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The Basis Monomial Ring of a Matroid*

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We define the basis monomial ring M_G of a matroid G and prove that it is Cohen–Macaulay for finite G. We then compute the Krull dimension of M_G, which is the rank over Q of the basis-point incidence matrix of G, and prove that $\dim B_G \geqslant \dim M_G$ under a certain hypothesis on coordinatizability of G, where B_G is the bracket ring of G.

There has been considerable interest recently in the applications of Cohen–Macaulay rings to combinatorics, especially in the work of Stanley. The question has been raised whether the bracket ring B_G of a matroid G is Cohen–Macaulay. As a step in this direction, we define the basis monomial ring M_G and prove that it is Cohen–Macaulay for all finite G. Since $B_G/\text{rad } B_G \cong M_G$ if G is unimodular, and rad $B_G = 0$ is conjectured in [13], we have settled the unimodular case modulo the conjecture. In the separate case of "skew-Schubert matroids," Stanley [10] proved B_G is Cohen–Macaulay. As a prelude to the Cohen–Macaulayness of M_G we prove that any monomial d on the elements of G, of degree a multiple of the rank of G, is factorable into bases of G provided some power d^p is factorable.

We proceed to a combinatorial computation of the Krull dimension of M_G. We then show $\dim B_G \geqslant \dim M_G$ for coordinatizable G, and relate this to results in [14] which characterize $\dim B_G$ in terms of the maximum transcendence degree of a coordinatization of G.

Although we use much of the notation and terminology of [1], we use the term "matroid" rather than the synonym "combinatorial pregeometry" because our Theorem 1 is completely matroidal in flavor.

Let $G(S)$ be a matroid (or combinatorial pregeometry) of rank n on the set S. If R is a commutative ring with 1, we consider the commutative polynomial ring $R[S]$. Let $\mathscr{M}$ be the multiplicative monoid (= semigroup with identity) consisting of all monomials $\prod_{s\in S} s^{e_s}$ in $R[S]$, where $e_s \in \mathbb{N} \cup \{0\}$ for all s, and $e_s = 0$ for all but a finite number of elements s. Let N be the set of all square-free monomials $s_1 s_2 \cdots s_n$ where $\{s_1, s_2, \ldots, s_n\}$ is a basis of G, and $\mathscr{M}_G$ the submonoid of $\mathscr{M}$ generated by N. The *basis monomial ring* $M_G{}^R$ is the ring $R[N] = R[\mathscr{M}_G]$, the subring of $R[S]$ generated by N over R.

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THEOREM 1. *For every matroid G, if rank $G = n$, $d \in \mathscr{M}$, degree $d = ln$ for some $l \in \mathbb{N} \cup \{0\}$, and $d^p \in \mathscr{M}_G$ for some $p \in \mathbb{N}$, then $d \in \mathscr{M}_G$.*

Proof. We must show that a monomial d of degree ln is factorable into bases of G, given that d^p is factorable into bases.

Let us change notation by omitting elements of exponent zero, so that $d = \prod_{i=1}^{N} s_i^{g_i}$ where $g_i > 0$, for $1 \leqslant i \leqslant N$. We construct sets $T = \{t_i^j \mid 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant g_i\}$ and $U = \{u_i^{j,k} \mid 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant g_i, 1 \leqslant k \leqslant p\}$ with all t_i^j distinct and all $u_i^{j,k}$ distinct. Let $G'(T)$ and $G''(U)$ be matroids defined by

$$A = \{t_{i_1}^{j_1}, \ldots, t_{i_m}^{j_m}\} \subseteq T \text{ is independent in } G'$$

$$\Leftrightarrow \{s_{i_1}, \ldots, s_{i_m}\} \text{ has } m \text{ distinct elements and is independent in } G$$

$$\Leftrightarrow B = \{u_{i_1}^{j_1,k_1}, \ldots, u_{i_m}^{j_m,k_m}\} \subseteq U \text{ is independent in } G'',$$

for all $i_1, \ldots, i_m, j_1, \ldots, j_m, k_1, \ldots, k_m$. Since $j \neq h$ implies $\{t_i^j, t_i^h\}$ is dependent, we have replaced s_i in G by g_i elements (resp., pg_i elements) all contained in the same closed point of G' (resp., G''). For any set $A = \{t_{i_1}^{j_1}, \ldots, t_{i_m}^{j_m}\} \subseteq T$, we associate the set $B_A = \{u_i^{j,k} \mid \text{there exists } l \leqslant m \text{ such that } i = i_l, j = j_l; k \leqslant p\}$. Thus $|B_A| = p|A|$, and $r''B_A = r'A$, where r'' (resp., r') denotes rank in G'' (resp., G').

