q-Binomials and the Grassmannian

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1. Binomial coefficients
2. q-binomial coefficients
3. A graded ring
4. Hilbert series
5. The PROBLEM

1. Binomial coefficients — 3 definitions

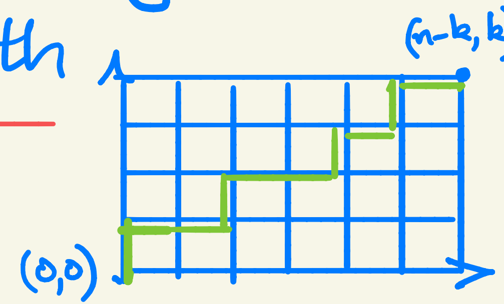
$\binom{n}{k}$ DEF. 1 $:= \dfrac{n!}{k!\,(n-k)!}$ where $n! = n(n-1)\cdots 3\cdot 2\cdot 1$

$0 \le k \le n$

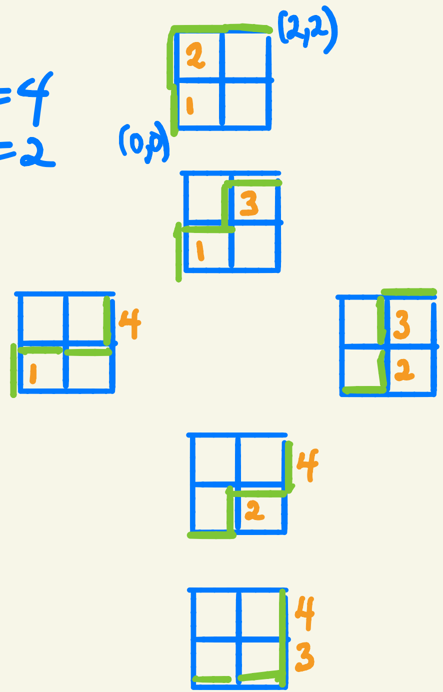
e.g. $n=4$, $k=2$: $\binom{4}{2} = \dfrac{4!}{2!\,2!} = \dfrac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 2\cdot 1} = \dfrac{4\cdot 3}{2\cdot 1} = 6$

$\binom{n}{k} \overset{\text{DEF 2}}{:=}$ # k-element subsets of $\{1, 2, \ldots, n\}$

$=$ # walks $(0,0) \longrightarrow (n-k, k)$ taking unit steps east or north



e.g. $n=4$, $k=2$



$$\binom{n}{k} \overset{\text{DEF. 3}}{:=} \begin{cases} 0 & \text{unless } 0 \le k \le n \\ 1 & \text{if } 0 = k \le n \\ \binom{n-1}{k-1} + \binom{n-1}{k} & \text{if } 1 \le k \le n \end{cases}$$

Pascal's triangle

$k=0$, $k=1$, $k=2$, $k=3$, $k=4$

$n=0$: $\binom{0}{0}$

$n=1$: $\binom{1}{0}$ $\binom{1}{1}$

$n=2$: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$n=3$: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$n=4$: $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

2. q-Binomial coefficients — 3 definitions

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \overset{\text{DEF. 1}}{:=} \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

$0 \le k \le n$

where

$$[n]!_q := [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q$$

and

$$[n]_q = 1+q+q^2+\ldots+q^{n-1} = \frac{1-q^n}{1-q}$$

e.g.

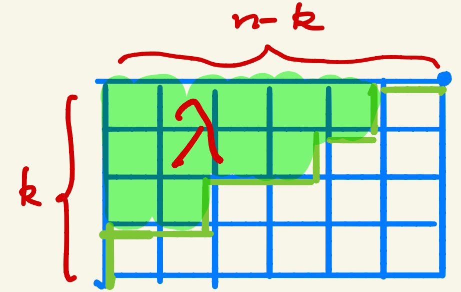
$n=4$
$k=2$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]!_q}{[2]!_q [2]!_q} = \frac{[4]_q [3]_q [2]_q [1]_q}{[2]_q [1]_q\, [2]_q [1]_q} = \frac{[4]_q [3]_q}{[2]_q\, [1]_q}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)\quad(1)} = (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \overset{\text{DEF 2}}{:=} \sum_{\substack{\text{Ferrers diagrams } \lambda \\ \text{fitting inside a } k \times (n-k) \text{ rectangle}}} q^{|\lambda|}$$

