Exercises on reflection group counting and q-counting
Summer School: Algebraic and Enumerative Combinatorics[1]
S. Miguel de Seide, Portugal, July 2012
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Contents

Problems marked with an asterisk * are particularly recommended.

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1. Invariant theory and Hilbert series

1*. (*Hilbert series warm-up*)
Recall that for a graded $\mathbb{F}$-vector space

$$V = \bigoplus_{d \geq 0} V_d$$

where each V_d is a finite-dimensional $\mathbb{F}$-vector space, the *Hilbert series* in variable t is the formal power series generating function in $\mathbb{Z}[[t]]$ for the dimensions of the V_d:

$$\mathrm{Hilb}(V, t) := \sum_{d \geq 0} t^d \cdot \dim_{\mathbb{F}} V_d.$$

(a) Show that a polynomial algebra $S := \mathbb{F}[x_1, \ldots, x_n]$ considered as a graded ring in which each variable x_i has $\deg(x_i) = 1$, so that a monomial $x^a := x_1^{a_1} \cdots x_n^{a_n}$ has degree $a_1 + \cdots + a_n$, will have Hilbert series

$$\mathrm{Hilb}(S, t) = \frac{1}{(1-t)^n}.$$

(b) Show more generally that a polynomial algebra $\mathbb{F}[f_1, \ldots, f_n]$ in which the algebraically independent generators $f_1, \ldots, f_n$ are have degrees $\deg(f_i) = d_i \geq 1$ will have Hilbert series

$$\mathrm{Hilb}(\ \mathbb{F}[f_1, \ldots, f_n]\ , t) = \prod_{i=1}^{n} \frac{1}{(1-t^{d_i})}.$$

(c) Show that the Hilbert series in part (b) has Laurent expansion about $t = 1$ whose pole is of order n and begins

$$\mathrm{Hilb}(\mathbb{F}[f_1, \ldots, f_n], t) = \frac{1}{d_1 \cdots d_n (1-t)^n} + O\left(\frac{1}{(1-t)^{n-1}}\right).$$

2*. (*Representation theory warm-up*)

(a) Let V be a finite-dimensional $\mathbb{C}$-vector space, and

$$V \xrightarrow{\pi} V$$

a $\mathbb{C}$-linear map which is *idempotent*: $\pi^2 = \pi$.

Show that one has a direct sum decomposition

$$V = \pi V \oplus (1 - \pi)V$$

in which $\pi V = \mathrm{im}\pi$ and $(1 - \pi)V = \ker\pi$.

(b) Deduce that

$$\dim_{\mathbb{C}}(\mathrm{im}\pi) = \mathrm{Tr}(\pi)$$

where Tr denotes the trace of π, that is, the sum $\pi_{11} + \cdots + \pi_{nn}$ where (π_{ij}) is the matrix expressing the action of π in any choice of basis for V.

(c) When G is a finite group acting $\mathbb{C}$-linearly on a $\mathbb{C}$-vector space V, show that the *averaging map*

$$\begin{array}{ccc} V & \xrightarrow{\pi_G} & V \\ v & \longmapsto & \frac{1}{|G|}\sum_{g \in G} g(v) \end{array}$$

is an idempotent, having image

$$\mathrm{im}(\pi_G) = V^G := \{v \in V : g(v) = v \text{ for all } g \text{ in } G\}.$$

(d) Deduce that, in the setting of part (c),

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|}\sum_{g \in G} \mathrm{Tr}(g|_V).$$

(e) Explain how (d) proves *Burnside's formula:* when G is a group of permutations acting on a finite set X, the number of G-orbits on X is given by

$$\frac{1}{|G|}\sum_{g \in G} |\{x \in X : g(x) = x\}|.$$

(*Hint:* Consider the $\mathbb{C}$-vector space V whose basis is indexed by X and permuted by G. How can one describe V^G? How can one compute $\mathrm{Tr}(g|_V)$?)