We now observe that d (resp., d^p) may be factored into l (resp., pl) bases of $G \Leftrightarrow T$ (resp., U) may be partitioned into bases of G' (resp., G''). We apply a theorem of Edmonds [2] that a matroid $H(V)$ may be partitioned into k independent subsets $\Leftrightarrow |A| \leqslant kr_H(A)$ for all $A \subseteq V$. Since d^p may be factored into pl bases of G, U may be partitioned into pl independent sets of G'', hence for all $B \subseteq U$, $|B| \leqslant plr''(B)$. Thus for all $A \subseteq T$, $|A| = |B_A|/p \leqslant lr''(B_A) = lr'(A)$ and hence T may be partitioned into l independent sets. Since $|T| = ln$, each of the independent sets is in fact a basis. Thus $d \in \mathscr{M}_{G'}$. Q.E.D.

THEOREM 2. *If $G(S)$ is finite and R is a Cohen-Macaulay domain, then M_G^R is a Cohen-Macaulay domain.*

Proof. We define a monoid $\mathscr{N}$ to be *normal* if $ab^p = c^p$ for $a, b, c \in \mathscr{N}$, $p \in \mathbb{N}$, implies that there exists $d \in \mathscr{N}$ such that $d^p = a$. We now verify that $\mathscr{M}_G$ is normal. Let $ab^p = c^p$ in $\mathscr{M}_G$, with $b = \prod_{s \in S} s^{e_s}$ and $c = \prod_{s \in S} s^{f_s}$. Thus $a = \prod_{s \in S} s^{p(f_s - e_s)}$, and $d = \prod_{s \in S} s^{f_s - e_s} \in \mathscr{M}$ and $d^p = a \in \mathscr{M}_G$. Since $db = c$ and b and c must each have degree a multiple of n, so must d, and by Theorem 1, $d \in \mathscr{M}_G$, proving that $\mathscr{M}_G$ is normal.

Now Hochster [3, Theorem 1] shows that if R is Cohen–Macaulay and $\mathscr{N}$ a finitely generated normal monoid of monomials then $R[\mathscr{N}]$ is Cohen–Macaulay. If G is finite, then $\mathscr{M}_G$ is finitely generated, hence M_G^R is Cohen–Macaulay if R is. Furthermore, M_G^R is a domain since it is a subring of the domain $R[S]$. Q.E.D.

In [13, Remark 6.6], we showed (using different notation) that if G is unimodular and B_G is the bracket ring of G, then $B_G/\text{rad}\, B_G \cong M_G{}^{\mathbb{Z}}$. Similarly, for any Cohen–Macaulay domain R, if $B_G{}^R$ is the bracket ring with coefficients from R, then $B_G{}^R/\text{rad}\, B_G{}^R \cong M_G{}^R$.

COROLLARY 3. *If R is a Cohen–Macaulay domain and $G(S)$ is a finite unimodular matroid, then $B_G{}^R/\text{rad}\, B_G{}^R$ is Cohen–Macaulay.*

We now proceed to compute the Krull dimension of $M_G{}^k$, where k is a field. Let $k[N]$ be the subring of a polynomial ring $k[x_1 ,..., x_u]$ generated over k by a finite set $N = \{n_1 ,..., n_v\}$ of monomials. We define the *monomial incidence matrix* to be the $v \times u$ rational matrix $I = (a_{ij})$, where $n_i = \prod_{j=1}^{u} x_j^{a_{ij}}$, $a_{ij} \in \mathbb{N} \cup \{0\}$.

