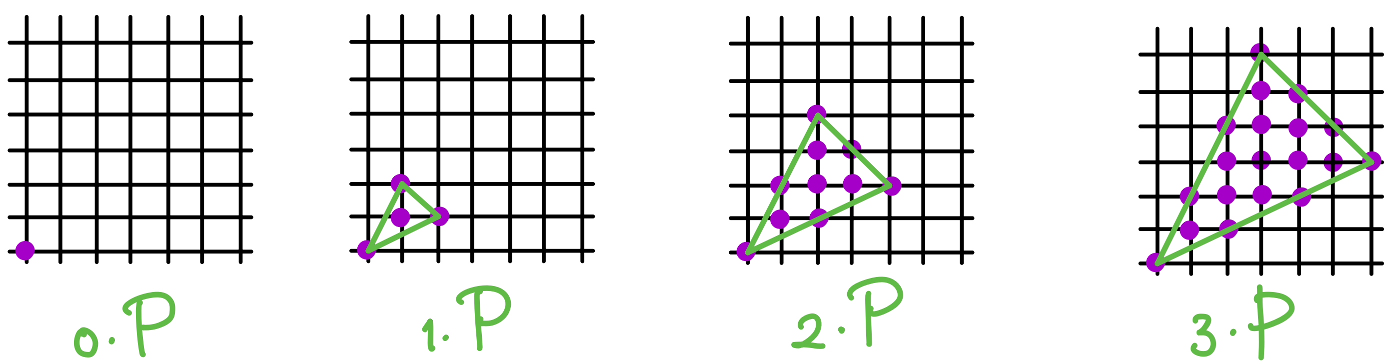
Macaulay inverse systems and graded Ehrhart theory

Vic Reiner - U. Minnesota
Brendon Rhoades - UC San Diego

(arXiv: 2407.06511)



Uwe Nagel 60th Birthday Fest

U. Notre Dame, August 12, 2024

1. Ehrhart theory review

2. q-analogues via
fat points, Macaulay inverse systems

3. q-Ehrhart theory CONJECTURE
& examples

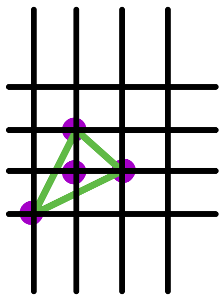
4. Harmonic algebra CONJECTURE

We need HELP, Uwe and friends?

1. Ehrhart theory review

vertices in $\mathbb{Z}^n$

$P \subset \mathbb{R}^n$ a lattice polytope



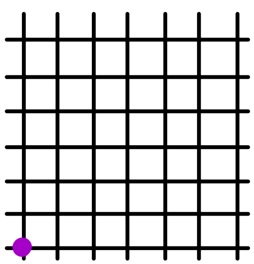
$\rightsquigarrow$ Ehrhart function/polynomial

$$i_P(m) := \#\, \mathbb{Z}^n \cap mP$$

for $m = 0, 1, 2, \ldots$

EXAMPLE

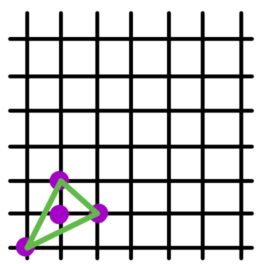
$0 \cdot P$



1

$i_P(0)$

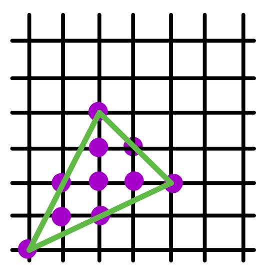
$1 \cdot P$



4

$i_P(1)$

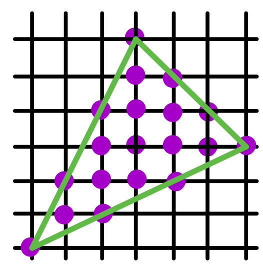
$2 \cdot P$



10

$i_P(2)$

$3 \cdot P$



19

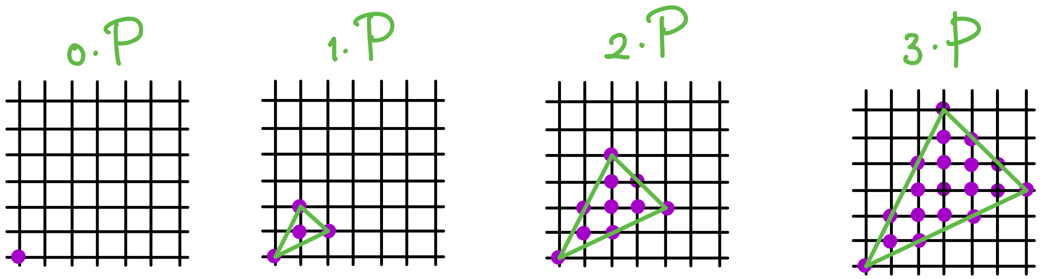
$i_P(3)$

$$i_P(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$$

... and Ehrhart series

$$E_P(t) := \sum_{m=0}^{\infty} t^m \cdot i_P(m) = \sum_{m=0}^{\infty} t^m \cdot \#(\mathbb{Z}^n \cap mP)$$

EXAMPLE



$$1 + 4t^1 + 10t^2 + 19t^3 + \dots$$

$$= \sum_{m=0}^{\infty} t^m \cdot \left(1 + \frac{3}{2}m + \frac{3}{2}m^2\right)$$

$$= \frac{1+t+t^2}{(1-t)^3}$$

THEOREM

(Ehrhart 1962)

