(1)

Math 8680 Topics in Combo

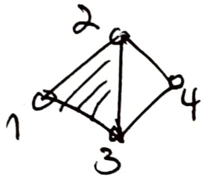
Feb. 19, 2025 Subbing for Anna Weigandt

Recall for a (fin. gen'd) $\mathbb{N}^n$-graded S-module M, $S = \Bbbk[x_1,\ldots,x_n]$

$\exists$ a minimal $\mathbb{N}^n$-graded S-free resolution (MFR)

$$0 \leftarrow M \leftarrow \bigoplus_{\underline{b}\in\mathbb{N}^n} S(-\underline{b})^{\beta_{0,\underline{b}}} \leftarrow \bigoplus_{\underline{b}\in\mathbb{N}^n} S(-\underline{b})^{\beta_{1,\underline{b}}} \leftarrow \cdots \leftarrow \bigoplus_{\underline{b}\in\mathbb{N}^n} S(-\underline{b})^{\beta_{n,\underline{b}}} \leftarrow 0$$

and $\beta_{i,\underline{b}}(M) = \dim_{\Bbbk} \mathrm{Tor}_i^S(\Bbbk, M)$ where $\Bbbk = S/(x_1,\ldots,x_n)$ as S-module.

EXAMPLE: $\Delta =$  has $I_\Delta = (x_1x_4, x_2x_3x_4) \subset S = \Bbbk[x_1,x_2,x_3,x_4]$

and MFR (by hand or Macaulay 2 - do demo!)

$$0 \leftarrow I_\Delta \xleftarrow{\begin{array}{c|cc} & \overset{1001}{14} & \overset{0111}{234} \\ \overset{0000}{\varnothing} & x_1x_4 & x_2x_3x_4 \end{array}} \underbrace{S(-\overset{14}{1001}) \oplus S(-\overset{234}{0111})}_{\mathrm{Tor}_0} \xleftarrow{\begin{array}{c|c} & \overset{1111}{1234} \\ \overset{1001}{14} & -x_2x_3 \\ \overset{0111}{234} & x_1 \end{array}} \underbrace{S(-\overset{1234}{1111})}_{\mathrm{Tor}_1} \leftarrow 0$$

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3 (related) results from Miller-Sturmfels's book that predict the $\beta_{i,\underline{b}}(I_\Delta)$:

(1) We saw THM 1.34: For a monomial ideal $I \subset S$,

$$\beta_{i,\underline{b}}(I) = \dim_{\Bbbk} \tilde{H}_{i-1}(K^{\underline{b}}(I); \Bbbk)$$

where $K^{\underline{b}}(I) \overset{\text{DEF}}{:=} \{\sigma \in \{0,1\}^n : \underline{x}^{\underline{b}-\sigma} \in I\}$ upper Koszul simplicial complex for I in degree $\underline{b}$

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② We stated …

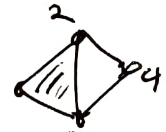
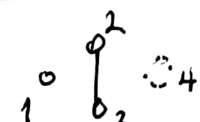
THM 1.40 (dual Hochster formula)

For squarefree monomial $I_\Delta \subset S$,

$$\beta_{i,\underline{b}}(I_\Delta) = \begin{cases} 0 & \text{if } \underline{b} \neq \sigma \in \{0,1\}^n \text{ where } \bar{\sigma} := \{1,2,\ldots,n\} - \sigma \text{ is a face of } \Delta^* \\ \dim_{\Bbbk} \tilde{H}_{i-1}(\mathrm{link}_{\Delta^*}(\bar{\sigma}); \Bbbk) \end{cases}$$

with Δ^* = Alexander dual to Δ on vertex set $\{1,2,\ldots,n\}$

$$:= \{\bar{\sigma} : \sigma \in \{0,1\}^n - \Delta\}$$

e.g. $\Delta =$  $\rightsquigarrow \Delta^* =$ 

(has facets $\bar{\sigma}$ corr. to minimal non-faces σ of Δ)

