SOME NON-KOSZUL ALGEBRAS

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The aim of this note is to show that the quadratic algebras $\mathcal{E}_n$ studied by S. Fomin and A. N. Kirillov in [FK] are *not* Koszul algebras for any $n \geq 3$. The algebra $\mathcal{E}_n$ (of type A) has generators τ_{ij}, for $1 \leq i < j \leq n$, subject to the following relations:

(i) $\tau_{ij}^2 = 0$ for $i < j$;

(ii) $\tau_{ij}\tau_{jk} = \tau_{jk}\tau_{ik} + \tau_{ik}\tau_{ij}$,
$\tau_{jk}\tau_{ij} = \tau_{ik}\tau_{jk} + \tau_{ij}\tau_{ik}$ for $i < j < k$;

(iii) $\tau_{ij}\tau_{kl} = \tau_{kl}\tau_{ij}$ whenever $\{i,j\} \cap \{k,l\} = \phi$, $i < j$, and $k < l$.

We will prove the following result (all algebras are over a field k).

Theorem. *$\mathcal{E}_n$ is an algebra retract of any $\mathcal{E}_m$ for $m \geq n$. The Hilbert series of $R = \mathcal{E}_3$ and its Koszul dual $R^!$ satisfy*

$$R(t)R^!(-t) = 1 - t^6 \; .$$

Hence $\mathcal{E}_3$ is not a Koszul algebra, and therefore the higher $\mathcal{E}_n$'s are not Koszul either.

Proof. Consider the algebra maps $in : \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$ and $pr : \mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n$, where pr is defined by dividing out by the two-sided ideal generated by the $\tau_{i,n+1}$ for $1 \leq i \leq n$, and in is the tautological inclusion $\tau_{ij} \mapsto \tau_{ij}$. The equation $pr \circ in = Id_{\mathcal{E}_n}$ is obvious. We now use one of the characterisations of Koszul algebras given in Löfwall [Lö, p. 305]: a quadratic k-algebra A is Koszul if and only if $\mathrm{Tor}^A_{i,j}(k,k) = 0$ for $i \neq j$. Since the induced map of in in Tor defines an injection of the bigraded Tor of $\mathcal{E}_n$ into the bigraded Tor of $\mathcal{E}_{n+1}$, it follows that if $\mathcal{E}_{n+1}$ is Koszul, then so is $\mathcal{E}_n$, and it remains to prove that the algebra $\mathcal{E}_3$ is not Koszul.

Let us denote $x = \tau_{1,2}$, $y = \tau_{1,3}$, $z = \tau_{2,3}$. Then

$$\mathcal{E}_3 = k\langle x,y,z\rangle/(x^2, y^2, z^2, xz - zy - yx, zx - yz - xy) \; ,$$

Mathematics Subject Classifications: 16E99, 18G15.

where $k\langle x, y, z\rangle$ stands for the free associative algebra generated by x, y, z. If X, Y, Z are the dual variables for x, y, z, respectively, then the Koszul dual $R^!$ of $R = \mathcal{E}_3$ (we use the notations of Manin [Ma]) is given by

$$R^! = k\langle X, Y, Z\rangle/(XZ + ZY, XZ + YX, ZX + YZ, ZX + XY) . \quad (*)$$

The Hilbert series of this algebra is

$$R^!(t) = 1 + 3t + 5t^2 + \frac{6t^3}{1-t} = (1+t)(1+t+t^2)/(1-t) . \quad (**)$$

This can either be proved by a direct hand calculation or with the help of Gröbner bases. For the background on Gröbner bases, we refer the reader to the book by V. Ufnarovskij [U], in particular its Chapter 2 and examples studied therein. A *finite* noncommutative Gröbner basis for the two-sided ideal in (*) and the natural term order induced from $X < Y < Z$ is given by

$$(YX + XZ, YZ - XY, ZX + XY, ZY + XZ, XZ^2 - XY^2, XY^3 - X^3Y) . \quad (***)$$

Thus the Hilbert series of (*) is the same as the one of the monomial ideal of leading terms of (***), i.e., the ideal $(YX, YZ, ZX, ZY, XZ^2, XY^3)$. The Hilbert series of this monomial ideal is (**). Since we already know (see [FK]) that $R(t) = (1+t)^2(1+t+t^2)$, it follows that $R(t)R^!(-t) = 1 - t^6$. Since $R(t)R^!(-t) = 1$ is a necessary (but *not* sufficient!—cf. [P.R2]) condition for R to be Koszul, the algebra $R = \mathcal{E}_3$ is not Koszul. □

