isomorphism and has the form

$$\cdots \to F_2 \xrightarrow{\begin{pmatrix}\text{a \textit{minimal}}\\ \text{generating}\\ \text{system of the}\\ \text{relations on the}\\ \text{relations in } d_1\end{pmatrix}} F_1 \xrightarrow{\begin{pmatrix}\text{a \textit{minimal}}\\ \text{generating}\\ \text{system of the}\\ \text{relations on the}\\ \text{generators of } M\end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix}\text{a \textit{minimal}}\\ \text{system of}\\ \text{generators}\\ \text{of } M\end{pmatrix}} M \to 0.$$

The properties of that resolution are closely related to the properties of M. A core area in commutative algebra is devoted to describing the properties of minimal free resolutions and relating them to the structure of the resolved modules. This area has many relations with and applications in other mathematical fields, especially algebraic geometry.

Free (or projective) resolutions exist over many rings (we can also consider noncommutative rings). However, the concept of a *minimal* free resolution needs in particular that each minimal system of generators of the module has the same number of elements, and that property follows from Nakayama's Lemma 4.3. For this reason, the theory of minimal free resolutions is developed in the local and in the graded cases where Nakayama's Lemma holds. This paper is focused on the graded case.

Definition 4.8. *Let $(\boldsymbol{F}, d)$ be a minimal graded free resolution of a finitely generated graded R-module M. Set $\operatorname{Syz}_0^R(M) = M$. For $i \geq 1$ the submodule*

$$\operatorname{Im}(d_i) = \operatorname{Ker}(d_{i-1}) \cong \operatorname{Coker}(d_{i+1})$$

of F_{i-1} is called the i-th syzygy module of M and is denoted $\operatorname{Syz}_i^R(M)$. Its elements are called i-th syzygies. Note that if $f_1, \dots, f_p$ is a basis of F_i, then the elements $d_i(f_1), \dots, d_i(f_p)$ form a minimal system of homogeneous generators of $\operatorname{Syz}_i^R(M)$.

Theorem 4.7 shows that the minimal graded free resolution is the smallest graded free resolution in the sense that the ranks of its free modules are less than or equal to the ranks of the corresponding free modules in an arbitrary graded free resolution of the resolved module. The i-th Betti number of M over R is

$$b_i^R(M) = \operatorname{rank}(F_i).$$

Observe that the differentials in the complexes $\mathbf{F} \otimes_R k$ and $\operatorname{Hom}_R(\mathbf{F}, k)$ are zero, and therefore

$$b_i^R(M) = \dim_k\big(\operatorname{Tor}_i^R(M, k)\big) = \dim_k\big(\operatorname{Ext}_R^i(M, k)\big) \quad \textit{for every } i.$$

Often it is very difficult to obtain a description of the differential. In such cases, we try to get some information about the numerical invariants of the resolution — the Betti numbers.

The length of the minimal graded free resolution is measured by the projective dimension, defined by

$$\operatorname{pd}_R(M) = \max\{i \mid b_i^R(M) \neq 0\}.$$

Hilbert introduced free resolutions motivated by invariant theory and proved the following important result.

Hilbert's Syzygy Theorem 4.9 (see [Peeva 2011, Theorem 15.2]). *The minimal graded free resolution of a finitely generated graded S-module is finite and its length is at most n.*

A more precise version of this is the Auslander–Buchsbaum formula, which states that

$$\mathrm{pd}_S(M) = n - \mathrm{depth}(M),$$

for any finitely generated graded S-module M (see [Peeva 2011, 15.3]).

It turns out that the main source of graded *finite* free resolutions are polynomial rings:

Auslander–Buchsbaum–Serre Regularity Criterion 4.10 (see [Eisenbud 1995, Theorem 19.12]). *The following are equivalent*:

(1) *Every finitely generated graded R-module has finite projective dimension.*

(2) $\mathrm{pd}_R(k) < \infty$.

(3) *$R = S/I$ is a polynomial ring, that is, I is generated by linear forms.*

This is a homological criterion for a ring to be regular. In the introduction to his book *Commutative Ring Theory*, Matsumura [1989] states that he considers Auslander–Buchsbaum–Serre's Criterion to be one of the top three results in commutative algebra.

Infinite minimal free resolutions appear abundantly over quotient rings. The simplest example of a minimal infinite free resolution is perhaps resolving $R/(x)$ over the quotient ring $R = k[x]/(x^2)$, which yields

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R.$$

A homological criterion for complete intersections was obtained by Gulliksen. We say that a Betti sequence $\{b_i^R(M)\}$ is *polynomially bounded* if there exists a polynomial $g \in \mathbb{N}[x]$ such that $b_i^R(M) \leq g(i)$ for $i \gg 0$.

Gulliksen's CI Criterion 4.11 [Gulliksen 1971; 1974; 1980]. *The following are equivalent*:

(1) *The Betti numbers of k are polynomially bounded.*

(2) *The Betti numbers of every finitely generated graded R-module are polynomially bounded.*

(3) *R is a complete intersection.*

One might hope to get similar homological criteria for Gorenstein rings and other interesting classes of rings. However, the type of growth of the Betti numbers of k cannot distinguish such rings: we will see in 6.12 that Serre's Inequality (6.11) implies that for every finitely generated graded R-module M, there exists a real number $\beta > 1$ such that $b_i^R(M) \le \beta^i$ for $i \ge 1$ (see [Avramov 1998, Corollary 4.1.5.]); thus, the Betti numbers grow at most exponentially. We say that $\{b_i^R(M)\}$ *grow exponentially* if there exists a real number $\alpha > 1$ such that $\alpha^i \le b_i^R(M)$ for $i \gg 0$. Avramov [1998] proved that the Betti numbers of k grow exponentially if R is not a complete intersection.

A general question on the Betti numbers is:

Open-Ended Problem 4.12. *How do the properties of the Betti sequence* $\{b_i^R(M)\}$ *relate to the structure of the minimal free resolution of* M, *the structure of* M, *and the structure of* R?

Auslander–Buchsbaum–Serre's Criterion 4.10 and Gulliksen's CI Criterion 4.11 are important results of this type.

The condition that an infinite minimal free resolution has bounded Betti numbers is very strong. Such resolutions do not occur over every quotient ring R, so one might ask which quotient rings admit such resolutions [Avramov 1992, Problem 4]. It is very interesting to explore what are the implications on the structure of the resolution.

Open-Ended Problem 4.13 [Eisenbud 1980]. *What causes bounded Betti numbers in an infinite minimal free resolution*?

Eisenbud [1980] conjectured that every such resolution is eventually periodic and its period is 2; we say that M is *periodic of period* p if $\mathrm{Syz}_p^R(M) \cong M$. The following counterexample was constructed:

Example 4.14 [Gasharov and Peeva 1990]. Let $0 \neq \alpha \in k$. Set

$$R = k[x_1, x_2, x_3, x_4]/(\alpha x_1 x_3 + x_2 x_3,\ x_1 x_4 + x_2 x_4,\ x_3 x_4,\ x_1^2,\ x_2^2,\ x_3^2,\ x_4^2),$$

and consider

$$\boldsymbol{T} : \cdots \to R(-3)^2 \xrightarrow{d_3} R(-2)^2 \xrightarrow{d_2} R(-1)^2 \xrightarrow{d_1} R^2 \to 0,$$

where

$$d_i = \begin{pmatrix} x_1 & \alpha^i x_3 + x_4 \\ 0 & x_2 \end{pmatrix}.$$

The complex $\boldsymbol{T}$ is acyclic and minimally resolves the module $M := \mathrm{Coker}(d_1)$. Moreover, M is shown to have period equal to the order of α in k^*, thus yielding resolutions with arbitrary period and with no period. Other counterexamples over Gorenstein rings are given in [Gasharov and Peeva 1990] as well. All of the examples have constant Betti numbers, supporting the following question which remains a mystery.

Problem 4.15 [Ramras 1980]. *Is it true that if the Betti numbers of a finitely generated graded* R*-module are bounded*, *then they are eventually constant*?

The following subproblem could be explored with the aid of computer computations.

Problem 4.16. *Does there exist a periodic module with nonconstant Betti numbers*?

Eisenbud [1980] proved that if the period is $p = 2$, then the Betti numbers are constant. The case $p = 3$ is open.

Problem 4.15 was extended by Avramov as follows:

Problem 4.17 [Avramov 1992, Problem 9]. *Is it true that the Betti numbers of every finitely generated graded R-module are eventually nondecreasing*?

In particular, Ramras [1980] asked whether $\{b_i^R(M)\}$ being unbounded implies that $\lim_{i\to\infty} b_i^R(M) = \infty$.

A positive answer to Problem 4.17 is known in some special cases: for example, for $M = k$ by a result of Gulliksen [1980], over complete intersections by a result of Avramov, Gasharov and Peeva [1997], when R is Golod by a result of Lescot [1990], for $R = S/I$ such that the integral closure of I is strictly smaller than the integral closure of $(I : (x_1, \dots, x_n))$ by a result of Choi [1990], and for rings with $(x_1, \dots, x_n)^3 = 0$ by a result of Lescot [1985].

For an infinite sequence of nonzero Betti numbers, one can ask how they change and how they behave asymptotically. Several such questions have been raised in [Avramov 1992; 1998].

5. Complete intersections

Throughout this section we assume that R is a graded complete intersection, that is, $R = S/(f_1, \dots, f_c)$ and $f_1, \dots, f_c$ is a homogeneous regular sequence.