3*. (*Molien's formula*)
This is a useful formula that lets one compute the Hilbert series for the subalgebra of G-invariant polynomials $S^G = \mathbb{C}[x_1, \ldots, x_n]^G$ when a finite subgroup G of $GL_n(\mathbb{C})$ acts on $S := \mathbb{C}[x_1, \ldots, x_n]$ by linear substitutions of the variables x_i.

(a) Explain why any g in a finite subgroup of $GL_n(\mathbb{C})$ is diagonalizable over $\mathbb{C}$.

(b) Assume g has eigenvalues $\lambda_1, \ldots, \lambda_n$ when it acts on $V^* = S_1 = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$. Explain why g acts on the d^{th} homogeneous component S_d of S with eigenvalues

$$\{\lambda_1^{a_1} \cdots \lambda_n^{a_n} : a_1 + \cdots a_n = d \text{ and } a_i \geq 0\}.$$

(*Hint:* why can one temporarily assume $g(x_i) = \lambda_i x_i$ for $i = 1, 2, \ldots, n$ in making this calculation?)

(c) Explain why, in the notation of part (b),

$$\frac{1}{\det(1 - tg)} = \prod_{i=1}^{n} \frac{1}{1 - t\lambda_i} = \sum_{d \geq 0} t^d \cdot \mathrm{Tr}(g|_{S_d}).$$

(d) Explain *Molien's formula:*

$$\mathrm{Hilb}(S^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - tg)}.$$

(e) Explain why the Laurent series for $\mathrm{Hilb}(S^G, t)$ about $t = 1$ has a pole of order n and begins

$$\mathrm{Hilb}(S^G, t) = \frac{1}{|G|(1 - t)^n} + O\left(\frac{1}{(1 - t)^{n-1}}\right).$$

(f) Explain how one can apply this together with Problem 1(c) to show that, whenever $S^G = \mathbb{C}[f_1, \ldots, f_n]$ is a polynomial algebra with homogeneous generators $f_1, \ldots, f_n$ having $\deg(f_i) = d_i$, then

$$|G| = d_1 d_2 \cdots d_n.$$

(g) Explain why for two nested subgroups $H \subset G$ in $GL_n(\mathbb{C})$, one can express their index as follows:

$$[G : H] = \lim_{t \to 1} \frac{\mathrm{Hilb}(S^H, t)}{\mathrm{Hilb}(S^G, t)}.$$

4. (*Exterior algebra variations on Molien's formula*)
One can think of $S = \mathbb{C}[x_1, \ldots, x_n]$ as the *symmetric algebra* $\mathrm{Sym}(V^*)$ on the vector space $V^* = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$. Similarly, one has the *exterior algebra*

$$\wedge V^* = \oplus_{d=0}^{n} \wedge^d V^*$$

and their tensor product

$$\mathrm{Sym}(V^*) \otimes_{\mathbb{C}} \wedge V^*.$$

(a) Assume g has eigenvalues $\lambda_1, \ldots, \lambda_n$ when it acts on $\wedge^1 V^* = V^* = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$. Explain why g acts on $\wedge^d V^*$ with eigenvalues

$$\{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_d} : 1 \leq i_1 < \cdots < i_d \leq n\}.$$

(b) Explain why

$$\det(1 + ug) = \prod_{i=1}^{n}(1 + u\lambda_i) = \sum_{d=0}^{n} u^d \cdot \mathrm{Tr}(g|_{\wedge^d V^*}).$$

(c) Explain why

$$\begin{aligned}\mathrm{Hilb}((\wedge V^*)^G, u) &:= \sum_{d=0}^{n} u^d \cdot \dim_{\mathbb{C}} \left(\wedge^d V^*\right)^G \\ &= \frac{1}{|G|}\sum_{g \in G} \det(1 + ug).\end{aligned}$$

(d) Explain why

$$\begin{aligned}\mathrm{Hilb}((S \otimes_{\mathbb{C}} \wedge V^*)^G, u, t) &:= \sum_{d \geq 0}\sum_{e \geq 0} u^e t^d \dim_{\mathbb{C}} \left(S_d \otimes \wedge^e V^*\right)^G \\ &= \frac{1}{|G|}\sum_{g \in G} \frac{\det(1 + ug)}{\det(1 - tg)}.\end{aligned}$$