LEMMA 4. $\dim k[N] = \text{rank}_Q I$, *where Q is the rational field.*

Proof. From [14, Theorem 2.8], $\dim k[N]$ = the maximum cardinality of a subset of N which is algebraically independent over k. Let $f(n_1 ,..., n_v) = 0$ for some polynomial f, $0 \neq f(Y_1 ,..., Y_v) \in k[Y_1 ,..., Y_v]$. However, if we consider the grading g on $k[Y_1 ,..., Y_v]$ induced by $g(Y_i) = n_i$, then each homogeneous component f_i of f must also satisfy $f_i(n_1 ,..., n_v) = 0$ in $k[x_1 ,..., x_u]$. Thus if $\beta \prod_{i=1}^{v} Y_i^{b_i}$ and $\gamma \prod_{i=1}^{v} Y_i^{c_i}$ are two terms of the homogeneous polynomial f_i, where $\beta, \gamma \in k$, we have $\prod_{i=1}^{v} n_i^{b_i} = \prod_{i=1}^{v} n_i^{c_i}$. If $\langle n_i \rangle = (a_{i1} ,..., a_{iu})$ is the row of I corresponding to n_i, then $\sum_{i=1}^{v} b_i \langle n_i \rangle = \sum_{i=1}^{v} c_i \langle n_i \rangle$. Hence if $n_{i_1} ,..., n_{i_l}$ are algebraically dependent over k, then the vectors $\langle n_{i_1} \rangle ,..., \langle n_{i_l} \rangle$ are linearly dependent over Q. The converse is easily seen to hold as well, and the lemma follows. Q.E.D.

The following theorem was proved by T. Brylawski and R. Stanley (unpublished).

THEOREM 5. *For every finite matroid $G(S)$ of positive rank,* $\dim M_G{}^k = | S | - c(G) + 1$, *where $c(G)$ is the number of connected components of G.*

Proof. $M_G{}^k = k[N]$ where N = the set of monomials which are bases of G. Thus we must show $\text{rank}_Q I(G) = | S | - c(G) + 1$, where $I(G)$ is the basis-point incidence matrix of G. We begin by inducting on $c(G)$. If $G(S) = G_1(S_1) \oplus G_2(S_2)$, suppose first that rank $G_1 > 0$ and rank $G_2 = 0$ in G. Then $| S_2 | = c(G_2)$, hence $\text{rank}_Q I(G) = \text{rank}_Q I(G_1) = | S_1 | - c(S_1) + 1 = | S | - c(S) + 1$, using the induction hypothesis on G_1. Thus we may assume that rank $G_1 > 0$ and rank $G_2 > 0$ in G. By the induction hypothesis, $\text{rank}_Q I(G_i) = | S_i | - c(G_i) + 1$, $i = 1, 2$. If $A_1 ,..., A_r$ are the rows of $I(G_1)$ corresponding to the r bases of G_1, and $B_1 ,..., B_t$ are the rows of $I(G_2)$, then the rows of $I(G)$ are the juxtaposed rows $A_i B_j$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant t$, since S_1 and S_2 are disjoint and the bases of G are precisely all unions of a basis of G_1 with a basis of G_2.

If $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are bases of the row spaces of $I(G_1)$ and $I(G_2)$, it is straightforward to verify that $A_1B_1, A_1B_2, \ldots, A_1B_q, A_2B_1, \ldots, A_pB_1$ is a basis of the row space of $I(G)$, hence

$$\begin{aligned}\operatorname{rank}_Q I(G) &= \operatorname{rank}_Q I(G_1) + \operatorname{rank}_Q I(G_2) - 1\\ &= |S_1| + |S_2| - c(G_1) - c(G_2) + 1\\ &= |S| - c(G) + 1.\end{aligned}$$

Thus we are reduced to proving the case $c(G) = 1$, i.e., G is connected.

We now prove the connected case by induction on $|S|$, the case for $|S| = 1$ being trivial. Let $x \in S$, and suppose $G - x$ is connected. Then $\operatorname{rank}_Q I(G - x) = |S| - 1$ by the induction hypothesis. Since G has positive rank and no loops, there exists a basis B such that $x \in B$. Then

$$I(G) = \begin{matrix} \\ B \\ \\ \end{matrix}\left\| \begin{array}{c|c} & x \\ I(G-x) & 0 \\ \hline * & 1 \\ \hline * & 1 \end{array} \right\|$$

The row corresponding to B is clearly independent of the rows above it, hence $\operatorname{rank}_Q I(G) \geqslant \operatorname{rank} I(G - x) + 1 = |S|$. Since $I(G)$ has $|S|$ columns, $\operatorname{rank}_Q I(G) = |S| = |S| - c(G) + 1$.