The numerical properties of minimal free resolutions over complete intersections are well-understood:

Theorem 5.1 [Gulliksen 1974; Avramov 1989; Avramov, Gasharov and Peeva 1997]. *Let M be a finitely generated graded R-module. The Poincarè series $\mathrm{P}_M^R(t) = \sum_{i\geq 0} b_i^R(M)t^i$ is rational and has the form*

$$\mathrm{P}_M^R(t) = \frac{g(t)}{(1-t^2)^c},$$

for some polynomial $g(t) \in \mathbb{Z}[t]$. The Betti numbers $\{b_i^R(M)\}$ are eventually nondecreasing and are eventually given by two polynomials (*one for the odd Betti numbers and one for the even Betti numbers*) *of the same degree and the same leading coefficient.*

Example 5.2. This is an example where the Betti numbers cannot be given by a single polynomial. Consider the complete intersection $R = k[x, y]/(x^3, y^3)$ and the module $M = R/(x, y)^2$. By [Avramov 1994, Section 2.1] we get

$$b_i^R(M) = \tfrac{3}{2}i + 1 \quad \text{for even } i \geq 0,$$
$$b_i^R(M) = \tfrac{3}{2}i + \tfrac{3}{2} \quad \text{for odd } i \geq 1.$$

The minimal free resolution of k has an elegant structure discovered by Tate. His construction provides the minimal free resolution of k over any R, but if R is not a complete intersection, then the construction is an algorithm building the resolution inductively on homological degree.

Tate's Resolution 5.3 [Tate 1957]. *We will describe Tate's resolution of k over a complete intersection. Write the homogeneous regular sequence*

$$f_j = a_{j1}x_1 + \cdots + a_{jn}x_n, \quad 1 \leq j \leq c,$$

with coefficients $a_{ij} \in S$. *Let* $\boldsymbol{F}' = R \otimes_S \boldsymbol{K}$, *where* $\boldsymbol{K}$ *is the Koszul complex resolving k over S. We may think of* $\boldsymbol{K}$ *as being the exterior algebra on variables* $e_1, \ldots, e_n$, *such that the differential maps* e_i *to* x_i. *In* F_1' *we have cycles*

$$a_{j1}e_1 + \cdots + a_{jn}e_n, \quad 1 \leq j \leq c.$$

For simplicity, we assume $\operatorname{char}(k) = 0$. *Set* $\boldsymbol{F} = \boldsymbol{F}'[y_1, \ldots, y_c]$ *and*

$$d(y_j) = a_{j1}e_1 + \cdots + a_{jn}e_n.$$

The minimal free resolution of k is

$$\boldsymbol{F} = R\langle e_1, \ldots, e_n\rangle[y_1, \ldots, y_c] = (R \otimes_S \mathbf{K})[y_1, \ldots, y_c],$$

with differential defined by

$$d(e_{i_1} \cdots e_{i_j} y_1^{s_1} \cdots y_c^{s_c}) = d(e_{i_1} \cdots e_{i_j}) y_1^{s_1} \cdots y_c^{s_c} + (-1)^j \sum_{\substack{1 \leq p \leq c \\ s_p \geq 1}} d(y_p) e_{i_1} \cdots e_{i_j} y_1^{s_1} \cdots y_p^{s_p - 1} \cdots y_c^{s_c}.$$

where $d(e_{i_1} \cdots e_{i_j})$ *is the Koszul differential. In particular, the Poincarè series of k over the complete intersection is*

$$\mathrm{P}_k^R(t) = \frac{(1+t)^n}{(1-t^2)^c}.$$

If $\operatorname{char}(k) \neq 0$, *then in the construction above instead of the polynomial algebra* $R[y_1, \ldots, y_c]$ *we have to take a divided power algebra.*

The study of infinite minimal free resolutions over complete intersections is focused on the asymptotic properties of the resolutions because for every $p > 0$, there exist examples where the first p steps do not agree with the asymptotic behavior:

Example 5.4 [Eisenbud 1980]. Consider the complete intersection

$$R = S/(x_1^2, \dots, x_n^2).$$

By Tate's Resolution 5.3, we have Tate's minimal free resolution $\boldsymbol{F}$ of k. It shows that the Betti numbers of k are strictly increasing. The dual $\boldsymbol{F}^* = \mathrm{Hom}(\mathbf{F}, R)$ is a minimal injective resolution of $\mathrm{Hom}(\mathbf{F}, R) \cong \mathrm{socle}(R) = (x_1 \cdots x_n) \cong k$. Gluing $\boldsymbol{F}$ and $\mathbf{F}^*$ we get a doubly infinite exact sequence of free R-modules

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow R \xrightarrow{x_1 \cdots x_n} R \longrightarrow F_1^* \longrightarrow F_2^* \longrightarrow \cdots .$$

Thus, for any p,

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow R \longrightarrow R \longrightarrow F_1^* \longrightarrow F_2^* \longrightarrow \cdots \longrightarrow F_p^*$$

is a minimal free resolution over R in which the first p Betti numbers are strictly decreasing, but after the $(p+1)$-st step the Betti numbers are strictly increasing.

Further examples exhibiting complex behavior of the Betti numbers at the beginning of a minimal free resolution are given in [Avramov, Gasharov and Peeva 1997]. Even though the beginning of a minimal free resolution can be unstructured and very complicated, the known results show that stable patterns occur eventually. Thus, instead of studying the entire resolution $\boldsymbol{F}$ we consider the *truncation*

$$\boldsymbol{F}_{\geq p}: \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_{p+1} \xrightarrow{d_{p+1}} F_p,$$

for sufficiently large p. From that point of view, Hilbert's Syzygy Theorem (Theorem 4.9) says that over S every minimal free resolution is eventually the zero-complex.

Eisenbud [1980] described the asymptotic structure of minimal free resolutions over a hypersurface. He introduced the concept of a *matrix factorization* for a homogeneous $f \in S$: it is a pair of square matrices (d, h) with entries in S such that

$$dh = hd = f\,\mathrm{id}.$$

The module $\mathrm{Coker}(d)$ is called the *matrix factorization module* of (d, h).

Theorem 5.5 [Eisenbud 1980]. *With the notation above, the minimal $S/(f)$-resolution of the matrix factorization module is*

$$\cdots \to R^a \xrightarrow{d} R^a \xrightarrow{h} R^a \xrightarrow{d} R^a \xrightarrow{h} R^a \xrightarrow{d} R^a,$$

where $R = S/(f)$ and a is the size of the square matrices d, h.

After ignoring finitely many steps at the beginning, every minimal free resolution over the hypersurface ring $R = S/(f)$ is of this type. More precisely, if $\boldsymbol{F}$ is a minimal graded free resolution, then for every $p \gg 0$ the truncation $\boldsymbol{F}_{\geq p}$ minimally resolves some matrix factorization module and so it is described by a matrix factorization for the element f.

Matrix factorizations have amazing applications in many areas. Kapustin and Li [2003] started the use of matrix factorizations in string theory following an idea of Kontsevich; see [Aspinwall 2013] for a survey. A major discovery was made by Orlov [2004], who showed that matrix factorizations can be used to study Kontsevich's homological mirror symmetry by giving a new description of singularity categories. Matrix factorizations also have applications in the study of Cohen–Macaulay modules and singularity theory, cluster algebras and cluster tilting, Hodge theory, Khovanov–Rozansky homology, moduli of curves, quiver and group representations, and other topics.

A new conjecture on the size of matrix factorizations is recently introduced in [Eisenbud and Peeva $\geq$ 2015b]. Part of the motivation is that it implies a version of the Buchsbaum–Eisenbud–Horroks conjecture and also the Betti Degree Conjecture 5.9.

Minimal free resolutions of high syzygies over a codimension two complete intersection $S/(f_1, f_2)$ were constructed by Avramov and Buchweitz [2000a, 5.5] as quotient complexes. Eisenbud and Peeva [$\geq$ 2015a] provide a construction without using a quotient and give an explicit formula for the differential.

Recently, Eisenbud and Peeva [2015, Definition 1.1] introduced the concept of matrix factorization (d, h) for a regular sequence $f_1, \dots, f_c$ of any length. They constructed the minimal free resolutions of the matrix factorization module $\operatorname{Coker}(R \otimes_S d)$ over S and over the complete intersection $R := S/(f_1, \dots, f_c)$. The infinite minimal free resolution over R is more complicated than the one over a hypersurface ring, but it is still nicely structured and exhibits two patterns — one pattern for odd homological degrees and another pattern for even homological degrees. They proved that asymptotically, every minimal free resolution over R is of this type. More precisely:

Theorem 5.6 [Eisenbud and Peeva 2015]. *If $\boldsymbol{F}$ is a minimal free resolution over the graded complete intersection $R = S/(f_1, \dots, f_c)$, then for every $p \gg 0$ the truncation $\boldsymbol{F}_{\geq p}$ resolves a matrix factorization module and so it is described by a matrix factorization.*

This structure explains and reproves the numerical results above.

One of the main tools in the study of free resolutions over a complete intersection are the CI operators. Let $(\boldsymbol{V}, \partial)$ be a complex of free modules over R.