5. (*Solomon's theorem and the Shephard-Todd formula*)
L. Solomon gave a beautiful derivation of the *Shephard-Todd formula*

$$\sum_{w \in W} q^{\dim(V^w)} = \prod_{i=1}^{n} (q + (d_i - 1)). \tag{1}$$

for a complex reflection group W with $S^W = \mathbb{C}[f_1, \ldots, f_n]$ and degrees $\deg(f_i) = d_i$, where $V^w = \{v \in V = \mathbb{C}^n : w(v) = v\}$.

He derived it from the following structural result on the W-invariant polynomial coefficient differential forms $(S \otimes_{\mathbb{C}} \wedge V^*)^W$, where one considers $\wedge^1 V^* \cong V^*$ to have a basis $dx_1, \ldots, dx_n$ carrying the same W-action as on $x_1, \ldots, x_n$. There is a natural structure of an S-module on these differential forms $S \otimes_{\mathbb{C}} \wedge V^*$ defined by

$$f \cdot \left(\sum_{1 \le i_1 < \cdots < i_k \le n} g_{i_1, \ldots, i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) := \sum f g_{i_1, \ldots, i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

It is easily seen that this endows the W-invariant differential forms $(S \otimes_{\mathbb{C}} \wedge V)^W$ with an S^W-module structure: if f in S and $\sum_I g_I \cdot dx_I$ in $S \otimes_{\mathbb{C}} \wedge V^*$ are both W-invariant, then $\sum_I f g_I \cdot dx_I$ is also.

Theorem 1.1. *(Solomon* [3]*)*
The W-invariant forms $(S \otimes_{\mathbb{C}} \wedge V^*)^W$ are a free S^W-module, on basis elements

$$\{df_{i_1} \wedge \cdots \wedge df_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$$

formed by taking all possible wedges of the exterior derivatives

$$df_i := \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j,$$

of the basic invariant polynomials $f_1, \ldots, f_n$.

(a) Explain how Problem 4(d) together with Solomon's theorem gives this formula:

$$\frac{1}{|W|} \sum_{w \in W} \frac{\det(1 + uw)}{\det(1 - tw)} = \prod_{i=1}^{n} \frac{1 + t^{d_i - 1} u}{1 - t^{d_i}}.$$

(b) After substituting $u = q(1-t) - 1$, show one can rewrite (a) as follows:

$$\sum_{w \in W} \prod_{\substack{\text{eigenvalues} \\ \lambda \text{ of } w}} \frac{1 + (q(1-t) - 1)\lambda}{1 - t\lambda} = |W| \prod_{i=1}^{n} \frac{(1 - t^{d_i - 1}) + q t^{d_i - 1}(1 - t)}{1 - t^{d_i}}.$$

(c) Show how formula (1) follows from (b) by taking the limit as t approaches 1.

6*. (*How to prove* $S^G = \mathbb{F}[f_1, \ldots, f_n]$ *is polynomial?*)
There is a converse to the statement in Problem 3(f), often used to prove that some group G has S^G polynomial. We give below a version that we will use in many examples, but we will not prove it here, as it requires a little commutative algebra; see [2, Prop. 5.5.5].

Lemma 1.2. *Let G be a finite subgroup of $GL_n(\mathbb{F})$ acting on $S = \mathbb{F}[x_1, \ldots, x_n]$ by linear substitutions, and suppose one has $f_1, \ldots, f_n$ homogeneous polynomials of degrees $d_1, \ldots, d_n$ with these three properties:*

- *$f_1, \ldots, f_n$ lie in S^G, that is, they are G-invariant,*
- *$|G| = d_1 \cdots d_n$, and*
- *Each variable x_i is integral[2] over the subalgebra $\mathbb{F}[f_1, \ldots, f_n]$.*

Then the G-invariant subalgebra S^G is a polynomial algebra, namely

$$S^G = \mathbb{F}[f_1, \ldots, f_n].$$

Let us apply this to several examples.

