Face numbers of nestohedra

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I. f-, h-, and γ-vectors

For a d-dimensional polytope P,
one has the f-vector

$$(f_0, f_1, \ldots, f_d)$$

where f_i is the number of i-dimensional faces.

For simple polytopes P,
one considers also the h-vector

$$(h_0, h_1, \ldots, h_d)$$

defined by

$$\sum_i h_i \, (t+1)^i = \sum_i f_i \, t^i.$$

For simple, flag polytopes P
(= those whose polar dual has boundary simplicial complex Δ a flag or clique complex)
one considers further the γ-vector

$$(\gamma_1, \gamma_2, \ldots, \gamma_{\lfloor d/2 \rfloor})$$

defined by

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

For flag simple polytopes,
S. Gal conjectured (2005) that

$$\gamma_i \geq 0 \text{ for all } i$$

generalizing the earlier conjecture of
R. Charney and M. Davis (1995) that

$$\gamma_{\lfloor \frac{d}{2} \rfloor} \geq 0.$$

II. EXAMPLE: Permutohedra

P := the $(n-1)$-dimensional permutohedron :=the convex hull of all permutations

$$\{(w(1),\ldots,w(n))\}_{w\in\mathfrak{S}_n}.$$

Define the descent number

$$\mathrm{des}(w) = \#\{i \mid w(i) > w(i+1)\}.$$

Let $\widehat{\mathfrak{S}}_n$ be permutations w with no consecutive descents $w(i) > w(i+1) > w(i+2)$ (with convention $w(n+1)=0$).

THEOREM(Getu-Shapiro-Woan 1983)

The permuothedron is a flag simple polytope, whose h-, γ-vectors have generating functions

$$\sum_i h_i t^i = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}(w)}$$

$$\sum_i \gamma_i t^i = \sum_{w\in\widehat{\mathfrak{S}}_n} t^{\mathrm{des}(w)}.$$

III. Buildings sets and nestohedra

DEFINITION (De Concini - Procesi 1995)

A connected building set $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$ is a collection of nonempty subsets in $[n]$ such that

- if $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$,

- $\mathcal{B}$ contains all singletons $\{i\}$, and $[n]$ itself.

The nestohedron $P_{\mathcal{B}}$ is the Minkowski sum

$$P_{\mathcal{B}} = \sum_{I \in \mathcal{B}} \Delta_I$$

of coordinate simplices

$$\Delta_I := \text{ConvexHull}(e_i \mid i \in I)$$

where the e_i are the coordinate vectors in $\mathbb{R}^n$.

Popular special case:(Carr-Devadoss 2006)
A connected graph G on vertex set $[n]$ gives rise to the graphical building set $\mathcal{B}(G)$, containing all nonempty subsets of vertices $I \subseteq [n]$ such that the induced graph $G|_I$ is connected.

PROPOSITION
Graphical building sets $\mathcal{B}(G)$ always have $P_{\mathcal{B}(G)}$ a flag simple polytope;
called a graph-associahedron.

EXAMPLES
For the complete graph,
one obtains the permutohedron.

For the path graph,
one obtains the associahedron.

For the cycle graph,
one obtains the cyclohedron.

PROPOSITION (Postnikov 2005)

For any connected building set $\mathcal{B}$, the nestohedron is a simple polytope, whose polar dual has boundary complex isomorphic to the complex of nested sets for $\mathcal{B}$.

A nested set for $\mathcal{B}$ is a collection $N \subset \mathcal{B} \setminus \{[n]\}$ satisfying these properties:

- If I, J both lie in N, then either they are nested, or disjoint, and

- for any collection of two or more of the sets in N which are all disjoint, their union does not lie in $\mathcal{B}$.

For the graphical building set case, Carr and Devadoss called the nested sets the tubings of the graph G.

h-vectors of nestohedra and $\mathcal{B}$-trees

For the building set $\mathcal{B}$, a $\mathcal{B}$-tree is a rooted tree T on $[n]$ such that

- For any i in $[n]$, one has $T_{\leq i}$ in $\mathcal{B}$.

- For incomparable nodes $i_1, \ldots, i_k$ in $[n]$, one has $\bigcup_{j=1}^{k} T_{\leq i_j} \notin \mathcal{B}$.

THEOREM For a connected building set $\mathcal{B}$, the h-vector of the nestohedron $P_{\mathcal{B}}$ has generating function

$$\sum_i h_i t^i = \sum_T t^{\mathrm{des}(T)}$$

where the sum is over $\mathcal{B}$-trees T, and a descent in T is an edge $i < j$ with i closer to the root than j.

$\mathcal{B}$-permutations, not $\mathcal{B}$-trees?

Each $\mathcal{B}$-tree T has a
lexicographically first linear extension $w(T)$.
Define the $\mathcal{B}$-permutations

$$\mathfrak{S}_n(\mathcal{B}) := \{w(T)\}_{\mathcal{B}\text{-trees } T}$$

The $\mathcal{B}$-permutations w in $\mathfrak{S}_n$ have several intrinsic characterizations, e.g. this one:

For each i there exists I in $\mathcal{B}$ such that $I \subseteq \{w(1), \ldots, w(i)\}$, and I contains both $w(i)$ and $\max\{w(1), w(2), \ldots, w(i)\}$.

In general, one has $\text{des}(w(T)) \leq \text{des}(T)$,
but for chordal building sets
this becomes an equality...

IV. Chordal building sets

Say a connected building set $\mathcal{B}$ is chordal if, for any of the sets $I = \{i_1 < \cdots < i_r\}$ in $\mathcal{B}$, all subsets $\{i_s, i_{s+1}, \ldots, i_r\}$ also belong to $\mathcal{B}$.

PROPOSITION

Graphical chordal building sets $\mathcal{B}(G)$
correspond to chordal graphs G
whose vertices $[n]$ have been labelled in a
simplicial/perfect elimination order.

Generalizing the permutohedron, one has...

THEOREM

For $\mathcal{B}$ a connected chordal building set, the nestohedron $P_{\mathcal{B}}$ is a flag simple polytope, whose h-,γ-vectors have generating functions

$$\sum_i h_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)}$$

$$\sum_i \gamma_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n} t^{\mathrm{des}(w)}.$$

So Gal's conjecture holds for chordal nestohedra.

V. QUESTION

When the building set $\mathcal{B}$ has $P_{\mathcal{B}}$ flag
(e.g. when $\mathcal{B}$ is a graphical building set),
but not necessarily chordal
can one find a subset of $\mathfrak{S}_n$ playing the role of $\mathfrak{S}_n(\mathcal{B})$, i.e. for which

$$\sum_i h_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)}$$

$$\sum_i \gamma_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n} t^{\mathrm{des}(w)} ?$$