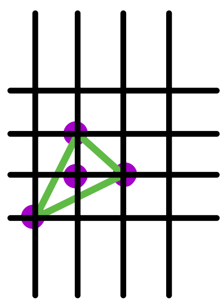
$i_P(m) := \# \mathbb{Z}^n \cap mP$ is a polynomial in m, of degree $d := \dim(P)$

$\Updownarrow$

$$E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m) = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}}$$

EXAMPLE

P



$d=2$

$$i_P(m) = 1 + \tfrac{3}{2} m + \tfrac{3}{2} m^2$$

$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

THEOREM
(Stanley 1980)

In the numerator of

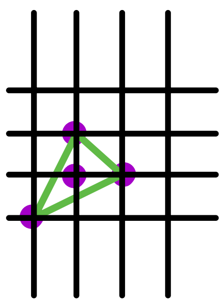
$$E_P(t) = \sum_{m=0}^{\infty} t^m i_P(m) = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}},$$

the h^*-vector entries $(h_0^*, h_1^*, \dots, h_d^*)$

are nonnegative.

EXAMPLE

P

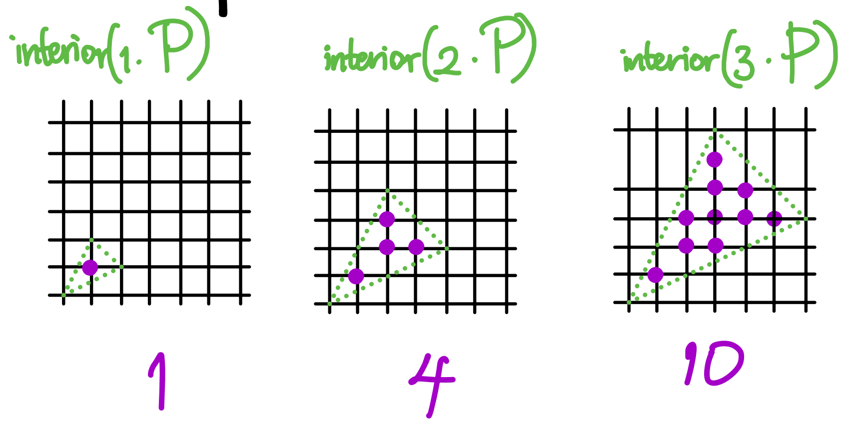


$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

One can also count interior lattice points...

$$\bar{i}_P(m) := \#\mathbb{Z}^n \cap \text{interior}(mP)$$

$$\bar{E}_P(t) := \sum_{m=1}^{\infty} t^m \cdot \bar{i}_P(m)$$



THEOREM (Ehrhart-Macdonald reciprocity)
CONJECTURE 1959 PROOF 1971

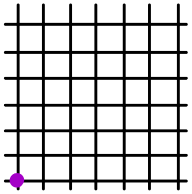
I.G. Macdonald
1928-2023

$$\bar{i}_P(m) = (-1)^d \, i_P(-m) \quad \text{for } m = 1,2,3,\dots$$

equivalently, $$\bar{E}_P(t) = (-1)^{d+1} E_P(1/t)$$

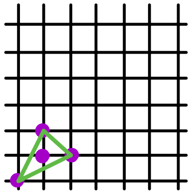
EXAMPLE

$i_P(0) = 1$



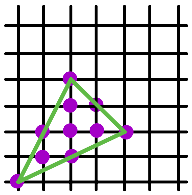
$0 \cdot P$

$i_P(1) = 4$



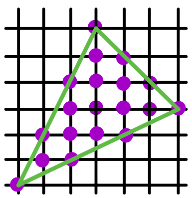
$1 \cdot P$

$i_P(2) = 10$



$2 \cdot P$

$i_P(3) = 19$

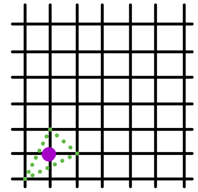


$3 \cdot P$

$$i_P(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$$

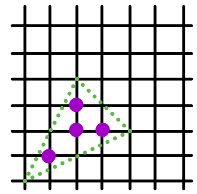
$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

interior$(1 \cdot P)$



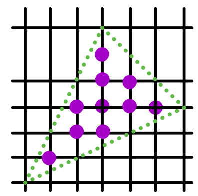
$\bar{i}_P(1) = 1$

interior$(2 \cdot P)$



$\bar{i}_P(2) = 4$

interior$(3 \cdot P)$



$\bar{i}_P(3) = 10$

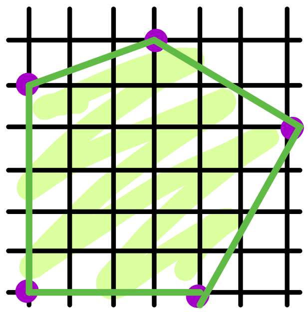
$$\bar{i}_P(m) = 1 - \frac{3}{2}m + \frac{3}{2}m^2 = (-1)^2\, i_P(-m)$$

$$\bar{E}_P(t) = \frac{t+t^2+t^3}{(1-t)^3} = (-1)^3 E_P(1/t)$$

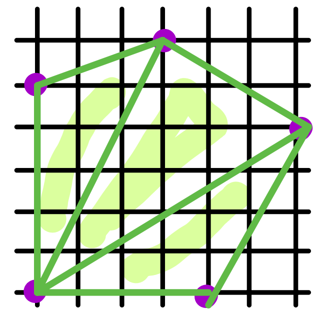
Two proof methods

Method 1 (Ehrhart Macdonald Stanley): Reduce to simplices via triangulations

P

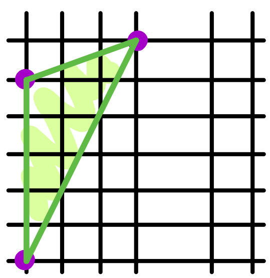


triangulate

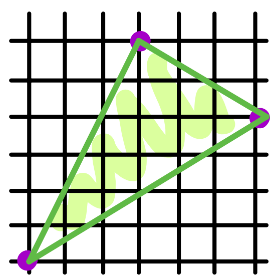


$E_P(t)$ is a valuative function of P:

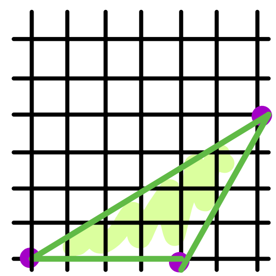
$$E_P(t) = E_{P_1}(t) + E_{P_2}(t) + E_{P_3}(t) - E_{P_1 \cap P_2}(t) - E_{P_2 \cap P_3}(t)$$



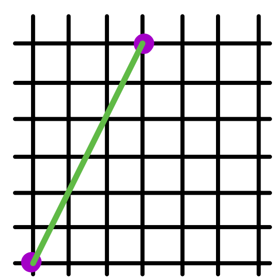
P_1



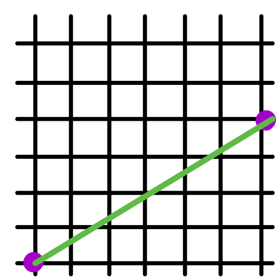
P_2



P_3



$P_1 \cap P_2$



$P_2 \cap P_3$

... and simplices have explicit formulas:

PROPOSITION: For a lattice d-simplex $P \subset \mathbb{R}^n$ with vertices $v^{(1)}, v^{(2)}, \dots, v^{(d+1)}$

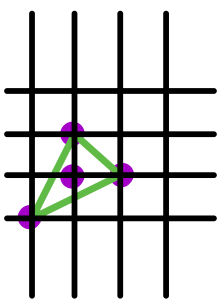
$$\mathcal{E}_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}} \quad \text{where } h_i^* = \#\left(\mathbb{Z}^n \times \{i\}\right) \cap \Pi$$

semi-open parallelepiped

$$\Pi := \sum_j [0,1) \cdot \begin{bmatrix} v^{(j)} \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^{n+1}$$

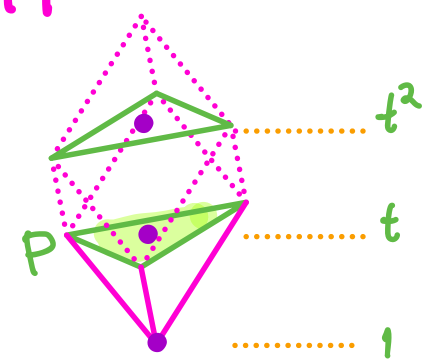
EXAMPLE

$P \subset \mathbb{R}^2$



$\rightsquigarrow$

$\Pi \subset \mathbb{R}^3$



$\rightsquigarrow$

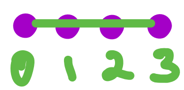
$$\mathcal{E}_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

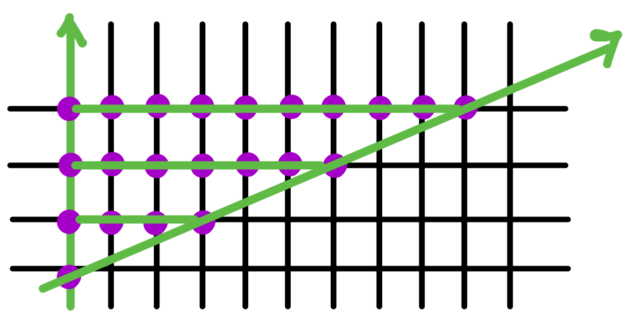
Method 2 (Stanley): Commutative algebra of the affine semigroup ring

$$\Bbbk[\Lambda_P] := \operatorname{span}_{\Bbbk}\left\{ y_0^{m} y_1^{a_1} \cdots y_n^{a_n} : \underline{a} \in \mathbb{Z}^n \cap mP \right\} \subset \Bbbk[y_0, y_1, \dots, y_n]$$

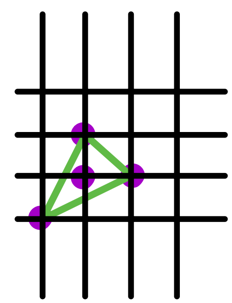
with $E_P(t) = \operatorname{Hilb}(\Bbbk[\Lambda_P], t)$

$P = [0,3] \subset \mathbb{R}^1$

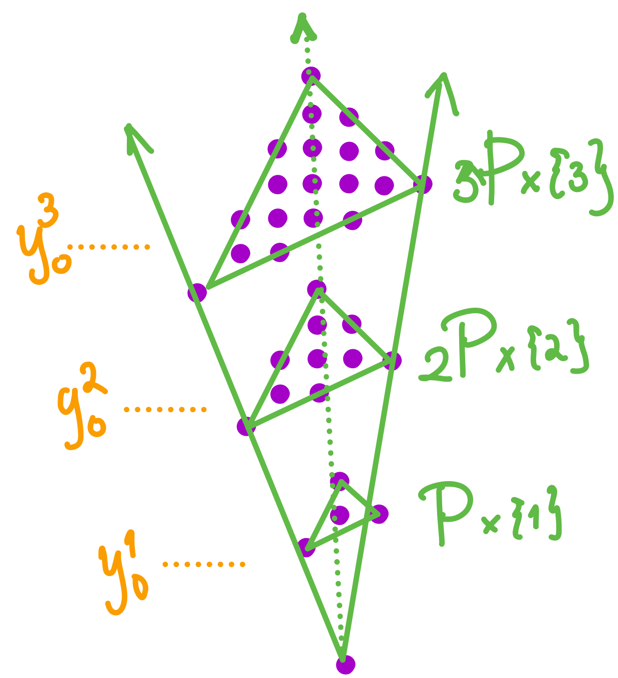




$\Bbbk[\Lambda_P] = \Bbbk[y_0, y_0y_1, y_0y_1^2, y_0y_1^3] \subset \Bbbk[y_0, y_1]$