Dihedral groups.
(a) Consider the *dihedral group* $G = I_2(m)$ of order $2m$ with presentation

$$I_2(m) \cong \langle s, r : s^2 = r^m = e, srs = r^{-1} \rangle$$

represented inside $GL_2(\mathbb{R})$ as the symmetries of a regular m-sided polygon in $\mathbb{R}^2$, with s being a reflection, and r a rotation through $\frac{2\pi}{m}$.

Show that one can extend scalars to $\mathbb{C}^2$ to diagonalize r so that one has simultaneously

- r scaling x by ζ, scaling y by ζ^{-1}, where $\zeta := e^{\frac{2\pi i}{m}}$,
- s swapping x, y.

(b) Prove that the latter version of $G = I_2(m)$ inside $GL_2(\mathbb{C})$ has

$$\begin{array}{rcl} S^G & = & \mathbb{C}[\quad xy, \quad x^m + y^m \quad] \\ \text{with degrees:} & & (\quad 2, \quad\quad m \quad\quad). \end{array}$$

(*Hint:* Apply Lemma 1.2, using the polynomial

$$F(t) = (t^m - x^m)(t^m - y^m) = t^{2m} - (x^m + y^m)t^m + (xy)^m.)$$

[2] Recall that this means there exists some monic polynomial $F_i(t) = t^N + c_{N-1}t^{N-1} + \cdots + c_0$ in t with coefficients c_k in $\mathbb{F}[f_1, \ldots, f_n]$ that vanishes when one plugs in $t = x_i$, that is, $F_i(x_i) = 0$.

Symmetric groups.
(c) Explain why the symmetric group $G = \mathfrak{S}_n$ inside $GL_n(\mathbb{C})$ has

$$\begin{array}{rcccccc} S^G & = \mathbb{C}[& e_1, & e_2 & ,\ldots, & e_n &] \\ \text{with degrees:} & (& 1, & 2, & ,\ldots, & n &). \end{array}$$

where e_i are the i^{th} *elementary symmetric functions*

$$\begin{aligned} e_1 &= x_1 + x_2 + \cdots + x_n \\ e_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \sum_{1 \le i < j \le n} x_i x_j \\ &\vdots \\ e_n &= x_1 x_2 \cdots x_n. \end{aligned}$$

(*Hint:* Apply Lemma 1.2, using the polynomial $F(t) = \prod_{i=1}^n (t - x_i)$.)

Wreath product groups.
(d) More generally, explain why the wreath product group

$$G = G(d, 1, n) = (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$$

whose elements are the matrices in $\mathbb{C}^{n \times n}$ having one nonzero entry in each row and column, with that entry a d^{th} root of unity, has

$$\begin{array}{rcccccc} S^G & = \mathbb{C}[& e_1(\mathbf{x}^d), & e_2(\mathbf{x}^d) & ,\ldots, & e_n(\mathbf{x}^d) &] \\ \text{with degrees:} & (& d, & 2d, & ,\ldots, & nd &). \end{array}$$

where $f(\mathbf{x}^d) := f(x_1^d, \ldots, x_n^d)$, and e_i are elementary symmetric functions as before. (*Hint:* Apply Lemma 1.2, using the polynomial $F(t) = \prod_{i=1}^n (t^d - x_i^d)$.)

The Shephard-Todd imprimitive groups $G(de, e, n)$.
(e) Recall that the subgroup of matrices $G(de, e, n)$ inside $G(de, 1, n)$ is defined by the property that the product of their nonzero entries is not just a $(de)^{th}$ root, but actually a d^{th} root of unity. Show that this group $G = G(de, e, n)$ has

$$\begin{array}{rccccccc} S^G & = \mathbb{C}[& e_1(\mathbf{x}^{de}), & e_2(\mathbf{x}^{de}) & ,\ldots, & e_{n-1}(\mathbf{x}^{de}), & e_n(\mathbf{x}^d) &] \\ \text{with degrees:} & (& de, & 2de, & ,\ldots, & (n-1)de, & nd &). \end{array}$$

(*Hint:* Apply Lemma 1.2 using the similar polynomial $F(t) = \prod_{i=1}^n (t^{de} - x_i^{de})$ as in part (d).)