Remark. Although the algebras $\mathcal{E}_n$ are not Koszul, they (and their Koszul duals) seem to have nice homological properties that show that they are rather close to being Koszul. Let us mention some results (without proofs) to show what we mean by our last assertion. Calculating the graded Betti numbers of R and $R^!$ (i.e., the dimensions of the $\mathrm{Tor}^R_{i,j}(k,k)$ and the $\mathrm{Tor}^{R^!}_{i,j}(k,k)$ for $R = \mathcal{E}_3$), we obtain the following two tables, where the second horizontal row is the list of the dimensions of the $\mathrm{Tor}^R_{i,i}(k,k)$, the third row is the list of the dimensions of the $\mathrm{Tor}^R_{i,i+1}(k,k)$, etc. (note that $\mathrm{Tor}^R_{i,j}(k,k) = 0$ for $j < i$):

Dimensions of $\mathrm{Tor}^R_{i,j}(k,k)$

$i \longrightarrow$	0	1	2	3	4	5	6	7	8	9
$j = i$	1	3	5	6	6	6	6	6	6	6
$j = i+1$	0	0	0	0	0	0	0	0	0	0
$j = i+2$	0	0	0	0	1	3	5	6	6	6
$j = i+3$	0	0	0	0	0	0	0	0	0	0
$j = i+4$	0	0	0	0	0	0	0	0	1	3

THE TABLE CONTINUES TO THE RIGHT AND BELOW.

Dimensions of $\mathrm{Tor}^{R^!}_{i,j}(k,k)$

$i \longrightarrow$	0	1	2	3	4	5	6	7	8	9
$j = i$	1	3	4	3	1	0	0	0	0	0
$j = i+1$	0	0	0	0	0	0	0	0	0	0
$j = i+2$	0	0	0	0	1	3	4	3	1	0
$j = i+3$	0	0	0	0	0	0	0	0	0	0
$j = i+4$	0	0	0	0	0	0	0	0	1	3

THE TABLE CONTINUES TO THE RIGHT AND BELOW.

Using the notation

$$P_R(x,y) = \sum_{i\geq 0, j\geq 0} \dim_k \left(\mathrm{Tor}^R_{i,j}(k,k)\right) x^i y^j$$

(and similarly for $P_{R^!}(x,y)$), we obtain that

$$P_R(x,y) = \frac{(1+xy)(1+xy+x^2y^2)}{(1-x^4y^6)(1-xy)}$$

and

$$P_{R^!}(x,y) = (1+xy)^2(1+xy+x^2y^2)/(1-x^4y^6)\ .$$

The first of these formulas shows that

$$1/P_R(x,y) = (1-1/x^2)/R^!(xy) + R(-xy)/x^2\ .$$

It is proved in [R1] that the latter identity holds if the ring R satisfies the $\mathcal{L}_4$ condition (we only study commutative local k-algebras there), i.e., if the

"Koszul complex" $\mathrm{Hom}_k(R,k)\otimes_k R^!$ has only two non-vanishing homology groups: H_3 and H_0. If all homology groups H_i for $i>0$ vanish, then we have a Koszul algebra. More details about this type of reasoning can be found in [R1-R5]. In this sense our algebra $R=\mathcal{E}_3$ is close to a Koszul algebra. The $\mathcal{E}_n$ for $n\geq 4$ should have similar but more complicated properties. The first non-vanishing Tor group $\mathrm{Tor}_{i,j}^{\mathcal{E}_n}(k,k)$ for $i\neq j$ occurs for $(i,j)=(4,6)$, and is 8-dimensional for $n=4$ and 30-dimensional for $n=5$. Using [B], theorem 3.3, (b), (iv) (cf. also L. Positselski [P]), we can now deduce similar results for the first non-vanishing $\mathrm{Tor}_{i,j}^{\mathcal{E}_n^!}(k,k)$.

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Received 4/18/97