$|\lambda| = \#\text{boxes in } \lambda$



$|\lambda| = 5+4+2 = 11$

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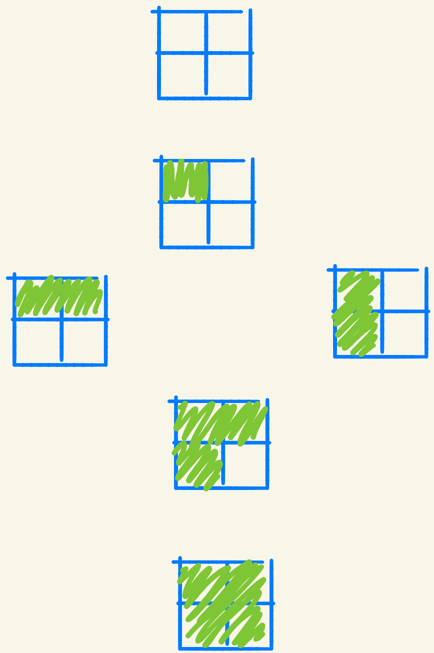
e.g. $n=4$, $k=2$



$$q^0 + q^1 + 2q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$$

DEF. 3

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} 0 & \text{unless } 0 \le k \le n \\ 1 & \text{if } 0 = k \le n \\ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q & \text{if } 1 \le k \le n \end{cases}$$

q-Pascal's triangle

k=0, k=1, k=2, k=3, k=4

$n = 0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q$

$n = 1$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q$

$n = 2$: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q$

$n = 3$: $\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q$

$n = 4$: $\begin{bmatrix} 4 \\ 0 \end{bmatrix}_q \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \quad \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q \quad \begin{bmatrix} 4 \\ 4 \end{bmatrix}_q$

1

$1 \quad 1$

$1 \quad 1+q^1 \quad 1$

$1 \quad 1+q^1(1+q) = 1+q+q^2 \quad 1+q+q^2 \quad 1$

$1 \quad 1+q^1(1+q+q^2) = 1+q+q^2+q^3 \quad 1+q+q^2 + q^2(1+q+q^2) = 1+q+2q^2+q^3+q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \quad 1+q+q^2+q^3 \quad 1$

WARM-UP EXERCISE #1:

(a) Prove DEFS 1, 2, 3 of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are all equivalent.

(b) Prove $\begin{bmatrix} n \\ k \end{bmatrix}_q$

- is a polynomial in q
- has nonnegative coefficients
- has evaluation at $q=1$ equal to $\binom{n}{k}$

((c) Prove or look up why whenever q is a prime power p^d then $\begin{bmatrix} n \\ k \end{bmatrix}_q = \#$ k-dimensional $\mathbb{F}_q$-linear subspaces of $(\mathbb{F}_q)^n$)

3. A graded ring

Let's now emphasize the symmetry between k and $n-k$ in $\binom{n}{k}$ or $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by setting

$\ell := n-k$, so $n = k + \ell$.

We'll re-interpret $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q \left(= \begin{bmatrix} k+\ell \\ \ell \end{bmatrix}_q \right)$

in terms of a certain graded ring.

$$R^{k,\ell} := \mathbb{Q}[e_1, e_2, \dots, e_k, h_1, h_2, \dots, h_\ell] \Big/ \left(\begin{array}{c} h_1 - e_1, \\ h_2 - e_1 h_1 + e_2, \\ h_3 - e_1 h_2 + e_2 h_1 - e_3, \\ \vdots \\ h_d - e_1 h_{d-1} + e_2 h_{d-2} - \dots + (-1)^d e_d \\ \vdots \end{array} \right)$$

for $k, \ell \geq 0$

a homogeneous ideal

if let $\deg(e_i) = \deg(h_i) = i$

interpreting $e_{k+1} = e_{k+2} = \dots = 0$

$h_{\ell+1} = h_{\ell+2} = \dots = 0$

$\left(R^{k,\ell} \cong H^*(\mathrm{Gr}(k, \mathbb{C}^{k+\ell}), \mathbb{Q}) \right.$ = cohomology of Grassmannian of k-planes in $\left. \mathbb{C}^{k+\ell} \right)$

e.g. $k=2$, $\ell=2$ (so $n=k+\ell=4$)

$$R^{2,2} = \mathbb{Q}[e_1,e_2,h_1,h_2] \Big/ \left(\begin{array}{l} h_1 - e_1, \\ h_2 - e_1h_1 + e_2, \\ \cancelto{0}{h_3} \boxed{-e_1h_2 + e_2h_1} - \cancelto{0}{e_3}, \\ \cancelto{0}{h_4} - \cancelto{0}{e_1h_3} \boxed{+e_2h_2} - \cancelto{0}{e_3h_1} + \cancelto{0}{e_4} \end{array} \right)$$

$$h_1 \mapsto e_1 \qquad h_2 \mapsto e_1 \cdot e_1 - e_2 = e_1^2 - e_2$$

$\wr\!\downarrow$ ring isomorphism

$$\mathbb{Q}[e_1,e_2] \Big/ \left(\overbrace{-e_1^3 + 2e_1e_2}, \ \overbrace{-e_1^2e_2 + e_2^2} \right)$$