$P \subset \mathbb{R}^2$



- $\mathbb{k}[\Lambda_P]$ is Noetherian (Gordan 1873)

- $\mathbb{k}[\Lambda_P]$ has a linear system of parameters (Noether 1926)

$$\Rightarrow E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

- $\mathbb{k}[\Lambda_P]$ is Cohen-Macaulay (Hochster 1972)

$\Rightarrow h_i^* \geq 0$ for $i=1,2,\ldots,d$

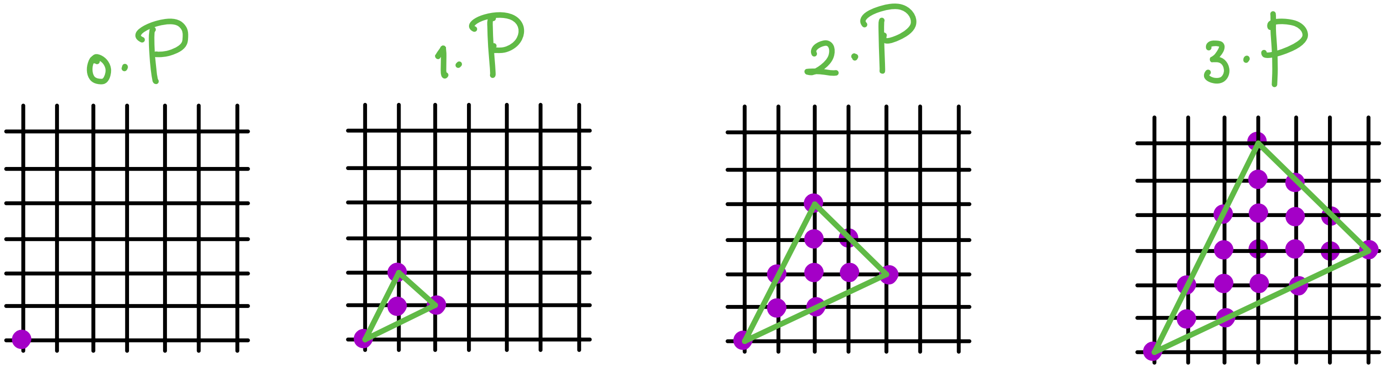
- $\Omega \, \mathbb{k}[\Lambda_P] \cong \mathbb{k}[\Lambda_{\text{interior}(P)}]$

canonical module

(Danilov 1978)

$\Rightarrow E_P(t) = (-1)^{d+1} E_P(1/t)$

2. q-analogues: fat points, Macaulay inverse systems



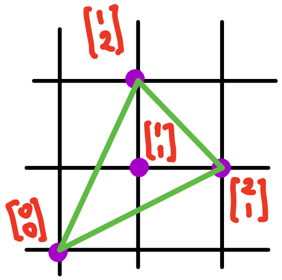
$$1 + 4t^1 + 10t^2 + 19t^3 + \dots = E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

$$\Big\downarrow \text{ q-analogue}$$

$$1 + (1+2q+q^2)t^1 + \begin{pmatrix} 1+2q+3q^2 \\ +3q^3+q^4 \end{pmatrix} t^2 + \begin{pmatrix} 1+2q+3q^2+4q^3 \\ +5q^4+3q^5+q^6 \end{pmatrix} t^3 + \dots = E_P(t,q)$$

$$= \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

Those q-analogues of $\#\mathbb{Z}^n \cap mP$ come from deformation ...



$Z := \mathbb{Z}^n \cap mP$

$\#Z = 4$

affine coordinate ring

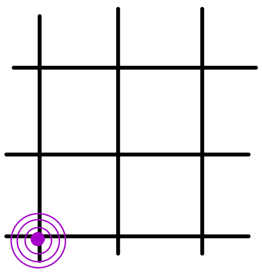
$$\mathbb{R}[Z] = \mathbb{R}[x,y]\Big/ I(Z)$$

$$I(Z) = \underset{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}{(x-0,y-0)} \cap \underset{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{(x-1,y-1)} \cap \underset{\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]}{(x-1,y-2)} \cap \underset{\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]}{(x-2,y-1)}$$

$$= (\underline{2xy - y^2} - 2x + y,\ \underline{x^2 - y^2} - 3x + y,\ \underline{y^3} - 3y^2 + 2y)$$

$\rightsquigarrow$ deform

(= by taking top degree forms)



fat point at $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ of multiplicity 4

associated graded ring

$$\mathrm{gr}\,\mathbb{R}[Z] = \mathbb{R}[x,y]\Big/ \mathrm{gr}\,I(Z)$$

$$\mathrm{gr}\,I(Z) = (\underline{2xy} - y^2,\ \underline{x^2} - y^2,\ \underline{y^3})$$

$$\mathrm{Hilb}(\mathrm{gr}\,\mathbb{R}[Z], q) = 1 + 2q + q^2 \overset{q=1}{\rightsquigarrow} 4 = \#Z$$

