Parking spaces

D. Armstrong,
V. Reiner,
B. Rhoades

U. Miami
U. Minnesota
UCSD

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Outline

1. Wonders of the classical parking space (type A)
2. Reflection group generalization
3. A conjecture (briefly)

What's a parking space?

The (by-now) classical parking space is a permutation represention of $W = \mathfrak{S}_n$, acting on the

$$(n+1)^{n-1}$$

different rearrangements of the

$$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$$

many increasing parking functions of length n.

Definition

Increasing parking functions of length n are sequences $(a_1, \ldots, a_n)$ with

- $a_1 \leq \cdots \leq a_n$
- $1 \leq a_i \leq i$.

Parking functions of length 3

Example

The $(3+1)^{3-1} = 16$ parking functions of length 3, grouped by W-orbit, increasing parking function leftmost:

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

Wondrous!

Just about every natural question about this W-permutation representation Park_n has a beautiful answer.

Many were noted by Haiman in his 1993 original paper "Conjectures on diagonal harmonics".

A starting point: the $(n+1)^{n-1}$ parking functions give coset representatives for the quotient

$$Q/(n+1)Q$$

where here Q is the rank $n-1$ lattice

$$Q := \mathbb{Z}^n/\mathbb{Z}[1,1,\ldots,1] \cong \mathbb{Z}^{n-1}.$$

The parking space character

Corollary

Each permutation w in $W = \mathfrak{S}_n$ acts on Park_n with character value = trace = number of fixed parking functions

$$\chi_{\mathrm{Park}_n}(w) = (n+1)^{\#(\text{cycles of } w)-1}.$$

Orbit structure?

We've seen the W-orbits in Park_n are parametrized by increasing parking functions, which are Catalan objects. The stabilizer of an orbit is always a Young subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$$

where λ are the multiplicities in any orbit representative.

Example

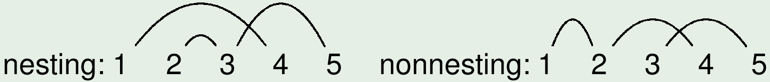
						λ
111						(3)
112	121	211				(2,1)
113	131	311				(2,1)
122	212	221				(2,1)
123	132	213	231	312	321	(1,1,1)

Two other Catalan objects

The stabilizer data $\mathfrak{S}_\lambda$ are predicted by two other Catalan objects: block sizes in these partitions of $\{1, 2, \ldots, n\}$:

- nonnesting partitions, or
- noncrossing partitions.

Example



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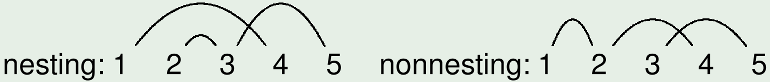

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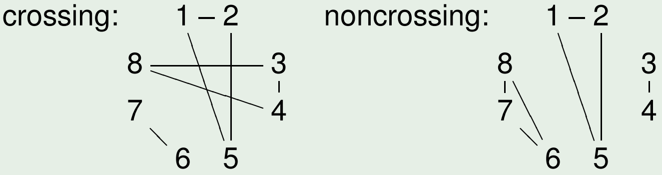
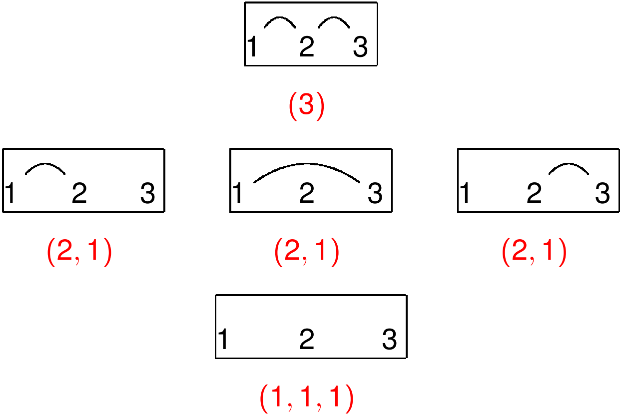