Consider a lifting $\widetilde{\boldsymbol{V}}$ of $\boldsymbol{V}$ to S, that is, a sequence of free modules $\widetilde{V}_i$ and maps $\tilde{\partial}_{i+1} : \widetilde{V}_{i+1} \to \widetilde{V}_i$ such that $\partial = R \otimes_S \tilde{\partial}$. Since $\partial^2 = 0$ we can choose maps $\tilde{t}_j : \widetilde{V}_{i+1} \to \widetilde{V}_{i-1}$, where $1 \le j \le c$, such that

$$\tilde{\partial}^2 = \sum_{j=1}^{c} f_j \tilde{t}_j.$$

The *CI operators*, sometimes called Eisenbud operators, are

$$t_j := R \otimes_S \tilde{t}_j.$$

This construction was introduced by Eisenbud in [1980]. Different constructions of the CI operators are discussed in [Avramov and Sun 1998]. Since

$$\sum_{j=1}^{c} f_j \tilde{t}_j \tilde{\partial} = \tilde{\partial}^3 = \sum_{j=1}^{c} f_j \tilde{\partial} \tilde{t}_j,$$

and the f_i form a regular sequence, it follows that each t_j commutes with the differential ∂, and thus each t_j defines a map of complexes $\boldsymbol{V}[-2] \to \boldsymbol{V}$; see [Eisenbud 1980, 1.1]. It was shown by Eisenbud [1980, 1.2 and 1.5] that the operators t_j are, up to homotopy, independent of the choice of liftings, and also that they commute up to homotopy.

Conjecture 5.7 [Eisenbud 1980]. *Let M be a finitely generated graded R-module, and let $\boldsymbol{F}$ be its graded minimal R-free resolution. There exists a choice of CI operators on a sufficiently high truncation $\boldsymbol{F}_{\ge p}$ that commute.*

The original conjecture was for the resolution $\boldsymbol{F}$ (not for a truncation), and a counterexample to that was provided in [Avramov, Gasharov and Peeva 1997].

If $\boldsymbol{V}$ is an R-free resolution of a finitely generated graded R-module M, then the CI operators t_j induce well-defined, commutative maps χ_j on $\mathrm{Ext}_R(M, k)$ and thus make $\mathrm{Ext}_R(M, k)$ into a module over the polynomial ring $\mathcal{R} := k[\chi_1, \dots, \chi_c]$, where the variables χ_j have degree 2. The χ_j are also called CI operators. By [Eisenbud 1980, Proposition 1.2], the action of χ_j can be defined using any CI operators on any R-free resolution of M. Since the χ_j have degree 2, we may split the Ext module into even degree and odd degree parts:

$$\mathrm{Ext}_R(N, k) = \mathrm{Ext}_R^{\mathrm{even}}(M, k) \oplus \mathrm{Ext}_R^{\mathrm{odd}}(M, k).$$

A version of the following result was first proved by Gulliksen [1974], who used a different construction of CI operators on Ext. A short proof using the above construction of CI operators is provided in [Eisenbud and Peeva 2015, Theorem 4.5].

Theorem 5.8 [Gulliksen 1974; Eisenbud 1980; Avramov and Sun 1998; Eisenbud and Peeva 2015, Theorem 4.5]. *Let M be a finitely generated graded R-module. The action of the CI operators makes $\mathrm{Ext}_R(M,k)$ into a finitely generated graded $k[\chi_1,\dots,\chi_c]$-module.*

The structure of the Ext module is studied in [Avramov and Buchweitz 2000b]. Avramov and Iyengar [2007] proved that the support variety (defined by the annihilator) of $\mathrm{Ext}_R(M,k)$ can be anything. Every sufficiently high truncation of $\mathrm{Ext}_R(M,k)$ is linearly presented, and its defining equations are described in terms of homotopies by Eisenbud and Peeva [$\geq$ 2015a].

By multiplicity $\mathrm{mult}(\mathrm{Ext}^*_R(N,k))$ we mean

$$\mathrm{mult}(\mathrm{Ext}^{\mathrm{even}}_R(M,k)) = \mathrm{mult}(\mathrm{Ext}^{\mathrm{odd}}_R(M,k)),$$

computed with respect to the standard grading of $k[\chi_1,\dots,\chi_c]$ with $\deg(\chi_i)=1$ for each i. The *Betti degree* of M is the multiplicity $\mathrm{mult}(\mathrm{Ext}^*_R(M,k))$.

The Betti Degree Conjecture 5.9 [Avramov and Buchweitz 2000a, Conjecture 7.5]. *Let M be a graded finitely generated R-module. The Betti degree of M satisfies the inequality*

$$\mathrm{mult}(\mathrm{Ext}^*_R(M,k)) \geq 2^{\dim(\mathrm{Ext}_R(M,k))-1}.$$

We close this section by bringing up that the Eisenbud–Huneke Question 9.10 has a positive answer for k over R, and also for any finitely generated graded module if the forms in the regular sequence $f_1,\dots,f_c$ are of the same degree, but is open otherwise (including in the codimension two case):

Question 5.10 [Eisenbud and Huneke 2005, Question A]. *Let M be a finitely generated graded R-module and suppose that the forms in the regular sequence $f_1,\dots,f_c$ do not have the same degree. Does there exists a number u and bases of the free modules in the minimal graded R-free resolution $\boldsymbol{F}$ of M, such that for all $i\geq 0$ the entries in the matrix of the differential d_i have degrees $\leq u$?*

6. Rationality and Golod rings

We now focus on resolving the simplest possible module, namely k. The next construction provides a free resolution.

The Bar Resolution 6.1 (see [Mac Lane 1963]). *The bar resolution is an explicit construction which resolves k over any ring R, but usually provides a highly nonminimal free resolution. Let $\widetilde{R}$ be the cokernel of the canonical inclusion of vector spaces $k\to R$. For $i\geq 0$ set $B_i = R\otimes_k \widetilde{R}\otimes_k\cdots\otimes_k\widetilde{R}$, where we have i factors $\widetilde{R}$. The left factor R gives B_i a structure of a free R-module. Fix a basis $\mathfrak{R}$ of R over k such that $1\in\mathfrak{R}$. Let $r\in R$ and $r_1,\dots,r_i\in\mathfrak{R}$. We* *denote by* $r[r_1 \mid \dots \mid r_i]$ *the element* $r \otimes_k r_1 \otimes_k \cdots \otimes_k r_i$ *in* B_i, *replacing* $\otimes_k$ *by a vertical bar*; *in particular*, $B_0 = R$ *with* $r[\,] \in B_0$ *identified with* $r \in R$. *Note that* $r[r_1 \mid \dots \mid r_i] = 0$ *if some* $r_j = 1$ *or* $r = 0$. *Consider the sequence*

$$\boldsymbol{B}: \quad \cdots \to B_i \to B_{i-1} \to \cdots \to B_0 = R \to k \to 0,$$

with differential d defined by

$$d_i(r[r_1 \mid \dots \mid r_i]) = rr_1[r_2 \mid \dots \mid r_i] + \sum_{1 \le j \le i-1} (-1)^j r[r_1 \mid \dots \mid r_j r_{j+1} \mid \dots \mid r_i].$$

The differential is well-defined since if $r_j = 1$ *for some* $j > 1$, *then the terms*

$$(-1)^j r[r_1 \mid \dots \mid r_j r_{j+1} \mid \dots \mid r_i],$$
$$(-1)^{j-1} r[r_1 \mid \dots \mid r_{j-1} r_j \mid \dots \mid r_i]$$

cancel and all other terms vanish; *similarly for* $j = 1$. *Exactness may be proved by constructing an explicit homotopy*; *see* [Mac Lane 1963].

The minimal free resolution of k over S is the Koszul complex. It has an elegant and simple structure. In contrast, the situation over quotient rings is complicated; the structure of the *minimal* free resolution of k is known in some cases, but has remained mysterious in general. We start the discussion of the properties of that resolution by focusing on its Betti numbers. When we have infinitely many Betti numbers of a module M, we may study their properties via the *Poincaré series*

$$\mathrm{P}^R_M(t) = \sum_{i \ge 0} b^R_i(M) t^i.$$

The first natural question to consider is:

Open-Ended Problem 6.2. *Are the structure and invariants of an infinite minimal graded free resolution encoded in finite data*?

The main peak in this direction was:

The Serre–Kaplansky Problem 6.3. Is the Poincaré series of the residue field k over R rational? The question was originally asked for finitely generated commutative local Noetherian rings.

A Poincaré series $\mathrm{P}^R_M(t)$ is a rational function of the complex variable t if $\mathrm{P}^R_M(t) = f(t)/g(t)$ for two complex polynomials $f(t)$, $g(t)$ with $g(0) \neq 0$. By Fatou's Theorem, we have that the polynomials can be chosen with integer coefficients.

The Serre–Kaplansky Problem 6.3 was a central question in commutative algebra for many years. The high enthusiasm for research on the problem was motivated on the one hand by the expectation that the answer is positive (the problem was often considered a conjecture) and on the other hand by a result of Gulliksen [1972] who proved that a positive answer for all such rings implies the rationality of the Poincaré series of any finitely generated module. Additional interest was generated by a result of Anick and Gulliksen [1985], who reduced the rationality question to rings with the cube of the maximal ideal being zero.

Note that Yoneda multiplication makes $\mathrm{Ext}_R(k,k)$ a graded (by homological degree) k-algebra, and the Hilbert series of that algebra is the Poincaré series $\mathrm{P}_k^R(t)$. Problems of rationality of Poincaré and Hilbert series were stated by several mathematicians: by Serre and Kaplansky for local Noetherian rings, by Kostrikin and Shafarevich for the Hochschild homology of a finite-dimensional nilpotent k-algebra, by Govorov for finitely presented associative graded algebras, by Serre and Moore for simply connected complexes; see the survey by Babenko [1986].

An example of an irrational Poincaré series was first constructed by Anick [1980].

Example 6.4 [Anick 1982]. The Poincaré series $\mathrm{P}_k^R(t)$ is irrational for

$$R = k[x_1,\dots,x_5]\big/\big(x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3+x_3x_4+x_2x_5, (x_1,\dots,x_n)^3\big)$$

if $\mathrm{char}(k) \neq 2$; in $\mathrm{char}(k) = 2$ we add x_3^2 to the defining ideal.