= the q-analogue of $\#Z$

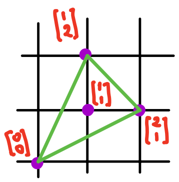
REMARK: Instead of Hilbert series for the fat point coordinate ring $\mathbb{R}[x_1,\dots,x_n]/\operatorname{gr} I(Z)$, can use its Macaulay inverse system

$$V_Z := \left\{ g(\underline{y}) \in \mathbb{R}[y_1,\dots,y_n] \; : \; f\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\right) g(\underline{y}) = 0 \quad \forall f(\underline{x}) \in \operatorname{gr} I(Z) \right\}$$

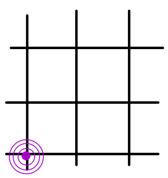
since $\quad \operatorname{Hilb}(V_Z, q) = \operatorname{Hilb}\left(\mathbb{R}[x_1,\dots,x_n]/\operatorname{gr} I(Z), q\right)$

EXAMPLE

Z



$$\mathbb{R}[Z] = \mathbb{R}[x_1,x_2]\Big/\left(\underline{2x_1x_2 - x_2^2} - 2x_1 + y_1,\ \underline{x_1^2 - x_2^2} - 3x_1 + x_2,\ \underline{x_2^3} - 3x_2^2 + 2x_2\right)$$



$$\operatorname{gr} \mathbb{R}[Z] = \mathbb{R}[x_1,x_2]\Big/\left(2x_1x_2 - x_2^2,\ x_1^2 - x_2^2,\ x_2^3\right)$$

$$V_Z = \operatorname{span}_{\mathbb{R}}\left\{ 1,\ y_1,\ y_2,\ y_1^2 + y_1y_2 + y_2^2 \right\} \subset \mathbb{R}[y_1,y_2]$$

$$\operatorname{Hilb}(V_Z, q) = 1 + 2q + q^2 \quad \overset{q=1}{\rightsquigarrow} \quad 4 = \#Z$$

MAIN DEFINITION: For a lattice polytope $P \subset \mathbb{R}^n$, define its

q-Ehrhart series $\boxed{E_P(t,q) := \sum_{m=0}^{\infty} t^m \cdot i_P(m;q)}$ $\xrightarrow{q=1}$ $E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m)$

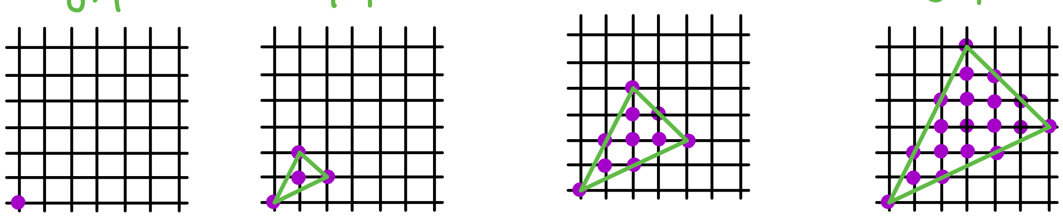
where $$i_P(m;q) = \mathrm{Hilb}(\mathrm{gr}\,\mathbb{R}[\mathbb{Z}^n \cap mP], q) = \mathrm{Hilb}(V_{\mathbb{Z}^n \cap mP}, q)$$

$$\Big\downarrow q=1$$

$$i_P(m) = \#\mathbb{Z}^n \cap mP$$

EXAMPLE

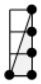
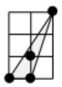
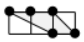
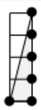
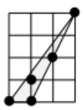
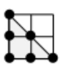
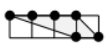
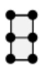
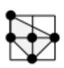
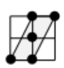
$0\cdot P$ $\quad$ $1\cdot P$ $\quad$ $2\cdot P$ $\quad$ $3\cdot P$

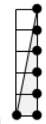
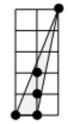
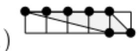
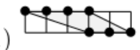
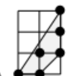
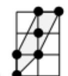


$$1 + 4t^1 + 10t^2 + 19t^3 + \dots = E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

$$1 + (1+2q+q^2)t^1 + \begin{pmatrix} 1+2q+3q^2 \\ +3q^3+q^4 \end{pmatrix} t^2 + \begin{pmatrix} 1+2q+3q^2+4q^3 \\ +5q^4+3q^5+q^6 \end{pmatrix} t^3 + \dots = E_P(t,q) = \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