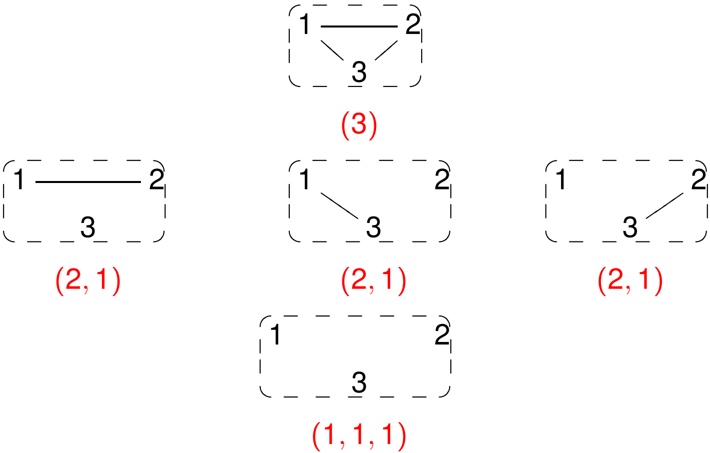
Nonnesting partitions $NN(3)$ of $\{1, 2, 3\}$



Theorem (Shi 1986, Cellini-Papi 2002)

$NN(n)$ bijects to increasing parking functions respecting λ.

Noncrossing partitions $NC(3)$ of $\{1, 2, 3\}$

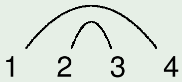


Theorem (Athanasiadis 1998)

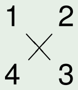
There is a bijection $NN(n) \to NC(n)$, respecting λ.

Example

For $n = 4$, among partitions of $\{1, 2, 3, 4\}$,
exactly one is nesting,



and exactly one is crossing,



and note that both correspond to $\lambda = (2, 2)$.

More wonders: Irreducible multiplicities in Park_n

W-irreducible characters are $\{\chi^\lambda\}$ indexed by partitions λ of n.
Haiman gave a formula for irreducible multiplicities

$$\langle \chi^\lambda, \mathrm{Park}_n \rangle.$$

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$$\langle \chi^\lambda, \mathrm{Park}_n \rangle.$$

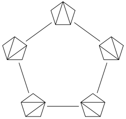
The special case of hook shapes $\lambda = (n-k, 1^k)$ becomes this .

Theorem (Pak-Postnikov 1997)

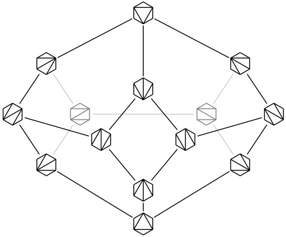
The multiplicity $\langle \chi^{(n-k,1^k)}, \chi_{\mathrm{Park}_n} \rangle_W$ is

- *the number of subdivisions of an $(n+2)$-gon using $n-1-k$ internal diagonals, or*
- *the number of k-dimensional faces in the $(n-1)$-dimensional associahedron.*

Example: n=4



$$\langle \chi^{(3)}, \chi_{\mathrm{Park}_3} \rangle_{\mathfrak{S}_3} = 5$$
$$\langle \chi^{(2,1)}, \chi_{\mathrm{Park}_3} \rangle_{\mathfrak{S}_3} = 5$$
$$\langle \chi^{(1,1,1)}, \chi_{\mathrm{Park}_3} \rangle_{\mathfrak{S}_3} = 1$$



$$\langle \chi^{(4)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 14$$
$$\langle \chi^{(3,1)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 21$$
$$\langle \chi^{(2,1,1)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 9$$
$$\langle \chi^{(1,1,1,1)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 1$$