Since then several other such examples have been found and they exist even over Gorenstein rings; see, for example, [Bøgvad 1983]. Surveys on nonrationality are given by Anick [1988] and Roos [1981]. At present, it is not clear how wide spread such examples are. We do not have a feel for which of the following cases holds:

(1) Most Poincaré series are rational, and irrational Poincaré series occur rarely in specially crafted examples.

(2) Most Poincaré series are irrational, and there are some nice classes of rings (for example, Golod rings, complete intersections) where we have rationality.

(3) Both rational and irrational Poincaré series occur widely.

One would like to have results showing whether the Poincaré series are rational generically, or are irrational generically. A difficulty in even posing meaningful problems and conjectures is that currently we do not know a good concept of "generic".

The situation is clear for generic Artinian Gorenstein rings by [Rossi and Şega 2014], and also when we have a lot of combinatorial structure: the cases of monomial and toric quotients. Backelin [1982] proved that the Poincaré series of k over $R = S/I$ is rational if I is generated by monomials. His result was extended to all modules:

Theorem 6.5 [Lescot 1988]. *The Poincaré series of every finitely generated graded module over $R = S/I$ is rational if I is generated by monomials.*

An ideal I is called *toric* if it is the kernel of a map

$$S \longrightarrow k[m_1, \ldots, m_n] \subset k[t_1, \ldots, t_r]$$

that maps each variable x_i to a monomial m_i; in that case, $R = S/I$ is called a *toric ring*. Gasharov, Peeva and Welker [2000] proved that the Poincaré series of k is rational for generic toric rings; see Theorem 6.16(4). However, in contrast to the monomial case, toric ideals with irrational Poincaré series were found:

Example 6.6 [Roos and Sturmfels 1998]. Set $S = k[x_0, \ldots, x_9]$ and let I be the kernel of the homomorphism

$$k[x_1, \ldots, x_9] \to k[t^{36}, t^{33}s^3, t^{30}s^6, t^{28}s^8, t^{26}s^{10}, t^{25}s^{11}, t^{24}s^{12}, t^{18}s^{18}, s^{36}],$$

that sends the variables x_i to the listed monomials in t and s. Computer computation shows that I is generated by 12 quadrics. It defines a projective monomial curve. Roos–Sturmfels showed that the Poincaré series of k is irrational over S/I.

Example 6.7 [Fröberg and Roos 2000; Löfwall, Lundqvist and Roos 2015]. Set $S = k[x_1, \ldots, x_7]$ and let I be the kernel of the homomorphism

$$k[x_1, \ldots, x_7] \to k[t^{18}, t^{24}, t^{25}, t^{26}, t^{28}, t^{30}, t^{33}],$$

that sends the variables x_i to the listed monomials in t. Computer computation shows that I is generated by 7 quadrics and 4 cubics. It defines an affine monomial curve. The Poincaré series of k is irrational over S/I.

Open-Ended Problem 6.8. *The above results motivate the question of whether there are classes of rings* (*other than toric rings*, *monomial quotients*, *and Artinian Gorenstein rings*) *whose generic objects are Golod.*

We now go back to the discussion of rationality over a graded ring $R = S/I$. Jacobsson, Stoltenberg and Hensen proved [1985] that the sequence of Betti numbers of any finitely generated graded R-module is primitive recursive. The class of primitive recursive functions is countable.

Theorem 6.16 provides interesting classes of rings for which $\mathrm{P}_k^R(t)$ is rational. It is also known that it is rational if R is a complete intersection by a result of Tate [1957], if R is one link from a complete intersection by a result of Avramov [1978], and in other special cases.

Inspired by Open-Ended Problem 6.2 one can consider the following problem:

Open-Ended Problem 6.9. *Relate the properties of the infinite graded minimal free resolution of k over S/I to the properties of the finite minimal graded free resolution of S/I over the ring S.*

One option is to explore in the following general direction:

Open-ended strategy 6.10. One can take conjectures or results on finite minimal graded free resolutions and try to prove analogues for infinite minimal graded free resolutions.

Another option is to study the relations between the infinite minimal free resolution of k over S/I and the finite minimal free resolution of S/I over the ring S. There is a classical Cartan–Eilenberg spectral sequence relating the two resolutions [Cartan and Eilenberg 1956]:

$$\operatorname{Tor}_p^R(M, \operatorname{Tor}_q^S(R,k)) \implies \operatorname{Tor}_{p+q}^S(M,k);$$

see [Avramov 1998, Section 3] for a detailed treatment and other spectral sequences.

Using that spectral sequence, Serre derived the inequality

$$\mathrm{P}_k^{S/I}(t) \preccurlyeq \frac{(1+t)^n}{1-t^2\,\mathrm{P}_I^S(t)}, \tag{6.11}$$

where $\preccurlyeq$ denotes coefficient-wise comparison of power series; see [Avramov 1998, Proposition 3.3.2]. Eagon constructed a free resolution of k over R whose generating function is the right-hand side of Serre's Inequality; see [Gulliksen and Levin 1969, Section 4.1]. We will see later in this section that the resolution is minimal over Golod rings.

Rationality and Growth of Betti Numbers 6.12. *Suppose that $\{b_i\}$ is a sequence of integer numbers and $\sum_i b_i t^i = f(t)/g(t)$ for some polynomials $f(t), g(t) \in \mathbb{Q}[t]$. Set $a = \deg(g)$. Let $h(t) = g(t^{-1})t^{\deg(g)}$; we may assume that the leading coefficient of h is 1 by scaling f if necessary, and write*

$$h(t) = t^a - h_1 t^{a-1} - h_2 t^{a-2} - \cdots - h_a.$$

Then the numbers b_i satisfy the recurrence relation

$$b_i = h_1 b_{i-1} + \cdots + h_a b_{i-a} \quad \textit{for } i \gg 0.$$

Thus, we have a recursive sequence. Let $r_1, \dots, r_s$ be the roots of $h(t)$ with multiplicities $m_1, \dots, m_s$ respectively. We have a formula for the numbers b_i in terms of the roots (see [Eisen 1969, Chapter III, Section 4] *and* [Markushevich

1975]), *namely*,

$$b_i = \sum_{1\le j\le s} r_j^i(c_{j,1} + c_{j,2}i + \cdots + c_{j,m_j} i^{m_j-1}) = \sum_{\substack{1\le j\le s\\ 0\le q\le m_j-1}} r_j^i c_{j,q+1} i^q, \quad (6.13)$$

where the coefficients c_{jq} *are determined in order to fit the initial conditions of the recurrence. It follows that the sequence* $\{b_i\}$ *is exponentially bounded, that is, there exists a real number* $\beta > 1$ *such that* $b_i \le \beta^i$ *for* $i \ge 1$. *Hence, Serre's Inequality* (6.11) *implies that for every finitely generated graded* R*-module* M *the sequence of Betti numbers* $\{b_i^R(M)\}$ *is exponentially bounded.*

Now, suppose that M *is a module with a rational Poincaré series, and set* $b_i := b_i^R(M)$. *By* (6.13) *it follows that one of the following two cases holds*:

- *If* $|r_j| \le 1$ *for all roots, then the Betti sequence* $\{b_i\}$ *is polynomially bounded, that is, there exists a polynomial* $e(t)$ *such that* $b_i \le e(i)$ *for* $i \gg 0$.
- *If there exists a root with* $|r_j| > 1$, *then the sequence* $\left\{\sum_{q\le i} b_q\right\}$ *grows exponentially* (*we say that* $\{b_q\}$ *grows weakly exponentially*), *that is, there exists a real number* $\alpha > 1$ *so that* $\alpha^i \le \sum_{q\le i} b_q$ *for* $i \gg 0$, *by* [Avramov 1978].

We may wonder how the Betti numbers grow if the Poincaré series is not rational. Such questions have been raised in [Avramov 1992; 1998].

In the rest of the section, we discuss Golod rings, which provide many classes of rings over which Poincaré series are rational.

Definition 6.14. *A ring is called Golod if equality holds in Serre's Inequality* (6.11). *In particular*, k *has a rational Poincaré series in that case. Sometimes, we say that* I *is Golod if* $R = S/I$ *is.*

Golodness is encoded in the finite data given by the Koszul homology

$$H(\boldsymbol{K} \otimes_S S/I),$$

where $\mathbf{K} := \mathbf{K}(\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}; \mathrm{S})$ is the Koszul complex resolving k over S, as follows. We define Massey operations on $\boldsymbol{K} \otimes_S S/I$ in the following way: Let $\mathcal{M}$ be a set of homogeneous (with respect to both the homological and the internal degree) elements in $\boldsymbol{K} \otimes S/I$ that form a basis of $H(\boldsymbol{K} \otimes_S S/I)$. For every $z_i, z_j \in \mathcal{M}$ we define $\mu_2(z_i, z_j)$ to be the homology class of $z_i z_j$ in $H(\boldsymbol{K} \otimes_S S/I)$ and call μ_2 the 2*-fold Massey operation*. This is just the multiplication in $H(\boldsymbol{K} \otimes_S S/I) \cong \mathrm{Tor}^S(S/I, k)$ and is sometimes called the *Koszul product*. Fix an $r > 2$. If all q-fold Massey operations vanish for all $q < r$, then we define the r*-fold Massey operation* as follows: for every $z_1, \dots, z_j \in \mathcal{M}$

with $j < r$ choose a homogeneous $y_{z_1,\dots,z_j} \in \boldsymbol{K} \otimes S/I$ such that

$$d(y_{z_1,\dots,z_j}) = \mu_j(z_1,\dots,z_j),$$

and note that the homological degree of $y_{z_1,\dots,z_j}$ is $-1 + \sum_{v=1}^{j}(\deg(z_v)+1)$; then set

$$\begin{aligned}\mu_r(z_1,\dots,z_r) := z_1 y_{z_2,\dots,z_r} &\\ &+ \sum_{2\le s\le r-2} (-1)^{\sum_{v=1}^{s}(\deg(z_v)+1)} y_{z_1,\dots,z_s} y_{z_{s+1},\dots,z_r}\\ &+(-1)^{\sum_{v=1}^{r-1}(\deg(z_v)+1)} y_{z_1,\dots,z_{r-1}} z_r.\end{aligned}$$

Note that Massey operations respect homological and internal degree. Massey operations are sometimes called *Massey products*. The r-fold products exist if and only if all lower products vanish. It was shown by Golod [1962] that all Massey products vanish (exist) exactly when the ring S/I is Golod; see [Gulliksen and Levin 1969].