(f) Explain the coincidence of degrees between the group $G(m, m, 2)$ and the dihedral group $I_2(m)$ by showing that these two groups are conjugate within $GL_2(\mathbb{C})$.

7. (*The Dickson polynomials and invariants of* $GL_n(\mathbb{F}_q)$)
Let q be a power of a prime, and $\mathbb{F}_q$ the finite field with q elements. Consider the *finite general linear group* $G = GL_n(\mathbb{F}_q)$ acting acting on $S = \mathbb{F}_q[x_1, \ldots, x_n]$ by linear substitutions. Our goal will be to show that the monic polynomial in t having coefficients in S defined by

$$F_n(t) := \prod_{\mathbf{c}=(c_1,\ldots,c_n)\in\mathbb{F}_q^n} (t - (c_1x_1 + \cdots c_nx_n))$$

has the property that its (nonzero) coefficients of powers of t give a set of polynomial generators for the G-invariants S^G.

(a) Explain why for $n = 1$ one has $F_1(t) = \prod_{c_1\in\mathbb{F}_q}(t - c_1x_1) = t^q - t^1x_1^{q-1}$.
(*Hint:* Recall from finite field theory why $\prod_{c_1\in\mathbb{F}_q}(t - c_1) = t^q - t$.)

(b) To gain more intuition, explicitly compute the coefficients of t^4, t^3, t^2, t^1, t^0 in

$$F_2(t) = t(t - x_1)(t - x_2)(t - (x_1 + x_2))$$

when working over $\mathbb{F}_2$, that is, $n = q = 2$. Note that $\pm$ signs don't matter!

For a ring R containing $\mathbb{F}_q$ as a subring, say that a polynomial $F(t)$ in $R[t]$ is an *Ore polynomial (in t)* if it has the form $\sum_i r_i t^{q^i}$ for some coefficients r_i in R.

(c) Explain why any Ore polynomial $F(t)$ has $F(ax + by) = aF(x) + bF(y)$ for all a, b in $\mathbb{F}_q$ and indeterminates x, y.

(d) Explain why for any Ore polynomial $F(t)$ in $R[t]$ the polynomial in $R[x][t]$

$$G(t) = \prod_{c\in\mathbb{F}_q} F(t - cx)$$

is again an Ore polynomial in t.

(e) Use (d) to explain why

$$F_n(t) = t^{q^n} - \sum_{i=0}^{n-1} t^{q^i} D_{n,i}(\mathbf{x})$$

for some $D_{n,0}, D_{n,1}, \ldots, D_{n,n-1}$ in S^G, called (up to $\pm$ signs) *Dickson polynomials*[3]

(f) Explain why

$$\begin{array}{rcccccccc} S^G = & \mathbb{C}[& D_{n,n-1}, & D_{n,n-2} & ,\ldots, & D_{n,1}, & D_{n,0} &] \\ \text{with degrees:} & (& q^n - q^{n-1}, & q^n - q^{n-2}, & ,\ldots, & q^n - q^1, & q^n - q^0 &). \end{array}$$

(*Hint:* Apply Lemma 1.2 to the polynomial $F_n(t)$.)

[3]The Dickson polynomial $D_{n,k}$ has an explicit formula (see e.g., Smith [2, Thm. 8.1.6]), making it analogous to the elementary symmetric function e_{n-k}:

$$D_{n,k} = \pm \sum_{\substack{k\text{-dimensional} \\ \text{subspaces } S\subset(\mathbb{F}_q^n)}} \prod_{\substack{\text{linear forms} \\ \ell(\mathbf{x})\notin S}} \ell(\mathbf{x})$$

$$e_{n-k} = \sum_{\substack{k\text{-element subsets} \\ S\subset\{1,2,\ldots,n\}}} \prod_{i\notin S} x_i.$$

8*. *Polytopes as arrangements of affine subspaces*
Recall that for vectors $\{v_i\}_{i \in I}$, their $\mathbb{C}$-*linear* and $\mathbb{C}$-*affine* spans are defined as
$\mathrm{Lin}\{v_i\}_{i\in I} = \{$ finite sums $\sum_i c_i v_i$ with $c_i \in \mathbb{C}\}$, and

$\mathrm{Aff}\{v_i\}_{i\in I} = \{$ finite sums $\sum_i c_i v_i$ with $c_i \in \mathbb{C}$ and $\sum_i c_i = 1\}$.