Last wonder: Cyclic symmetry and q-Catalan

The noncrossings $NC(n)$ have a $\mathbb{Z}/n\mathbb{Z}$-action via rotations, interacting well with MacMahon's q-Catalan number

$$\mathrm{Cat}_n(q) := \frac{(1-q^{n+2})(1-q^{n+3})\cdots(1-q^{2n})}{(1-q^2)(1-q^3)\cdots(1-q^n)}.$$

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Theorem (Stanton-White-R. 2004)

For d dividing n, the number of noncrossing partitions of n with d-fold rotational symmetry is

$$[\mathrm{Cat}_n(q)]_{q=\zeta_d}$$

where ζ_d is any primitive d^{th} root of unity in $\mathbb{C}$.

$NC(4), \mathrm{Cat}_4(q)$ and rotational symmetry

Example

$$\mathrm{Cat}_4(q) = \frac{(1-q^6)(1-q^7)(1-q^8)}{(1-q^2)(1-q^3)(1-q^4)} = \begin{cases} 14 & \text{if } q = +1 = \zeta_1 \\ 6 & \text{if } q = -1 = \zeta_2 \\ 2 & \text{if } q = \pm i = \zeta_4. \end{cases}$$

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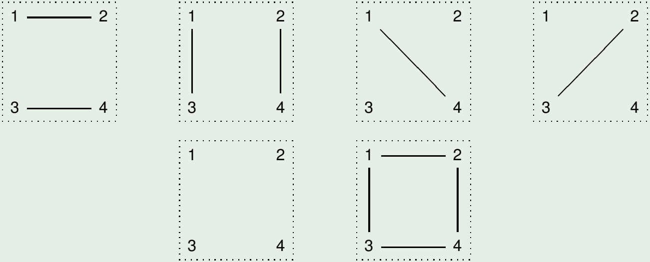

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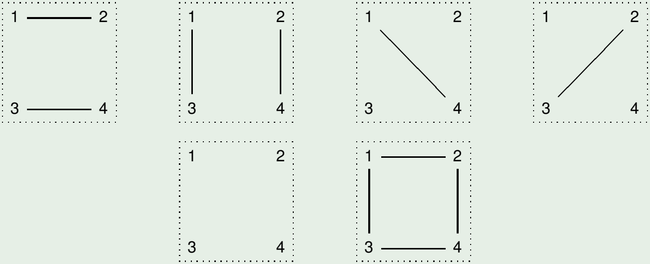


2 of which have 4-fold rotational symmetry.

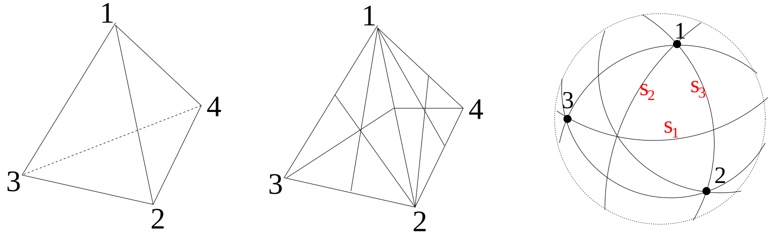
On to the reflection group generalization

Generalize to irreducible real ref'n groups W acting on $V = \mathbb{R}^n$.

Example

$W = \mathfrak{S}_n$ acts irreducibly on $V = \mathbb{R}^{n-1}$,
realized as $x_1 + x_2 + \cdots + x_n = 0$ within $\mathbb{R}^n$.

It is generated transpositions (i, j),
which are reflections through the hyperplanes $x_i = x_j$.



Invariant theory enters the picture

Theorem (Chevalley, Shephard-Todd 1955)

When W acts on polynomials $S = \mathbb{C}[x_1, \ldots, x_n] = \mathrm{Sym}(V^*)$, its W-invariant subalgebra is again a polynomial algebra

$$S^W = \mathbb{C}[f_1, \ldots, f_n]$$

One can pick $f_1, \ldots, f_n$ homogeneous, with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, and define $h := d_n$ the Coxeter number.