There remains the case where G is connected and $G - x$ is disconnected. Let $G - x = G_1(T_1) \oplus \cdots \oplus G_k(T_k)$, and let $B^* = B_1 \oplus \cdots \oplus B_k$ be a basis of $G - x$, B_i is a basis of $G_i(T_i)$. Let C be the basic circuit of x with respect to B^* in G. Then $C \cap B_i \neq \varnothing$ for all i, $1 \leqslant i \leqslant k$, since G is connected. In G/x, $G_i(T_i)$ are all connected subgeometries and $C - \{x\}$ is a circuit intersecting each $G_i(T_i)$. Hence by [1, Proposition 14.1] G/x is connected. Since G is connected, there exists a basis B such that $x \notin B$. Then

$$I(G) = \begin{matrix} \\ B \\ \\ \end{matrix}\left\| \begin{array}{c|c} & x \\ I(G/x) & 1 \\ \hline * & 0 \\ \hline * & 0 \end{array} \right\|$$

Let $v = (v^j)$ denote the row corresponding to B and $v_i = (v_i^j)$, $1 \leqslant i \leqslant s$, the preceding rows, which correspond to the bases of G containing x. Since $\operatorname{rank}_Q I(G/x) = |S| - 1$ by the induction hypotheses, and $I(G)$ has $|S|$ columns, it suffices to show that v is linearly independent of $v_1, \ldots, v_s$. Suppose to the contrary that $v = \sum_{i=1}^{s} \alpha_i v_i$, $\alpha_i \in Q$. From the column corresponding to x we see that $\sum_{i=1}^{s} \alpha_i = 0$. But since each row is the incidence row of a basis of G,

$\sum_j v_i^j = n$ for all i. Thus $\sum_i \sum_j \alpha_i v_i^j = 0 = \sum_j v^j = n$, a contradiction. Thus v is independent of $v_1, \ldots, v_s$ and $\operatorname{rank}_Q I(G) = |S| = |S| - c(G) + 1$. Q.E.D.

COROLLARY 6. *$\dim M_G^k$ is independent of the field k.*

PROPOSITION 7. *If $G(S)$ is a finite matroid and is coordinatizable over an extension field of k, then*

$$\dim B_G^k \geqslant \dim M_G^k.$$

Proof. By [6, Theorem 4] there exists a finite algebraic extension K/k such that G is coordinatizable over K. Thus by [13, Propositions 2.1 and 6.1] there exists a prime ideal P in B_G^K such that no bracket is an element of P and the residue field is $K(P) \cong K$. If the canonical homomorphism is η: $B_G^K \to B_G^K/P \to K$, we define a K-algebra homomorphism $\eta_0 : B_G^K \to M_G^K$ by $\eta_0[x_1, \ldots, x_n] = \eta[x_1, \ldots, x_n]\, x_1 \cdots x_n$. If n is a monomial in M_G^K, n factors into bases $X_1, \ldots, X_l$, hence $\eta_0((1/(\eta[X_1] \cdots \eta[X_l]))[X_1] \cdots [X_l]) = n$. Since η is an epimorphism, it follows that η_0 is an epimorphism, and hence $\dim B_G^K \geqslant \dim M_G^K$. However, from [14], $\dim B_G^K = \dim B_G^k$, and from Corollary 6, $\dim M_G^K = \dim M_G^k$, and the proposition follows. Q.E.D.

COROLLARY 8. *Let $G(S)$ be a finite matroid coordinatizable over a field k. Then G has a weak coordinatization matrix in echelon form with at least $|S| - c(G)$ entries which are algebraically independent$/k$. If G is unimodular, this result is the best possible.*

Proof. This follows from the result in [14] that $\dim B_G^k - 1 =$ the maximum transcendence degree$/k$ of a weak coordinatization of G in echelon form. Q.E.D.

Remarks. If G is unimodular, then $\dim B_G^k = \dim (B_G^k/\operatorname{rad} B_G^k) = \dim M_G^k$ for all k. The inequality in Proposition 7 is strict, for example, if G is the four-point line, and perhaps it is strict if and only if G is nonbinary. The hypothesis that G be coordinatizable over an extension of k is necessary, as the following example shows. Let G be the 7-point Fano plane. Then $\dim M_G^k = 7$ for any field k, and it is not difficult to show that if char $k = 2$, the maximum transcendence degree of a weak coordinatization of G in echelon form is 6, hence $\dim B_G^k = 7$. However if char $k \neq 2$, G cannot be coordinatized over an extension of k. Thus any prime P in B_G^k yields a coordinatization of a proper weak-map image F of G. By results of Lucas [5], F is a disconnected unimodular matroid. Thus

$$\dim B_G^k = \max_P \operatorname{coht} P = \max_F \dim B_F^k = \max_F \dim M_F^k \leqslant 6,$$

since F is on the same set S as G, and $c(F) \geqslant 2$. In fact $\dim B_G^k = 6$, since there exists a proper weak-map image F of G with $c(F) = 2$.

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