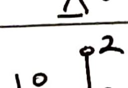
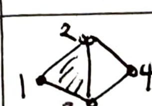
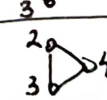
③ COR 5.12 (Hochster's formula)

$$\beta_{i,\underline{b}}(I_\Delta) = \begin{cases} 0 & \text{if } \underline{b} \neq \sigma \in \{0,1\}^n \\ \dim_{\Bbbk} \tilde{H}^{|\sigma|-i-2}(\Delta|_\sigma; \Bbbk) \end{cases}$$

where $\Delta|_\sigma$ = vertex-induced subcomplex of Δ on σ

$$:= \{\tau \in \Delta : \tau \subseteq \sigma\}$$

EXAMPLE (of dual Hochster & Hochster formula)

face $\bar{\sigma}$ of Δ^*	$\mathrm{link}_{\Delta^*}(\bar{\sigma})$	$\tilde{H}_*(\mathrm{link}_{\Delta^*}(\bar{\sigma}); \Bbbk)$	$\underline{b} = \sigma$	$\Delta\|_\sigma$	$\tilde{H}^*(\Delta\|_\sigma; \Bbbk)$
$\emptyset$		$\tilde{H}_0 = \Bbbk^1$ Tor$_1$	1234		$\tilde{H}^1 = \Bbbk^1$ Tor$_1$
1	$\{\emptyset\}$	$\tilde{H}_{-1} = \Bbbk^1$ Tor$_0$	234		$\tilde{H}^1 = \Bbbk^1$ Tor$_0$
2	∘3	—	134		—
3	∘2	—	124		—
23	$\{\emptyset\}$	$\tilde{H}_{-1} = \Bbbk^1$ Tor$_0$	14	1∘ ∘4	$\tilde{H}^0 = \Bbbk^1$ Tor$_0$

Why do (1), (2), (3) hold?

We saw (1) arose because $\mathrm{Tor}_i^S(k, I) \cong H_i(\overset{\text{Koszul resolution of } \mathbb{k}=S/(x_1,\ldots,x_n)}{\widetilde{\mathbb{K}_\bullet}} \otimes_S I)$

and one can check $(\mathbb{K}_\bullet \otimes_S I)_{\underline{b}} \underset{\text{as chain complexes}}{\cong} \tilde{C}_{\bullet-1}(K^{\underline{b}}(I); \mathbb{k})$

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(1)$\Rightarrow$(2) because for $I = I_\Delta$ and any $\underline{b} \in \mathbb{N}^n$, one can check that

$$K^{\underline{b}}(I_\Delta) = \begin{cases} \text{a \underline{cone}, with cone vertex } i, \text{ if any } b_i \geq 2 \\ \varnothing \quad \text{if } \underline{b} = \sigma \in \{0,1\}^n \text{ but } \bar{\sigma} \notin \Delta^* \\ \mathrm{link}_{\Delta^*}(\bar{\sigma}) \text{ if } \underline{b} = \sigma \in \{0,1\}^n \text{ with } \bar{\sigma} \in \Delta^* \end{cases}$$

$$\overset{\shortparallel}{\left(\{\sigma \in \{0,1\}^n : \underline{x}^{\underline{b}-\sigma} \in I_\Delta\}\right)}$$

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(2) $\Leftrightarrow$ (3) would follow if we understood why

$$\tilde{H}_{i-1}(\underbrace{\mathrm{link}_{\Delta^*}(\bar{\sigma})}; \mathbb{k}) \overset{?}{\cong} \tilde{H}^{|\sigma|-i-2}(\underbrace{\Delta|_\sigma}; \mathbb{k}) \ \forall \text{ faces } \bar{\sigma} \in \Delta^*$$

both simplicial complexes <u>on vertex set σ</u>,
and can check they are <u>Alexander duals</u> on $V = \sigma$

i.e. $(\Delta|_\sigma)^* = \mathrm{link}_{\Delta^*}(\bar{\sigma})$

Thus we need to understand why

$$\tilde{H}_{i-1}(\Delta^*; \mathbb{k}) \cong \tilde{H}^{n-i-2}(\Delta; \mathbb{k})$$

for any simplicial complex Δ on $\{1, 2, \ldots, n\}$.

Miller & Sturmfels give an algebraic proof in §5.1.

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But it also follows from (~~topological~~) Alexander duality:

THM: For a subspace $X \subset \mathbb{S}^d$ a d-sphere,

one has $\tilde{H}_i(X;\Bbbk) \cong \check{H}^j(\mathbb{S}^d - X;\Bbbk)$ for $\boxed{i+j=d-1}$

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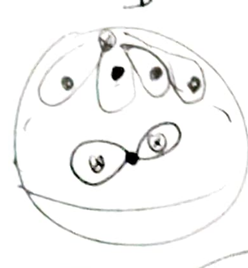
e.g. $X =$  $\hookrightarrow$ $\mathbb{S}^2$  has $\mathbb{S}^2 \setminus X \approx$ 

so $\tilde{H}_0(X;\Bbbk) \cong \Bbbk^4$ $\check{H}^0(\mathbb{S}^2 \setminus X;\Bbbk) \cong \Bbbk^2$