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Example

For $W = \mathfrak{S}_n$, one has

$$S^W = \mathbb{C}[e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})],$$

so the degrees are $(2, 3, \ldots, n)$, and $h = n$.

Weyl groups and the first W-parking space

When W is a Weyl (crystallographic) real finite reflection group, it preserves a full rank lattice

$$Q \cong \mathbb{Z}^n$$

inside $V = \mathbb{R}^n$. One can choose a root system Φ of normals to the hyperplanes, in such a way that the root lattice $Q := \mathbb{Z}\Phi$ is a W-stable lattice.

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Definition (Haiman 1993)

We should think of the W-permutation representation on the set

$$\mathrm{Park}(W) := Q/(h+1)Q$$

as a W-analogue of parking functions.

Wondrous properties of $\text{Park}(w) = Q/(h+1)Q$

Theorem (Haiman 1993)

For a Weyl group W,

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$$\chi_{\mathrm{Park}(W)}(w) = (h+1)^{\dim V^w}.$$

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- *The W-orbit count $\#W \backslash Q/(h+1)Q$ is the W-Catalan:*

$$\langle \mathbf{1}_W, \chi_{\text{Park}(W)} \rangle = \prod_{i=1}^{n} \frac{h+d_i}{d_i} =: \text{Cat}(W)$$

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- $\mathrm{Park}(W)$ *contains one copy of the sign/det character for W:*

$$\langle \det{}_W, \chi_{\mathrm{Park}(W)} \rangle = 1.$$

W-Catalan example: $W = \mathfrak{S}_n$

Example

Recall that $W = \mathfrak{S}_n$ acts irreducibly on $V = \mathbb{R}^{n-1}$ with degrees $(2, 3, \ldots, n)$ and $h = n$.

One can identify the root lattice $Q \cong \mathbb{Z}^n/(1, 1, \ldots, 1)\mathbb{Z}$.

One has $\#Q/(h+1)Q = (n+1)^{n-1}$, and

$$\begin{aligned}\mathrm{Cat}(\mathfrak{S}_n) &= \#W\backslash Q/(h+1)Q \\ &= \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n} \\ &= \frac{1}{n+1}\binom{2n}{n} \\ &= \mathrm{Cat}_n.\end{aligned}$$

Exterior powers of V

One can consider multiplicities in $\text{Park}(W)$ not just of

$$\mathbf{1}_W = \wedge^0 V$$
$$\det{}_W = \wedge^n V$$

but all the exterior powers $\wedge^k V$ for $k = 0, 1, 2, \ldots, n$, which are known to all be W-irreducibles (Steinberg).

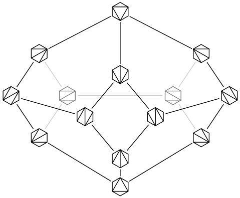
Example

$W = \mathfrak{S}_n$ acts irreducibly on $V = \mathbb{R}^{n-1}$ with character $\chi^{(n-1,1)}$, and on $\wedge^k V$ with character $\chi^{(n-k,1^k)}$.

Theorem (Armstrong-Rhoades-R. 2012)

For Weyl groups W, the multiplicity $\langle \chi_{\wedge^k V}, \chi_{\mathrm{Park}(W)} \rangle$ is

- *the number of $(n-k)$-element sets of compatible cluster variables in a cluster algebra of finite type W,*
- *or the number of k-dimensional faces in the W-associahedron of Chapoton-Fomin-Zelevinsky (2002).*



Two W-Catalan objects: $NN(W)$ and $NC(W)$

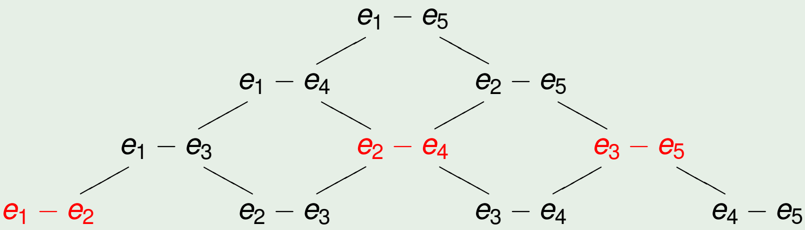
The previous result relies on an amazing coincidence for two W-Catalan counted families generalizing $NN(n), NC(n)$.