WARM-UP EXERCISE #2

Prove there is a ring isomorphism

$$R^{k,\ell} = \mathbb{Q}[e_1,\dots,e_k,h_1,\dots,h_\ell] \Big/ \left(\sum_{i=0}^{d} (-1)^i e_i h_{d-i} \right)_{d=0,1,2,\dots} \xrightarrow{\sim} \mathbb{Q}[e_1,\dots,e_k] \Big/ \left(h_{\ell+1}, h_{\ell+2}, \dots, h_{\ell+k} \right)$$

by re-interpreting

$$h_d = \det \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & \\ 1 & e_1 & e_2 & e_3 & \cdots \\ 0 & 1 & e_1 & e_2 & \\ 0 & 0 & 1 & \ddots & \\ \vdots & & \ddots & & e_1 \end{bmatrix}$$

a (special) Jacobi-Trudi determinant

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e.g. $k=2$, $\ell=2$

$h_1 = \det[e_1] = e_1$

$h_2 = \det\begin{bmatrix} e_1 & e_2 \\ 1 & e_1 \end{bmatrix} = e_1^2 - e_2$

$h_3 = \det\begin{bmatrix} e_1 & e_2 & 0 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{bmatrix} = e_1^3 - 2e_1e_2$

$h_4 = \det\begin{bmatrix} e_1 & e_2 & 0 & 0 \\ 1 & e_1 & e_2 & 0 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{bmatrix} = e_1^4 - 3e_1^2e_2 + e_2^2$

4. Hilbert Series

For a graded ring $R = \bigoplus_{d=0}^{\infty} R_d$ its Hilbert Series

$$\mathrm{Hilb}(R, q) := \dim_{\mathbb{Q}} R_0 + \dim_{\mathbb{Q}} R_1 \cdot q^1 + \dim_{\mathbb{Q}} R_2 \cdot q^2 + \dots$$
$$= \sum_{d=0}^{\infty} (\dim_{\mathbb{Q}} R_d) \cdot q^d$$

$$\boxed{\text{What is } \mathrm{Hilb}(R^{k,l}, q)\,?}$$

e.g. $\begin{matrix} k=2 \\ l=2 \\ n=4 \end{matrix}$ $\quad R^{2,2} \cong \mathbb{Q}[e_1, e_2] \Big/ \left(-\underline{\underline{e_1^3}} + 2e_1e_2 ,\ -e_1^2e_2 + \underline{\underline{e_2^2}} \right)$

has homogeneous $\mathbb{Q}$-basis

	$\{1,$	$e_1,$	$e_1^2, e_2,$	$e_1e_2, \cancel{e_1^3},$	$e_1^2e_2, \cancel{e_2^2}, \cancel{e_1^4}\}$
degree:	0	1	2	3	4

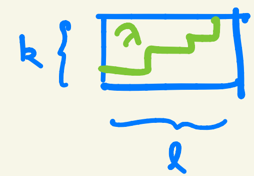
and hence

$$\mathrm{Hilb}(R^{2,2}, q) = q^0 + q^1 + 2q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$$

THEOREM

$$\mathrm{Hilb}(R^{k,\ell}, q) = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$$

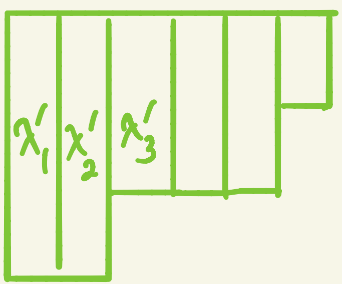
e.g., because $R^{k,\ell}$ has a $\mathbb{Q}$-basis $\{s_\lambda\}$ λ a Ferrers diagram inside $k \times \ell$ rectangle



where $$s_\lambda = \det \begin{bmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & e_{\lambda'_1+2} & \cdots \\ e_{\lambda'_2-1} & e_{\lambda'_2} & e_{\lambda'_2+1} & \\ \vdots & & e_{\lambda'_3} & \ddots \end{bmatrix}$$

Schur function corresponding to λ

$\lambda =$



5. The PROBLEM

... is trying to answer this topologically motivated ...