$\tilde{H}_1(X;\Bbbk) \cong \Bbbk^2$ $\check{H}^1(\mathbb{S}^2 \setminus X;\Bbbk) \cong \Bbbk^4$

This shows $\tilde{H}_{i-1}(\Delta^*) \cong \check{H}^{n-i-2}(\Delta)$

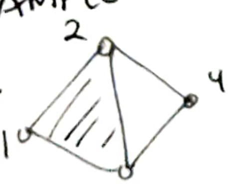
because after taking their barycentric subdivisions, one

can check that $Sd(\Delta^*) \hookrightarrow Sd\left(\begin{array}{c}\text{boundary of}\\ (n-1)\text{-simplex}\\ \text{on vertices } \{1,2,\dots,n\}\end{array}\right) = \mathbb{S}^{n-2}$ an $(n-2)$-sphere

$Sd(\Delta) \hookrightarrow$

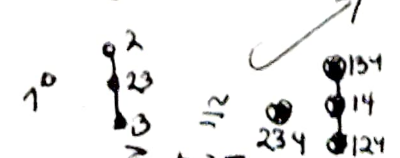
and they are each deformation retracts of each other's complement within that $\mathbb{S}^{n-2}$?

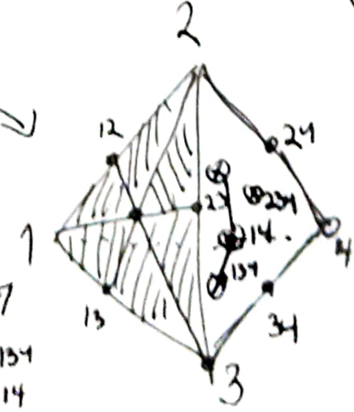
EXAMPLE

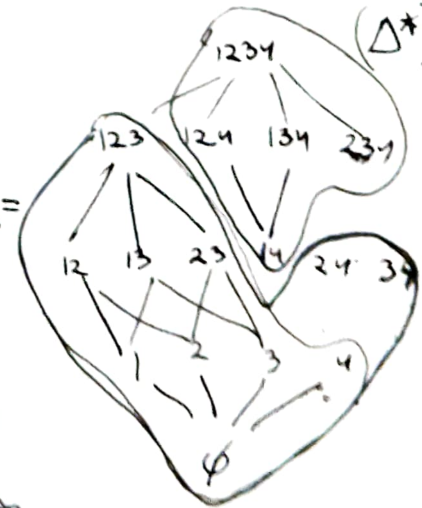
$\Delta =$ 

$sd\,\Delta =$ 

$\Delta^* =$ 

$sd\,\Delta^* =$ 



$\Delta =$ 

$\overline{(\Delta^*)}$ $\cong$ opposite of  Δ^*

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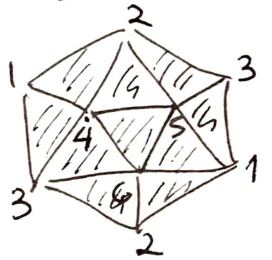
REMARK: Hochster & dual Hochster both show that $\beta_{i,\underline{b}}(I_\Delta)$ can depend on the field $\Bbbk$ in $S=\Bbbk[x_1,\dots,x_n]$ once n gets large enough to create Δ with torsion in $\tilde{H}_i(\Delta;\mathbb{Z})$, so $\tilde{H}_i(\Delta;\Bbbk)$ depends on $\Bbbk$.

Need $\boxed{n\geq 6}$, where smallest example is famous minimal triangulation of $\mathbb{RP}^2 = \mathbb{P}^2_{\mathbb{R}}$:



$$= \underbrace{\left(\text{boundary of icosahedron}\right)}_{S^2} \Big/ \begin{matrix}\text{antipodal}\\ \text{map } x\sim -x\end{matrix}$$

$$\tilde{H}_i(\mathbb{RP}^2;\Bbbk) = 0 \quad \forall i \text{ if } \operatorname{char}(\Bbbk)\neq 2$$

$$\left.\begin{array}{ll}\tilde{H}_i(\mathbb{RP}^2;\Bbbk)=0 & \text{for } i\neq 1,2\\ \tilde{H}_1(\mathbb{RP}^2;\Bbbk)=\Bbbk^1 & \\ \tilde{H}_2(\mathbb{RP}^2;\Bbbk)=\Bbbk^1 & \end{array}\right\} \text{ if } \operatorname{char}(\Bbbk)=2$$