Definition (Postnikov 1997)

For Weyl groups W, define W-nonnesting partitions $NN(W)$ to be the antichains in the poset of positive roots Φ_+.

Example

corresponds to this antichain A:



W-noncrossing partitions

Definition (Bessis 2003, Brady-Watt 2002)

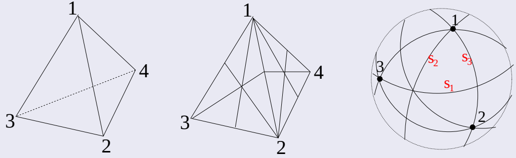
W-noncrossing partitions $NC(W)$ are the interval $[e, c]_{\text{abs}}$ from identity e to any Coxeter element c in absolute order $\leq_{\text{abs}}$ on W:

$$x \leq_{\text{abs}} y \quad \text{if} \quad \ell_T(x) + \ell_T(x^{-1}y) = \ell_T(y)$$

where the absolute (reflection) length is

$$\ell_T(w) = \min\{w = t_1 t_2 \cdots t_\ell : t_i \text{ reflections}\}$$

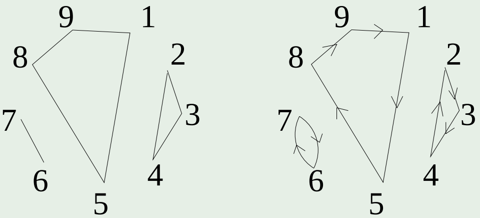
and a Coxeter element $c = s_1 s_2 \cdots s_n$ is any product of a choice of simple reflections $S = \{s_1, \ldots, s_n\}$.



Example

For $W = \mathfrak{S}_n$, the n-cycle $c = (1, 2, \ldots, n)$ is one choice of a Coxeter element.

And permutations w in $NC(W) = [e, c]_{\text{abs}}$ come from orienting clockwise the blocks of the noncrossing partitions $NC(n)$.



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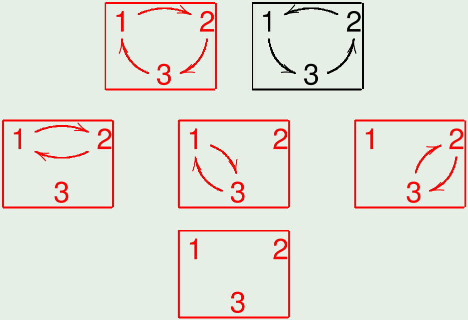

Generalizing NN, NC block size coincidence

We understand why $NN(W)$ is counted by $\mathrm{Cat}(W)$.

We do not really understand why the same holds for $NC(W)$.

Worse, we do not really understand why the following holds– it was checked case-by-case.

Theorem (Athanasiadis-R. 2004)

The W-orbit distributions coincide[a] for subspaces arising as

- *intersections $X = \cap_{\alpha \in A} \alpha^{\perp}$ for A in $NN(W)$, and as*
- *fixed spaces $X = V^w$ for w in $NC(W)$.*

[a] ...and have a nice product formula via Orlik-Solomon exponents.

What about a q-analogue of $\mathrm{Cat}(W)$?