Frank Moore pointed out that there are no known examples of rings which are not Golod and for which the second Massey product vanishes. Berglund and Jöllenbeck [2007] showed that the vanishing of the Koszul product (second Massey operation) is equivalent to Golodness if I is generated by monomials.

Golod's Resolution 6.15. If R is Golod, then Eagon's free resolution of k is minimal. This was proved by Golod [1962] (see [Gulliksen and Levin 1969]) who also provided an explicit formula for the differential using Massey products; see [Avramov 1998, Theorem 5.2.2]. It is known that the Ext-algebra over a Golod ring R is finitely presented by a result of Sjödin [1985].

The following is a list of some Golod rings.

Theorem 6.16.

(1) *Since Massey operations respect internal degree, it is easy to see that if an ideal I is generated in one degree and its minimal free resolution over S is linear* (*that is, the entries in the differential matrices are linear forms*), *then S/I is a Golod ring. In particular, $S/(x_1,\dots,x_n)^p$ is a Golod ring for all $p > 1$.*

(2) *Craig Huneke* (*personal communication*) *observed that using degree-reasons, one can show that if I is an ideal such that $I_i = 0$ for $i < r$ and* $\operatorname{reg}(I) \le 2r-3$, *then S/I is a Golod ring.*

(3) [Herzog, Reiner and Welker 1999] *If I is a componentwise linear ideal* (*that is, I_p has a linear minimal free resolution for every p*), *then S/I is Golod. This includes the class of Gotzmann ideals.*

(4) [Gasharov, Peeva and Welker 2000] *If I is a generic toric ideal, then S/I is a Golod ring.*

(5) [Herzog and Steurich 1979] *If for two proper graded ideals we have $JJ' = J \cap J'$, then JJ' is Golod.*

(6) [Aramova and Herzog 1996; Peeva 1996] *An ideal L generated by monomials in S is called* 0*-Borel fixed (also referred to as strongly stable) if whenever m is a monomial in L and x_i divides m, then $x_j(m/x_i) \in L$ for all $1 \leq j < i$. The interest in such ideals comes from the fact that generic initial ideals in characteristic zero are* 0*-Borel fixed. If L is a* 0*-Borel fixed ideal contained in $(x_1, \ldots, x_n)^2$, then S/L is a Golod ring.*

(7) [Berglund and Jöllenbeck 2007] *If I is an ideal generated by monomials, then S/I is Golod if and only if the product on $H(\boldsymbol{K} \otimes S/I) = \mathrm{Tor}^S(S/I, k)$ is trivial.*

(8) [Fakhary and Welker 2012] *For any two proper monomial ideals I and J in S, the ring S/IJ is Golod.*

(9) [Herzog and Huneke 2013] *Let I be a graded ideal. For every $q \geq 2$ the rings S/I^q, $S/I^{(q)}$ and $S/\widetilde{I^q}$ are Golod, where $I^{(q)}$ and $\widetilde{I^q}$ denote the q-th symbolic and saturated powers of I, respectively. The proof hinges on a new definition, whereby the authors call an ideal I strongly Golod if $\partial(I)^2 \subset I$, where $\partial(I)$ denotes the ideal which is generated by all partial derivatives of the generators of I, and proceed to show that strongly Golod ideals are Golod. For large powers of ideals the result was previously proved by Herzog, Welker and Yassemi* [2011].

Open-Ended Problem 6.17. *It is of interest to find other nice classes of rings which are Golod.*

Theorem 6.16(5), (8) and (9) suggests the following open problem, which was first raised by Welker (personal communication).

Problem 6.18. *Is the product of any two proper graded ideals Golod*?

Recent work on the topic leads to the following question:

Problem 6.19 (Craig Huneke, personal communication). *If I is (strongly) Golod, then is the integral closure of I Golod as well*?

Over a Golod ring, we have rationality not only for the Poincaré series of k but for all Poincaré series:

Theorem 6.20 (Lescot 1990; see [Avramov 1998, Theorem 5.3.2]). *Let $R = S/I$ be a Golod ring. If M is a graded finitely generated R-module, then its Poincaré series is*

$$\mathrm{P}_M^R(t) = h(t) + \frac{\mathrm{P}_{M'}^S(t)}{1 - t^2\,\mathrm{P}_I^S(t)},$$

where $h(t)$ is a polynomial in $\mathbb{N}[t] \cup 0$ of degree $\leq n$, and the polynomial $\mathrm{P}^S_{M'}(t)$ of degree $\leq n$ is the Poincaré series over S of a syzygy M' of M over R. In particular, the Poincaré series of all graded finitely generated modules over R have common denominator

$$1 - t^2\, \mathrm{P}^S_I(t).$$

The property about the common denominator in the above theorem does not hold in general:

Example 6.21 [Roos 2005]. There exists a ring R defined by quadrics, such that the rational Poincaré series over R do not have a common denominator. Such examples are provided in [Roos 2005, Theorem 2.4]. For example,

$$R = k[x, y, z, u]/(x^2, y^2, z^2, u^2, xy, zu),$$
$$R = k[x, y, z, u]/(x^2, z^2, u^2, xy, zu),$$
$$R = k[x, y, z, u]/(x^2, u^2, xy, zu).$$

7. Regularity

Definition 7.1. *We will define the graded Betti numbers, which are a refined version of the Betti numbers. Let $\boldsymbol{F}$ be the minimal graded free resolution of a finitely generated graded R-module M. We may write*

$$F_i = \bigoplus_{p \in \mathbb{Z}} R(-p)^{b_{i,p}},$$

for each i. Therefore, the resolution is

$$\boldsymbol{F} : \quad \cdots \longrightarrow \bigoplus_{p \in \mathbb{Z}} R(-p)^{b_{i,p}} \xrightarrow{d_i} \bigoplus_{p \in \mathbb{Z}} R(-p)^{b_{i-1,p}} \longrightarrow \cdots .$$

The numbers $b_{i,p}$ are called the graded Betti numbers of M and denoted $b^R_{i,p}(M)$. We say that $b^R_{i,p}(M)$ is the Betti number in homological degree i and internal degree p. We have that

$$b^R_{i,p}(M) = \dim_k(\mathrm{Tor}^R_i(M, k)_p) = \dim_k(\mathrm{Ext}^i_R(M, k)_p).$$

The graded Poincaré series of M over R is

$$\mathrm{P}^R_M(t, z) = \sum_{i \geq 0,\, p \in \mathbb{Z}} b^R_{i,p}(M) t^i z^p.$$

There is a graded version of Serre's Inequality (6.11):

$$\mathrm{P}^R_M(t, z) \preccurlyeq \frac{\mathrm{P}^S_M(t, z)}{1 - t^2\, \mathrm{P}^S_I(t, z)}. \tag{7.2}$$

Often we consider the *Betti table*, defined as follows: The columns are indexed from left to right by homological degree starting with homological degree zero. The rows are indexed increasingly from top to bottom starting with the minimal degree of an element in a minimal system of homogeneous generators of M. The entry in position i, j is $b^R_{i,i+j}(M)$. Note that the Betti numbers $b^R_{i,i}(M)$ appear in the top row if M is generated in degree 0. This format of the table is meaningful since the minimality of the resolution implies that $b^R_{i,p}(M) = 0$ for $p < i + c$ if c is the minimal degree of an element in a minimal system of homogeneous generators of M. For example, a module M generated in degrees ≥ 0 has Betti table of the form

	0	1	2	…
0:	$b_{0,0}$	$b_{1,1}$	$b_{2,2}$	…
1:	$b_{0,1}$	$b_{1,2}$	$b_{2,3}$	…
2:	$b_{0,2}$	$b_{1,3}$	$b_{2,4}$	…
3:	$b_{0,3}$	$b_{1,4}$	$b_{2,5}$	…
⋮	⋮	⋮	⋮	

In Example 3.4, the Betti table of S/J is

	0	1	2
0:	1	-	-
1:	-	-	-
2:	-	2	1
3:	-	-	-
4:	-	-	-
5:	-	-	-
6:	-	1	1

where we put - instead of zero.

We may ignore the zeros in a Betti table and consider the shape in which the nonzero entries lie. In Example 3.4 the shape of the Betti table is determined by

	0	1	2
0:	*		
1:			
2:		*	*
3:			
4:			
5:			
6:		*	*

Open-Ended Problem 7.3. *What are the possible shapes of Betti tables either over a fixed ring, or of a fixed class of modules*?