MORE EXAMPLES of $E_P(t,q)$ for lattice polygons

normalized area of P	vertices of P	$h^*_P(t) = 1 + h^*_1 t + h^*_2 t^2$	$\mathrm{E}_P(t,q)$
1	$(0,0),(1,0),(0,1)$ 	1	$\frac{1}{(1-t)(1-qt)^2}$
2	$(0,0),(1,0),(1,2)$ 	$1+t$	$\frac{1+qt}{(1-t)(1-qt)(1-q^2t)}$
2	$(0,0),(1,0),(0,1),(1,1)$ 	$1+t$	$\frac{1+qt}{(1-t)(1-qt)(1-q^2t)}$
3	$(0,0),(1,0),(1,3)$ 	$1+2t$	$\frac{1+qt+q^2t}{(1-t)(1-qt)(1-q^3t)}$
3	$(0,0),(1,0),(2,3)$ 	$1+t+t^2$	$\frac{(1+qt)(1+qt+q^2t^2)}{(1-t)(1-q^2t)(1-q^3t^2)}$
3	$(0,0),(1,0),(0,1),(-2,1)$ 	$1+2t$	$\frac{1+qt-q^2t^2-q^3t^2}{(1-t)(1-qt)(1-q^2t)^2}$
4	$(0,0),(1,0),(1,4)$ 	$1+3t$	$\frac{1+t(q+q^2+q^3)}{(1-t)(1-qt)(1-q^4t)}$
4	$(0,0),(1,0),(3,4)$ 	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2}$
4	$(0,0),(2,0),(0,2)$ 	$1+3t$	$\frac{1+2qt+q^2t}{(1-t)(1-q^2t)^2}$
4	$(0,0),(1,0),(0,1),(-3,1)$ 	$1+3t$	$\frac{1+qt+q^2t-q^2t^2-q^3t^2-q^4t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}$
4	$(0,0),(1,0),(0,2),(1,2)$ 	$1+3t$	$\frac{1+qt+q^2t-q^2t^2-q^3t^2-q^4t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}$
4	$(0,0),(2,0),(0,1),(1,-1)$ 	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2}$
4	$(0,0),(1,0),(1,2),(2,2)$ 	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2)}$

vertices of P	$h^*_P(t) = 1 + h^*_1 t + h^*_2 t^2$	$\mathrm{E}_P(t,q)$
$(0,0),(1,0),(1,5)$ 	$1+4t$	$\frac{1+t(q+q^2+q^3+q^4)}{(1-t)(1-qt)(1-q^5t)}$
$(0,0),(1,0),(2,5)$ 	$1+2t+2t^2$	$\frac{(1+qt)(1+t(q+q^2)+t^2(q^3+q^4))}{(1-t)(1-q^2t)(1-q^5t^2)}$
$(0,0),(1,0),(0,1),(-4,1)$ 	$1+4t$	$\frac{1+qt+q^2t+q^3t-q^2t^2-q^3t^2-q^4t^2-q^5t^2}{(1-t)(1-qt)(1-q^2t)(1-q^4t)}$
$(0,0),(2,0),(0,1),(-3,1)$ 	$1+4t$	$\frac{1+qt+2q^2t-q^2t^2-q^3t^2-q^4t^2-q^5t^2}{(1-t)(1-qt)(1-q^3t)^2}$
$(0,0),(1,0),(2,3),(2,1)$ 	$1+3t+t^2$	$\frac{1+2qt+2q^2t+2q^3t^2+2q^4t^2+q^5t^3}{(1-t)(1-q^2t)(1-q^5t^2)}$
$(0,0),(1,0),(1,2),(2,2),(0,-1)$ 	$1+3t+t^2$	$\frac{1+2qt+2q^2t+2q^3t^2+2q^4t^2+q^5t^3}{(1-t)(1-q^2t)(1-q^5t^2)}$

Since $E_P(t,q)$ is an $\mathrm{Aff}(\mathbb{Z}^n)$-invariant of P, can use database of lattice polytopes by Balletti 2021

3. q-Ehrhart theory CONJECTURE

First, recall the

CLASSICAL Ehrhart Theorems: For d-dimensional lattice polytopes P,

- $$E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$ (RATIONALITY)

- $$E_P\left(\frac{1}{t}\right) = (-1)^{d+1} \bar{E}_P(t)$$ (RECIPROCITY)

- For lattice simplices, $$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$$ (SIMPLEX NONNEGATIVITY)

- $$h_i^* \geq 0 \quad \text{for } i = 1, 2, \ldots, d$$ (GENERAL NONNEGATIVITY)

Classical Ehrhart THEOREMS

CONJECTURES (R.-Rhoades 2024)

RATIONALITY

- $$E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

- $$E_P(t,q) = \frac{N_P(t,q)}{\prod_{i=1}^{\nu}(1-q^{a_i}t^{b_i})}$$ with $N_P(t,q)$ in $\mathbb{Z}[t,q]$ and $d+1 \leq \nu \leq$ #vertices of P

RECIPROCITY

- $$E_P\left(\tfrac{1}{t}\right) = (-1)^{d+1}\,\overline{E}_P(t)$$

- $$E_P\left(\tfrac{1}{t},\tfrac{1}{q}\right) = (-1)^{d+1} q^{\dim P}\,\overline{E}_P(t,q)$$

SIMPLEX NONNEGATIVITY

- For lattice simplices, $$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$$

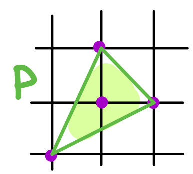
- For lattice simplices, $N_P(t,q)$ lies in $\mathbb{N}[t,q]$

GENERAL NONNEGATIVITY

- $h_i^* \geq 0$ for $i=1,2,\ldots,d$

- ??

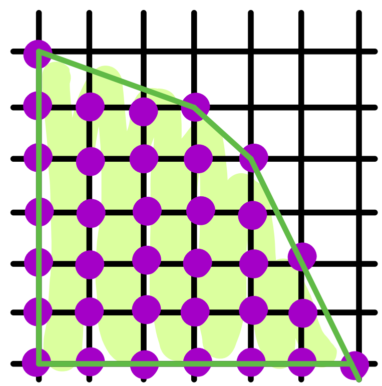
EXAMPLE P



$$E_P(t,q) = \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

EXAMPLES: These conjectures are proven for

antiblocking polytopes $:=$ polytopes P inside $(\mathbb{R}_{\geq 0})^n$ with $\underbrace{0 \leq z \leq z'}_{\text{componentwise comparison}}$ and $z' \in P \Rightarrow z \in P$