QUESTION : What is the Hilbert series for each of

$R^{k,\ell,m} :=$ subalgebra of $R^{k,\ell}$ generated by $e_1, e_2, \dots, e_m$?

e.g. $R^{k,\ell,0} \subset R^{k,\ell,1} \subset R^{k,\ell,2} \subset \dots \subset R^{k,\ell,k}$

$R^{k,\ell,0} = \mathbb{Q}$

$R^{k,\ell,1} =$ subalg. gen'd by e_1

$R^{k,\ell,2} =$ subalg gen'd by e_1, e_2

$R^{k,\ell,k} = R^{k,\ell}$

e.g. $\begin{matrix} k=2 \\ \ell=2 \end{matrix}$ $\quad R^{2,2} \cong \mathbb{Q}[e_1,e_2] \big/ \left(-e_1^3+2e_1e_2,\ -e_1^2e_2+e_2^2\right)$

$$\cong \mathbb{Q}[e_1,e_2] \big/ \left(\underline{e_1^5},\ \underline{2e_2e_1}-e_1^3,\ \underline{2e_2^2}-e_1^4\right)$$

Used Macaulay2 to compute a Gröbner basis with respect to lexicographic monomial order with $e_2 > e_1$ eliminating variable e_2.

$$\begin{array}{lccccc}
\text{has} & R^{2,2,0} & \subset & R^{2,2,1} & \subset & R^{2,2,2} \\
 & \| & & \cong & & \| \\
 & \mathbb{Q} & & \mathbb{Q}[e_1]/(e_1^5) & & R^{2,2} \\
 & & & \| & & \\
 & & & \mathbb{Q}\text{-span of } \{1, e_1, e_1^2, e_1^3, e_1^4\} & & \\
\mathrm{Hilb}(\cdot, q) & \downarrow & & \downarrow & & \downarrow \\
 & 1 & & 1+q+q^2+q^3+q^4 & & 1+q+2q^2+q^3+q^4
\end{array}$$

In 2003, Geanina Tudose and I made the following...

CONJECTURE

$$\mathrm{Hilb}\left(R^{k,\ell,m}/R^{k,\ell,m-1}, q\right) \quad \left(= \mathrm{Hilb}(R^{k,\ell,m},q) - \mathrm{Hilb}(R^{k,\ell,m-1},q)\right)$$

$$= q^{m} \cdot \begin{bmatrix} \ell \\ m \end{bmatrix}_q \cdot \left(\sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q \right)$$

CONJECTURE

$$\mathrm{Hilb}\left(R^{k,\ell,m}/R^{k,\ell,m-1}, q\right) = q^{m} \cdot \begin{bmatrix} \ell \\ m \end{bmatrix}_q \cdot \left(\sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q \right)$$

WARM-UP EXERCISE #3

(a) Letting $f_m^{k,\ell}(q) := \sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q$ in parentheses above,

show $\left[f_m^{k,\ell}(q) \right]_{q=1} = \binom{k}{m}$.

(b) Prove $\sum_{m=0}^{k} \binom{\ell}{m}\binom{k}{m} = \binom{k+\ell}{k}$ $\left(= \left[\mathrm{Hilb}(R^{k,\ell}, q) \right]_{q=1} \right)$

POLYMATH PROBLEM:

(a) (not easy)

Prove that **CONJECTURE**

(b) (likely easier)

Formulate the analogous conjecture for ...

… the graded ring

$$R^n_{LG} := \mathbb{Q}[e_1, e_2, \ldots, e_n] \Big/ \begin{pmatrix} e_1^2 - 2e_2, \\ e_2^2 - 2(e_3e_1 - e_4), \\ e_3^2 - 2(e_4e_2 - e_5e_1 + e_6), \\ \vdots \\ e_d^2 + 2\sum_{k=1}^{n-d}(-1)^k e_{d+k}e_{d-k} \\ \vdots \\ e_{n-1}^2 - 2(e_ne_{n-2}) \\ e_n^2 \end{pmatrix}$$

where $\deg(e_i) := i$

and $e_0 := 1$

$e_j := 0$ if $j \notin \{0, 1, \ldots, n\}$

$\cong$ cohomology of Lagrangian Grassmannian of Type C_n

with

$$\mathrm{Hilb}(R^n_{LG}, q) = (1+q)(1+q^2)(1+q^3)\cdots(1+q^n)$$

QUESTION: What is $\mathrm{Hilb}(R^{n,m}_{LG}, q)$?

subalg. of R^n_{LG} gen'd by $e_1, e_2, \ldots, e_m$

Thanks for your attention!