Two basic invariants measuring the shape of a Betti table are the projective dimension and the regularity: The projective dimension $\mathrm{pd}_R(M)$ is the index of the last nonzero column of the Betti table, and thus it measures the length of the table. The width of the table is measured by the index of the last nonzero row of the Betti table, and it is another well-studied numerical invariant: the *Castelnuovo–Mumford regularity* of M, which is

$$\mathrm{reg}_R(M) = \sup\{ j \mid b^R_{i,\,i+j}(M) \neq 0 \}.$$

In Example 3.4 we have

	0	1	2
0:	1	-	-
1:	-	-	-
2:	-	2	1
3:	-	-	-
4:	-	-	-
5:	-	-	-
6:	-	1	1

$\mathrm{reg}(S/J) = 6$ (rows 1–6)

$\mathrm{pd}(S/J) = 2$ (columns 1–2)

Hilbert's Syzygy Theorem 4.9 implies that every finitely generated graded module over the polynomial ring S has finite regularity. If the module M has finite length, then

$$\mathrm{reg}_R(M) = \sup\{ j \mid M_j \neq 0 \};$$

see [Eisenbud 2005, Section 4B]. Regularity is among the most interesting and important numerical invariants of M, and it has attracted a lot of attention and work both in commutative algebra and algebraic geometry.

It is natural to ask for an analogue of Auslander–Buchsbaum–Serre's Criterion 4.10 to characterize the rings over which all modules have finite regularity. It is given by the following two results.

Theorem 7.4 [Avramov and Eisenbud 1992]. *If R is a Koszul algebra*, *that is*, $\mathrm{reg}_R(k) = 0$, *then for every graded R-module M we have*

$$\mathrm{reg}_R(M) \leq \mathrm{reg}_S(M).$$

Theorem 7.5 [Avramov and Peeva 2001]. *The following are equivalent*:

(1) *Every finitely generated graded R-module has finite regularity.*

(2) *The residue field k has finite regularity.*

(3) *R is a Koszul algebra.*

As noted above in Theorem 7.4, Koszul algebras are defined by the vanishing of the regularity of k. They are the topic discussed in the next section.

Open-Ended Problem 7.6. *It would be interesting to find analogues over Koszul rings of conjectures/results on regularity over a polynomial ring.*

We give an example: In a recent paper, Ananyan and Hochster [2012] showed that the projective dimension of an ideal generated by a fixed number r of quadrics in a polynomial ring is bounded by a number independent of the number of variables. This solved a problem of Stillman [Peeva and Stillman 2009, Problem 3.14] in the case of quadrics. Then by a result of Caviglia (see [Peeva 2011, Theorem 29.5]), it follows that the regularity of an ideal generated by r quadrics in a polynomial ring is bounded by a number independent of the number of variables. One can ask for an analogue to Stillman's conjecture (which is for polynomial rings) for infinite free resolutions over a Koszul ring. Caviglia (personal communication) observed that if we fix integer numbers r and q, and consider an ideal J generated by r quadrics in a Koszul algebra $R = S/I$ defined by q quadrics, then by Theorem 7.4 we have $\operatorname{reg}_R(J) \le \operatorname{reg}_S(I + J)$, which is bounded by the result of Ananyan–Hochster since $I + J$ is generated by $r + q$ quadrics. Thus, the regularity of an ideal generated by r quadrics in a Koszul algebra with a fixed number of defining equations is bounded by a formula independent of the number of variables. If the number of defining equations of the Koszul algebra is not fixed, then the property fails to hold by an example constructed by McCullough [2013].

8. Koszul rings

Definition 8.1. *Following* [Priddy 1970], *we say that R is Koszul if* $\operatorname{reg}_R(k) = 0$. *Equivalently*, *$R$ is Koszul if the minimal graded free resolution of k over R is linear*, *that is*, *the entries in the matrices of the differential are linear forms. These rings have played an important role in several mathematical fields. It is easy to see that if $R = S/I$ is Koszul*, *then the ideal I is generated by quadrics and linear forms*; *we ignore the linear forms by assuming $I \subseteq (x_1, \dots, x_n)^2$. We say that I is Koszul when S/I is Koszul.*

Example 8.2. If $f_1, \dots, f_c$ is a regular sequence of quadrics, then by Tate's Resolution 5.3 it follows that $S/(f_1, \dots, f_c)$ is a Koszul ring.

Example 8.3. Suppose I is generated by quadrics and has a linear minimal free resolution over S. Then R is Golod by Theorem 6.16(1). It follows that R is Koszul.

Theorem 7.4 implies:

Corollary 8.4 [Avramov and Eisenbud 1992]. *A sufficiently high truncation $M_{\geq p}$ of a graded finitely generated module M over a Koszul algebra R has a linear minimal R-free resolution*, *that is*, *the entries in the matrices of the differential are linear forms.*

Note that R_i is a k-vector space since $R_0 = k$ and $R_0 R_i \subseteq R_i$. The generating function

$$i \mapsto \dim_k(R_i)$$

is called the *Hilbert function* of R and is studied via the *Hilbert series*

$$\mathrm{Hilb}_R(t) = \sum_{i \geq 0} \dim_k(R_i) t^i.$$

The Hilbert function encodes important information about R, for example, its dimension and multiplicity. Hilbert introduced resolutions in order to compute Hilbert functions; see [Peeva 2011, Section 16]. The same kind of computation works over a Koszul ring and yields the following result.

Theorem 8.5 [see Polishchuk and Positselski 2005, Chapter 2, Section 2; Fröberg 1999]. *If the ring R is Koszul*, *then the Poincaré series of k is related to the Hilbert series of R as follows*

$$\mathrm{P}_k^R(t) = \frac{1}{\mathrm{Hilb}_R(-t)}.$$

Example 8.6. Not all ideals generated by quadrics define Koszul rings. The relation in Theorem 8.5 can be used to show that particular rings are not Koszul. Consider

$$R = k[x, y, z, w]/(x^2, y^2, z^2, w^2, xy + xz + xw).$$

This is an Artinian ring with Hilbert series $\mathrm{Hilb}_R(t) = 1 + 4t + 5t^2 + t^3$. One computes

$$\frac{1}{\mathrm{Hilb}_R(-t)} = 1 + 4t + 11t^2 + 25t^3 + 49t^4 + 82t^5 + 108t^6 + 71t^7 - 174t^8 \cdots.$$

Hence, R cannot be Koszul; if it were, the previous expression would be its Poincaré series, which cannot have any negative coefficients.

Next we will see that Theorem 8.5 is an expression of duality. Suppose that I is generated by quadrics; in that case we say that the algebra *R is quadratic*. Let $y_1, \dots, y_n$ be indeterminates (recall that n is the number of variables in S), and denote by V the vector space spanned by them. Write $R = k\langle V\rangle/(W)$, where $k\langle V\rangle = k \oplus V \oplus (V \otimes_k V) \oplus \cdots$ is the tensor algebra on V and $W \subset V \otimes_k V$ is the vector space spanned by the quadrics generating I and the commutator relations $y_i \otimes y_j - y_j \otimes y_i$ for $i \neq j$. The *dual algebra* of R is the quadratic algebra $R^! = k\langle V^*\rangle/(W^\perp)$, where V^* is the dual vector space of V and $W^\perp \subset (V \otimes_k V)^*$ is the two-sided ideal of forms that vanish on W; see [Polishchuk and Positselski 2005, Chapter I, Section 2]. For example, the dual algebra of the polynomial ring S is an exterior algebra. We denote $z_1, \dots, z_n$ the basis of V^* dual to the basis $y_1, \dots, y_n$ of V. Computing the generators of $W^\perp$ amounts to linear algebra computations: Set $[z_i, z_i] = z_i^2$, and $[z_i, z_j] = z_i z_j + z_j z_i$ for $i \neq j$. If $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$, where

$$f_p = \sum_{i \le j} a_{pij} x_i x_j,$$

then choose a basis (c_{qij}) of the solutions to the linear system of equations

$$\sum_{i \le j} a_{pij} X_{ij} = 0, \quad p = 1, \dots, r,$$

and then $R^! = k\langle z_1, \dots, z_n\rangle/(g_1, \dots, g_s)$, where

$$g_q = \sum_{i \le j} c_{qij} [z_i, z_j], \quad q = 1, \dots, s.$$

Example 8.7. Let $R = S/I$, where I is generated by quadratic monomials. Then $R^! = k\langle z_1, \dots, z_n\rangle/T$, where T is generated by all z_i^2 such that $x_i^2 \notin I$ and $z_i z_j + z_j z_i$ such that $i \neq j$ and $x_i x_j \notin I$.

Example 8.8. Let $R = k[x, y, z]/I$ with

$$I = (x^2, y^2, xy + xz, xy + yz).$$

Then $R^! = k\langle X, Y, Z\rangle/T$ with

$$T = (Z^2, XY + YX - XZ - ZX - YZ - ZY).$$

While it is not obvious that the ideal I has a quadratic Gröbner basis, it is apparent that T is generated by a noncommutative quadratic Gröbner basis. It follows that both R and $R^!$ are Koszul. The role of Gröbner bases is explained after Example 8.13.

The dual algebra can be defined for any (not necessarily commutative) graded k-algebra generated by finitely many generators of degree 1 and with relations generated in degree 2; it is easy to see that $(R^!)^! \cong R$. The notions of grading, resolution and Koszulness extend to the noncommutative setting. It then follows that a quadratic algebra is Koszul if and only if its dual algebra is Koszul [Priddy 1970]. Furthermore, Löfwall [1986] proved that $R^!$ is isomorphic to the *diagonal subalgebra* $\sum_i \operatorname{Ext}^i_R(k,k)_i$ of the Yoneda algebra $\operatorname{Ext}_R(k,k)$. If R is Koszul, then $\operatorname{Ext}^i_R(k,k)_j = 0$ for $i \neq j$, so $R^!$ is the entire Yoneda algebra $\operatorname{Ext}_R(k,k)$ and hence $\mathrm{P}^R_k(t) = \operatorname{Hilb}_{R^!}(t)$. Therefore, the formula in Theorem 8.5 can be written

$$\operatorname{Hilb}_R(t)\operatorname{Hilb}_{R^!}(-t) = 1. \tag{8.9}$$

Examples of non-Koszul quadratic algebras for which the above formula holds were constructed by Roos [1995] (see [Positselksi 1995] and [Piontkovskiĭ 2001] for noncommutative examples).