PROPOSITION: For P antiblocking, every ideal $\mathrm{gr}\, I(Z)$ for $Z := \mathbb{Z}^n \cap mP$ is a monomial ideal $\mathrm{gr}\, I(Z) = \mathrm{span}_{\mathbb{R}}\{\underline{x}^{\underline{a}} : \underline{a} \notin Z\}$,

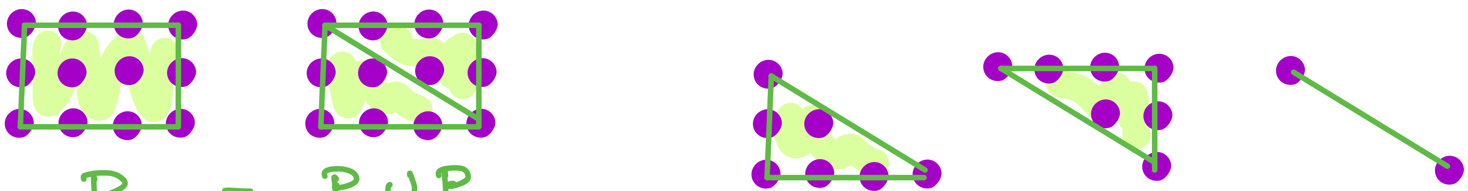
and
$$E_P(t,q) = \sum_{m=0}^{\infty} t^m \sum_{\underline{a} \in \mathbb{Z}^n \cap mP} q^{a_1 + \ldots + a_n}$$
$$= \Big[\mathrm{Hilb}\big(\underbrace{\mathbb{k}[\Lambda_P]}_{\text{affine semigroup ring for } P}, y_0, y_1, \ldots, y_n\big)\Big]_{\substack{y_0 = t \\ y_1 = y_2 = \ldots = y_n = q}}$$

4. Harmonic algebra CONJECTURE

Method 1 of Ehrhart theory (= reduce to simplices via triangulation) seems elusive, because

$$i_P(m;q) = \mathrm{Hilb}(\mathrm{gr}\, R[\mathbb{Z}^n \cap mP], q)$$ is not valuative as a function of P.

EXAMPLE



$$P = P_1 \cup P_2$$

$$i_P(1) = i_{P_1}(1) + i_{P_2}(1) - i_{P_1 \cap P_2}(1)$$

$$12 = 7 + 7 - 2$$

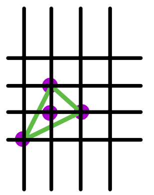
But

$$i_P(1;q) \neq i_{P_1}(1;q) + i_{P_2}(1;q) - i_{P_1 \cap P_2}(1;q)$$

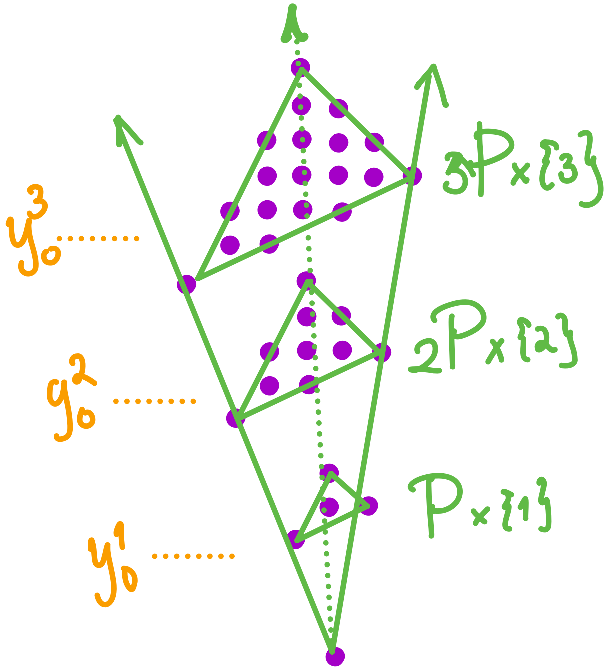
$$i_P(1;q) = (1+q+q^2)\cdot(1+q+q^2+q^3) = 1+2q+3q^2+3q^3+2q^4+q^5$$

$$i_{P_1}(1;q) = 1+2q+3q^2+q^3, \quad i_{P_2}(1;q) = 1+2q+3q^2+q^3, \quad i_{P_1 \cap P_2}(1;q) = 1+q$$

Method 2 (= commutative algebra) looks promising...



$P \subset \mathbb{R}^2$



Recall

$$\Bbbk[\Lambda_P] := \bigoplus_{m=0}^{\infty} \mathrm{span}_{\Bbbk}\left\{ y_0^m \, \underline{y}^{\underline{a}} \right\}_{\underline{a} \in \mathbb{Z}^n \cap mP}$$

affine semigroup ring

$$\subset \Bbbk[y_0, y_1, \dots, y_n]$$

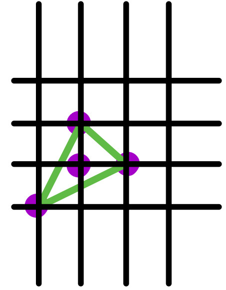
DEFINITION:

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP} \subset \mathbb{R}[y_0, y_1, \dots, y_n]$$

Harmonic algebra

where V_Z = harmonic space/Macaulay inverse system for $\mathbb{R}[y_1, \dots, y_n] / \mathrm{gr}\, I(Z)$