Theorem (Gordon 2002, Berest-Etingof-Ginzburg 2003)

For irreducible real reflection groups W,

$$\mathrm{Cat}(W,q) := \prod_{i=1}^{n} \frac{1-q^{h+d_i}}{1-q^{d_i}}$$

turns out to lie in $\mathbb{N}[q]$, as it is a Hilbert series

$$\mathrm{Cat}(W,q) = \mathrm{Hilb}(\ (S/(\Theta))^W\ ,q)$$

where $\Theta = (\theta_1, \ldots, \theta_n)$ is a magical hsop in $S = \mathbb{C}[x_1, \ldots, x_n]$

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- their $\mathbb{C}$-span carries W-rep'n V^*, like $\{x_1, \ldots, x_n\}$, and
- $S/(\Theta)$ is finite-dim'l (=: the graded W-parking space).

Do you believe in magic?

These magical hsop's do exist, and they're not unique.

Example

For $W = B_n$, the hyperoctahedral group of signed permutation matrices, acting on $V = \mathbb{R}^n$, one has $h = 2n$, and one can take

$$\Theta = (x_1^{2n+1}, \ldots, x_n^{2n+1}).$$

Example

For $W = \mathfrak{S}_n$ they're tricky. A construction by Kraft appears in Haiman (1993), and Dunkl (1998) gave another.

For general real reflection groups, Θ comes from rep theory of the rational Cherednik algebra for W, with parameter $\frac{h+1}{h}$.

$NC(W)$ and cyclic symmetry

$\mathrm{Cat}(W, q)$ interacts well with a cyclic $\mathbb{Z}/h\mathbb{Z}$-action on $NC(W) = [e, c]_{\mathrm{abs}}$ that comes from conjugation

$$w \mapsto cwc^{-1},$$

generalizing rotation of noncrossing partitions $NC(n)$.

Theorem (Bessis-R. 2004)

For any d dividing h, the number of w in $NC(W)$ that have d-fold symmetry, meaning that $c^{\frac{h}{d}} w c^{-\frac{h}{d}} = w$, is

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But the proof again needed some of the case-by-case facts!

A conjecture that explains it all

Conjecture

For any irreducible real reflection group W and magical hsop Θ, chosen so the map $x_i \mapsto \Theta_i$ is W-equivariant, the set

$$V^{\Theta} := \{\mathbf{x} \in V = \mathbb{C}^n : \theta_i(\mathbf{x}) = \mathbf{x} \text{ for } i = 1, 2, \ldots, n\}$$

A conjecture that explains it all

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For any irreducible real reflection group W and magical hsop Θ, chosen so the map $x_i \mapsto \Theta_i$ is W-equivariant, the set

$$V^{\Theta} := \{\mathbf{x} \in V = \mathbb{C}^n : \theta_i(\mathbf{x}) = \mathbf{x} \text{ for } i = 1, 2, \ldots, n\}$$

- *contains $(h+1)^n$ distinct points, permuted by W.*

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- *has the orbit $\mathcal{O}_w$ described as W/W' where W' is the reflection subgroup pointwise-stabilizing $X = V^w$.*
- *has $\mathbb{Z}/h\mathbb{Z}$-action from scaling $\mathbf{x} \mapsto \zeta_h \mathbf{x}$ easily described using conjugation $w \mapsto cwc^{-1}$ on $NC(W)$.*

Example

For $W = B_n$ the hyperoctahedral group,
if we pick the magical hsop $\Theta = (x_1^{2n+1}, \ldots, x_n^{2n+1})$, then

$$\begin{aligned} V^\Theta &= \{\mathbf{x} \in V = \mathbb{C}^n : x_i^{2n+1} = x_i \text{ for } i = 1, 2, \ldots, n\} \\ &= \left(\{0\} \cup \{\sqrt[2n]{1}\}\right)^n \end{aligned}$$

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Type B_n

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The two actions on V^Θ of

- W via signed permutations, and
- $\mathbb{Z}/2n\mathbb{Z}$ via scalings $\mathbf{x} \mapsto \zeta_{2n}\mathbf{x}$

both have a simple description in terms of $NC(B_n)$,
proven via a slightly non-trivial bijection.

Question

Can one resolve the conjecture case-free?

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Thanks for listening!