Example 8.10 [Roos 1995, Case B]. Consider the ring

$$R = k[x, y, z, u, v, w]/(x^2+xy,\ x^2+yz,\ xz,\ z^2,\ zu+yv,\ zv,\ uw+v^2).$$

Roos proved that (8.9) holds for R, but R is not Koszul.

It would have been very helpful if one could recognize whether a ring is Koszul or not by just looking at the beginning of the infinite minimal free resolution of k (for example, by computing the beginning of the resolution by computer). Unfortunately, this does not work out. Roos constructed for each integer $q \geq 3$ a quotient $Q(q)$ of a polynomial ring in 6 variables subject to 11 quadratic relations, so that the minimal free resolution of k over $Q(q)$ is linear for the first q steps and has a nonlinear q-th Betti number:

Example 8.11 [Roos 1993]. Choose a number $2 \leq q \in \mathbb{N}$. Let the ring R be

$$\frac{\mathbb{Q}[x, y, z, u, v, w]}{\big(x^2, xy, yz, z^2, zu, u^2, uv, vw, w^2, xz+qzw-uw, zw+xu+(q-2)uw\big)}.$$

Roos proved that

$$b^R_{i,j}(k) = 0 \quad \text{for } j \neq i \text{ and } i \leq q,$$
$$b^R_{q+1,q+2}(k) \neq 0.$$

Generalized Koszul Resolution 8.12 ([Priddy 1970]; also see [Beilinson et al. 1996, 2.8.1] and [Manin 1987]). *If R is Koszul, then the minimal free resolution of k over R can be described by the generalized Koszul complex* (*also called the Priddy complex*) *constructed by Priddy* [1970]. *The j-th term of the complex is*

$$\boldsymbol{K}_j(R,k) := R \otimes_k (R^!_j)^*,$$

where $-^*$ *stands for taking a vector space dual. The differential is defined by*

$$\boldsymbol{K}_{j+1}(R,k) = R \otimes_k (R^!_{j+1})^* \to \boldsymbol{K}_j(R,k) = R \otimes_k (R^!_j)^*,$$
$$r \otimes \varphi \mapsto \sum_{1 \le i \le n} r x_i \otimes \varphi \tilde{z}_i,$$

where $\varphi\tilde{z}_i \in (R^!_j)^*$ *is defined by* $\varphi\tilde{z}_i(e) = \varphi(ez_i)$ *for* $e \in R^!_j$ (*thus,* $\varphi\tilde{z}_i$ *is the composition of* φ *after multiplication by* z_i *on the right*).

In Example 8.7, we considered the case when $R = S/I$ and I is generated by quadratic monomials. In that case, the generalized Koszul complex is described in [Fröberg 1975].

Example 8.13. Let $R = k[x_1, x_2]/(x_1^2, x_1x_2)$. Then R is Koszul and

$$R^! = k\langle z_1, z_2\rangle/(z_2^2).$$

Hence we can resolve k minimally over R by the generalized Koszul complex. Here we compute the first few terms in the resolution. We fix k-bases for $(R^!_i)^*$:

$$R^!_0 = \operatorname{span}\langle 1^*\rangle, \quad R^!_1 = \operatorname{span}\langle z_1^*, z_2^*\rangle, \quad R^!_2 = \operatorname{span}\big\langle (z_1^2)^*, (z_2z_1)^*, (z_1z_2)^*\big\rangle,$$

which we identify with R-bases of $\boldsymbol{K}_i(R,k)$. We then compute the beginning of $\boldsymbol{K}(R,k)$ as

$$\cdots \longrightarrow R^3 \xrightarrow{\begin{pmatrix} x_1 & 0 & x_2 \\ 0 & x_1 & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} R.$$

Koszul rings were introduced by Priddy and he also introduced an approach very similar to using quadratic Gröbner bases. The following result is well-known and often used in proofs that a ring is Koszul:

Theorem 8.14. *An ideal with a quadratic Gröbner basis is Koszul.*

This follows from the fact that the Betti numbers of k over R are less or equal than the Betti numbers of k over $S/\operatorname{in}(I)$ for any initial ideal I (see [Peeva 2011, Theorem 22.9]) and the following result.

Theorem 8.15 [Fröberg 1975]. *Every ideal generated by quadratic monomials is Koszul.*

Two important examples using a quadratic Gröbner basis are described below. The *r-th Veronese ring* is

$$V_{c,r} = \bigoplus_{i=0}^{\infty} T_{ir} = k[\text{all monomials of degree } r \text{ in } c \text{ variables}],$$

and it defines the r-th *Veronese embedding of* $\boldsymbol{P}^{c-1}$. Bărcănescu and Manolache [1981] showed that the defining toric ideal of every Veronese ring has a quadratic Gröbner basis. Thus, the Veronese rings are Koszul.

The toric ideal of the Segre embedding of $\boldsymbol{P}^p \times \boldsymbol{P}^q$ in $\boldsymbol{P}^{pq+p+q}$ is generated by the (2×2)-minors of a $\big((p+1) \times (q+1)\big)$-matrix of indeterminates

$$\{x_{i,j} \mid 1 \le i \le p+1,\ 1 \le j \le q+1\}.$$

Bărcănescu and Manolache [1981] showed that there exists a quadratic Gröbner basis. Thus, the Segre rings are also Koszul.

There exist examples of Koszul rings for which there is no quadratic Gröbner basis:

Example 8.16 (Conca, personal communication). The ring

$$k[x, y, z, w]/(xz,\ x^2 - xw,\ yw,\ yz + xw,\ y^2)$$

is Koszul. However, it has no quadratic Gröbner basis even after change of coordinates because there is no quadratic monomial ideal with the same Hilbert function as the ideal $(xz,\ x^2 - xw,\ yw,\ yz + xw,\ y^2)$, which can be easily verified by computer.

Example 8.17 [Caviglia 2009]. The ideal

$$\begin{aligned} I = \big(&x_8^2 - x_3x_9, x_5x_8 - x_6x_9, x_1x_8 - x_9^2, x_5x_7 - x_2x_8, x_4x_7 - x_6x_9,\\ &x_1x_7 - x_4x_8, x_6^2 - x_2x_7, x_4x_6 - x_2x_9, x_3x_6 - x_7^2, x_1x_6 - x_5x_9, x_3x_5 - x_6x_8,\\ &x_3x_4 - x_7x_9, x_4^2 - x_1x_5, x_2x_4 - x_5^2, x_2x_3 - x_6x_7, x_1x_3 - x_8x_9, x_1x_2 - x_4x_5\big) \end{aligned}$$

is the toric ideal, that is the kernel of the homomorphism

$$k[x_1, \dots, x_9] \longrightarrow k\big[\text{all monomials of degree 3 except } abc \text{ in } k[a, b, c]\big]$$

sending the variables $x_1, \dots, x_9$ to the cubic monomials

$$a^3, b^3, c^3, a^2b, ab^2, b^2c, c^2b, c^2a, a^2c,$$

respectively. This example was introduced by Sturmfels and is very similar to the cubic Veronese. It is called the pinched Veronese. But in contrast to the cubic Veronese, the ideal I has no quadratic Gröbner basis in these variables (it is not known if a quadratic Gröbner basis exists after a change of variables), which Sturmfels verified by computer computation. Caviglia proved that I is Koszul. The example was revisited in two papers: Caviglia and Conca [2013] classify the projections of the Veronese cubic surface to $\mathbb{P}^8$ whose coordinate rings are Koszul, and Vu [2013] proved Koszulness for a more general class of ideals.

Proving cases with no known quadratic Gröbner basis can be challenging. Recently, Nguyen and Vu [2015] introduced a method using Fröbenius-like epimorphisms. Another possibility might be to use filtrations. There are various versions in which this method can be used. The method was formally introduced by Conca, Trung and Valla [2001] with the name "Koszul filtration", although it had been used by other authors previously. If there exists a Koszul filtration of R, then R is Koszul. We define a version of a Koszul filtration: Fix a graded ideal I in S. Let $\mathcal{H}$ be a set of tuples $(L; l)$, where L is a linear ideal (that is, L is generated by linear forms) in $R = S/I$ and l is a linear form in L. Denote by $\overline{\mathcal{H}}$ the set of linear ideals appearing in the tuples in $\mathcal{H}$. A *Koszul filtration* of R is a set $\mathcal{H}$ such that the following two conditions are satisfied:

(1) $(x_1, \dots, x_n) \in \overline{\mathcal{H}}$.

(2) If $(L; l) \in \mathcal{H}$ and $L \neq 0$, then there exists a proper subideal $N \subset L$ such that $L = (N, l)$, $(N : l) \in \overline{\mathcal{H}}$ and $N \in \overline{\mathcal{H}}$.

Note that we do not assume that the ideal I is generated by quadrics.

Open-Ended Problem 8.18. *It is an ever tantalizing problem to find more classes of Koszul rings and to develop new approaches that can be used to show that a ring is Koszul in the absence of a quadratic Gröbner basis.*

Here is a sample conjecture:

Conjecture 8.19 [Bøgvad 1994]. *The toric ring of a smooth projectively normal toric variety is Koszul.*

The idea to consider linear minimal free resolutions of k naturally leads to the consideration of linear minimal free resolutions of other modules. We say that M has a *linear* (or a p-linear) minimal free resolution if $b_{ij}^R(M) = 0$ for all i and $j \neq i + p$; in particular, M is generated in degree p in this case. Equivalently, M is generated in one degree and has linear entries in the matrices of the differentials (in any basis) of its minimal free resolution. As in Theorem 8.5, a straightforward computation shows that

$$t^p \, \mathrm{P}_M^R(t) = (-1)^p \frac{\mathrm{Hilb}_M(-t)}{\mathrm{Hilb}_R(-t)},$$

for such modules.