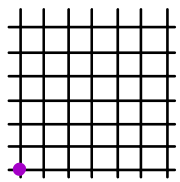
EXAMPLE $P \subset \mathbb{R}^2$

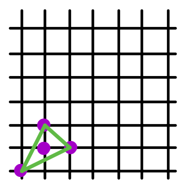


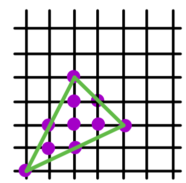
has harmonic algebra

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m V_{\mathbb{Z}^n \cap mP} \subset \mathbb{R}[y_0, y_1, y_2]$$

$$= \mathbb{R} \cdot 1 \quad \oplus \quad y_0 V_{\mathbb{Z}^2 \cap 1P} \quad \oplus \quad y_0^2 V_{\mathbb{Z}^2 \cap 2P} \quad \oplus \quad \cdots$$







$$= \operatorname{span}_{\mathbb{R}} \Big\{ 1, \quad y_0, \; y_0 y_1, \; y_0 y_2, \; y_0(y_1^2 + y_1 y_2 + y_2^2) \quad y_0^2, \; y_0^2 y_1, \; y_0^2 y_2, \; y_0^2 y_1^2, \; y_0^2 y_1 y_2, \; y_0^2 y_2^2, \; y_0^2 y_1^3, \; y_0^2(y_1^2 y_2 + y_1 y_2^2), \; y_0^2 y_2^3, \; y_0^2(y_1^4 + 2y_1^3 y_2 + 3y_1^2 y_2^2 + 2y_1 y_2^3 + y_2^4) \quad \cdots \Big\}$$

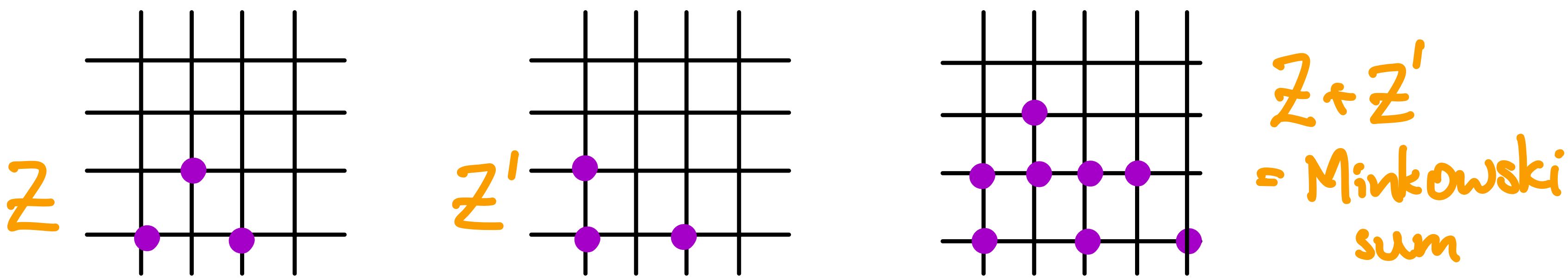
Why is $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP}$ even an algebra ?!

For $\Bbbk[\Lambda_P] := \bigoplus_{m=0}^{\infty} \mathrm{span}_{\Bbbk}\{ y_0^m \, y^{\underline{a}} \}_{\underline{a} \in \mathbb{Z}^n \cap mP}$ it came from

$$\mathbb{Z}^n \cap mP \; + \; \mathbb{Z}^n \cap m'P \; \subset \; \mathbb{Z}^n \cap (m+m')P$$

THEOREM (R.-Rhoades 2024) For finite point sets Z and $Z' \subset \Bbbk^n$,

$$V_Z \cdot V_{Z'} \subseteq V_{Z+Z'}$$



By construction, $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP}$

has $\mathrm{Hilb}(\mathcal{H}_P, t, q) = \sum_{m=0}^{\infty} t^m \, i_P(m;q) = E_P(t,q)$

(Classical) THEOREMS	CONJECTURES (R.-Rhoades 2024)
• $\Bbbk[\Lambda_P]$ is Noetherian (Gordan 1873)	• $\mathcal{H}_P$ is Noetherian (with degree bounds on generators)
• $\Bbbk[\Lambda_P]$ is Cohen-Macaulay (Hochster 1972)	• $\mathcal{H}_P$ is Cohen-Macaulay
• $\Omega \Bbbk[\Lambda_P] \cong \Bbbk[\Lambda_{\mathrm{interior}(P)}]$ (Danilov 1978)	• $\Omega \mathcal{H}_P \cong \mathcal{H}_{\mathrm{interior}(P)}$

CONJECTURES on $\mathcal{H}_P$ $\Longrightarrow$ CONJECTURES on $E_P(t,q)$

- $\mathcal{H}_P$ is Noetherian (with degree bounds on generators) $\Longrightarrow$ (RATIONALITY)
  - $E_P(t,q) = \dfrac{N_P(t,q)}{\prod_{i=1}^{\nu}(1-q^{a_i}t^{b_i})}$ with $N_P(t,q)$ in $\mathbb{Z}[t,q]$ and $d+1 \leq \nu \leq$ #vertices of P

- $\mathcal{H}_P$ is Cohen-Macaulay $\Longrightarrow$ (SIMPLEX NONNEGATIVITY)
  - For lattice simplices, $N_P(t,q)$ lies in $\mathbb{N}[t,q]$

- $\Omega\mathcal{H}_P \cong \mathcal{H}_{\text{interior}(P)}$ $\Longrightarrow$ (RECIPROCITY)
  - $E_P\left(\frac{1}{t},\frac{1}{q}\right) = (-1)^{d+1} q^{\dim P} E_P(t,q)$

I repeat:

HELP, Uwe and friends!

... and Happy Birthday, Uwe!