We close this section by outlining a problem on Koszul rings coming from the theory of hyperplane arrangements. A set $\mathcal{A} = \bigcup_{i=1}^s H_i \subseteq \mathbb{C}^r$ is a *central hyperplane arrangement* if each H_i is a hyperplane containing the origin. Arnold considered the case when $\mathcal{A}$ is a braid arrangement and constructed the cohomology algebra of the complement. For any central hyperplane arrangement, Orlik and Solomon [1980] provided a description of the cohomology algebra

$A := \mathbf{H}^*(\mathbb{C}^r \setminus \mathscr{A}; \mathbb{C})$ of the complement of $\mathscr{A}$; it is a quotient of an exterior algebra by a combinatorially determined ideal. Namely, if E is the exterior algebra on n variables $e_1, \dots, e_n$ over $\mathbb{C}$, then the *Orlik-Solomon algebra* is $A = E/J$, where J is generated by the elements

$$\partial(e_{i_1 \dots i_p}) = \sum_{\substack{1 \le q \le p \\ \operatorname{codim}(H_{i_1} \cap \dots \cap H_{i_p}) < p}} (-1)^{q-1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_q}} \wedge \dots \wedge e_{i_p},$$

and $\widehat{e_{i_q}}$ means that e_{i_q} is omitted. In the introduction to [Hirzebruch 1983], Hirzebruch wrote: "The topology of the complement of an arrangement of lines in the projective plane is very interesting, the investigation of the fundamental group of the complement very difficult." The fundamental group $\pi_1(X)$ of the complement $X = \mathbb{C}^r \setminus \mathscr{A}$ is interesting, complicated, and few results are known about it. Let

$$Z_1 = \pi_1(X), \dots, Z_{i+1} = [Z_i, \pi_1(X)], \dots$$

be the *lower central series* and set $\varphi_i = \operatorname{rank}(Z_i/Z_{i+1})$. For supersolvable arrangements, Falk and Randell [1985] have shown that these numbers are determined by the Orlik–Solomon algebra A through the LCS (Lower Central Series) Formula

$$\prod_{j=1}^{\infty} (1 - t^j)^{\varphi_j} = \sum_{i \ge 0} (-t)^i \dim A_i.$$

It was first noted by Shelton and Yuzvinsky [1997] that the formula holds precisely when the algebra A is Koszul, that is, when $b^A_{i,i+j}(\mathbb{C})$ vanish for $j \neq 0$. As described in [Falk and Randell 2000], much progress has been made on the investigation of the fundamental group $\pi_1(X)$ of the complement, but the following challenging problem [Falk and Randell, Problem 2.2] remains open: *Does there exist a nonsupersolvable central hyperplane arrangement for which the LCS Formula holds*? Peeva showed [2003] that a central hyperplane arrangement is supersolvable if and only if J has a quadratic Gröbner basis with respect to some monomial order. Thus, the above problem is equivalent to:

Problem 8.20. *Does there exist a central hyperplane arrangement for which A is Koszul but J does not have any quadratic Gröbner basis*?

9. Slope and shifts

The following simple example shows that infinite regularity can occur if R is not a polynomial ring: resolving $R/(x^2)$ over $R = k[x]/(x^4)$ we get

$$\dots \longrightarrow R(-6) \xrightarrow{x^2} R(-4) \xrightarrow{x^2} R(-2) \xrightarrow{x^2} R.$$

The infinite Betti table is

	0	1	2	3	…
0:	1	-	-	-	…
1:	-	1	-	-	…
2:	-	-	1	-	…
3:	-	-	-	1	…
⋮	⋮	⋮	⋮	⋮	⋱

(9.1)

If the regularity is infinite, we study another numerical invariant, called slope. The concept was introduced by Backelin [1988], who defined and studied a different version called rate. It is easier to visualize the slope. Consider the *maximal shift at step i*

$$t_i(M) = \max\{j \mid \operatorname{Tor}^{S/I}_{i,j}(M,k) \neq 0\}$$

and the *adjusted maximal shift*

$$r_i(M) = \max\{j \mid \operatorname{Tor}^{S/I}_{i,i+j}(M,k) \neq 0\},$$

so $r_i(M) = t_i(M) - i$. Note that $r_0(M)$ is then the maximal degree of an element in a minimal system of generators of M. Following Eisenbud (personal communication), we consider the *slope*

$$\operatorname{slope}_R(M) = \sup\left\{\frac{r_i(M) - r_0(M)}{i} \;\middle|\; i \geq 1\right\}, \tag{9.2}$$

which is the minimal absolute value of the slope of a line in the Betti table through position $(0, r_0(M))$ and such that there are only zeros below it. For example, in the Betti table (9.1) above we consider the following line with slope -1:

	0	1	2	3	…
0:	1	-	-	-	…
1:	-	1	-	-	…
2:	-	-	1	-	…
3:	-	-	-	1	…
⋮	⋮	⋮	⋮	⋮	⋱

In (9.2) we start measuring the slope at homological degree 1 because if we start in homological degree 0 then we can make a dramatic change of the invariant by simply increasing by a large number q the degrees of the elements in a minimal system of generators of M, while the structure of the minimal free resolution will remain the same (the graded Betti numbers will get shifted by q). Note that our definition of slope is slightly different than the one introduced by

Avramov, Conca and Iyengar [2010] which is measuring the slope in a different Betti table with entries $b_{i,j}$ instead of our entries $b_{i,i+j}$.

Straightforward computation using Serre's Inequality (7.2) implies the following result.

Theorem 9.3. *Every finitely generated graded R-module has finite slope over R.*

In some situations it might be helpful to consider the slope of a syzygy module

$$\operatorname{slope}_R(\operatorname{Syz}_s(M)) = \sup\left\{\frac{r_{i+s}(M) - r_s(M)}{i} \;\middle|\; i \geq 1\right\} \quad \text{for a fixed } s,$$

or measure the slope starting at a later place v by

$$\operatorname{slope}_R(M, v) = \sup\left\{\frac{r_i(M) - r_0(M)}{i} \;\middle|\; i \geq v\right\}.$$

Both concepts lead to the following open problem, recently raised by Conca:

Open-Ended Problem 9.4 (Conca, personal communication, 2012). *Describe the asymptotic properties of slope for particular classes of rings.*

The first version of the concept slope was introduced by Backelin [1988] and it is the *rate* of a module defined by

$$\operatorname{rate}_R(M) = \sup\left\{\frac{t_i(M) - 1}{i - 1} \;\middle|\; i \geq 2\right\}.$$

Clearly,

$$\operatorname{rate}_R(k) = \operatorname{slope}_R\big((x_1, \ldots, x_n)\big) + 1.$$

He also considered

$$\operatorname{slant}_S(R) = \sup\left\{\frac{t_i(R)}{i} \;\middle|\; i \geq 1\right\},$$

which he denoted by $\operatorname{rate}(\varphi)$ for $\varphi : S \to R$. Backelin proved some inequalities, which are usually not sharp.

Theorem 9.5 [Backelin 1988, Theorem 1; Avramov, Conca and Iyengar 2010, Proposition 1.2].

$$\begin{aligned} \operatorname{slope}_R(M) &\leq \max\big\{\operatorname{slope}_S(M), \operatorname{slope}_S(R)\big\}, \\ \operatorname{slant}_S(R) &\leq \operatorname{rate}_R(k) + 1. \end{aligned}$$

Note that $\text{rate}_R(k) = 1$ is equivalent to R being Koszul. Thus, if R is Koszul then Theorem 9.5 shows that

$$t_i^S(S/I) \le 2i,$$

for every $i \ge 1$. The following inequality is conjectured:

Conjecture 9.6 [Avramov, Conca and Iyengar 2015, Introduction]. *Suppose that $R = S/I$ is Koszul. Then*

$$t_{i+j}^S(S/I) \le t_i^S(S/I) + t_j^S(S/I),$$

for all $i \ge 1,\ \ j \ge 1$.

See [Herzog and Srinivasan 2013] and [McCullough 2012] for related results.

Open-Ended Problem 9.7. *It would be interesting to study the properties of the shifts over R.*

The rate is known in very few cases, for example:

Theorem 9.8. *Let I be a graded ideal generated in degrees $\le r$ and such that it has a minimal generator in degree r.*

(1) [Eisenbud, Reeves and Totaro 1994] *If I is generated by monomials then* $\text{rate}_{S/I}(k) = r - 1$.

(2) [Gasharov, Peeva and Welker 2000] *If I is a generic toric ideal*, *then we have* $\text{rate}_{S/I}(k) = r - 1$.

In both cases, *the rate is achieved at the beginning of the free resolution.*

Open-Ended Problem 9.9. *Determine the slope* (*rate*) *for other nice classes of quotient rings*, *or obtain upper bounds on it.*

We close this section in a related direction with the following interesting problem.

Eisenbud–Huneke Question 9.10 [Eisenbud and Huneke 2005, Question A]. Let M be a finitely generated graded R-module. Does there exists a number u and bases of the free modules in the minimal graded R-free resolution $\boldsymbol{F}$ of M, such that for all $i \ge 0$ the entries in the matrix of the differential d_i have degrees $\le u$?

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