The affine span of a subset of vectors is called an *affine subspace*. Given a finite collection $\mathcal{P}$ of complex affine subspaces inside $V = \mathbb{C}^n$, let us say that $\mathcal{P}$ is an *abstract polytope* if it satisfies these axioms:

(i) $\mathcal{P}$ contains the (-1)-dimensional empty affine subspace $\varnothing$ and the n-dimensional ambient space $V = \mathbb{C}^n$.
(ii) For any two nested subspaces $F \subset F''$ of $\mathcal{P}$ with $\dim F'' - \dim F \geq 2$, there are at least two intermediate subspaces F_1', F_2' of $\mathcal{P}$ with $F \subset F_1', F_2', \subset F''$.
(iii) For any two nested subspaces $F \subset F''$ of $\mathcal{P}$ with $\dim F'' - \dim F \geq 3$, the open interval $(F, F'') := \{F' \in \mathcal{P} : F \subsetneq F' \subsetneq F''\}$ ordered by inclusion is a connected poset.

Given an abstract polytope $\mathcal{P}$, call the subspaces F of $\mathcal{P}$ *faces* of $\mathcal{P}$. Call a sequence of nested subpsaces $F_1 \subset \cdots \subset F_\ell$ in $\mathcal{P}$ a *flag* from F_1 to F_ℓ, and call it a *maximal flag* if $\dim(F_{i+1}) = \dim(F_i) + 1$ for $i = 1, 2, \ldots, \ell - 1$.

(a) In an abstract polytope $\mathcal{P}$, show every face lies in a maximal flag from $\varnothing$ to V.

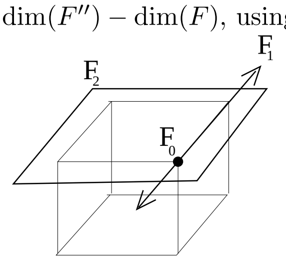
(b) Show that in an abstract polytope $\mathcal{P}$, every closed interval

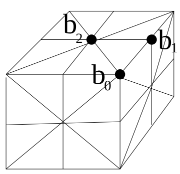
$$[F, F''] := \{F' \in \mathcal{P} : F \subseteq F' \subseteq F''\}$$

regarded as a poset is again isomorphic to the poset of faces of an abstract polytope. (*Hint:* First explain how one can reduce to the case where (a) $F" = V = \mathbb{C}^n$, (b) F is a vertex, and (c) the only face of $\mathcal{P}$ containing the origin is $V = \mathbb{C}^n$ itself, that is, the proper faces are all truly affine and not linear. In this case, let U be any linear subspace complementary to the line $\mathbb{C}F$ spanned by the vertex F, so that $V = \mathbb{C}F \oplus U$, and show that the collection of subspaces $\{F' \cap U : F' \in [F, F'']\}$ forms an abstract polytope inside U whose face poset is isomorphic to $[F, F'']$)

Say that two maximal flags from F to F'' are *adjacent (at dimension d)* if they differ in only one face F' and $\dim(F') = d$, and say that a closed interval $[F, F'']$ is *gallery-connected* if for every pair of maximal flags from F to F'' can be connected by a sequence of flags in which consecutive flags in the sequence are adjacent.

(c) In an abstract polytope $\mathcal{P}$, show closed intervals $[F, F'']$ are gallery-connected. (*Hint:* Induct on $\dim(F'') - \dim(F)$, using part (b) and Axiom (iii)).





Remark 2.1. One can show that an n-dimensional *convex polytope* in $\mathbb{R}^n$ gives rise to an abstract polytope $\mathcal{P}$ satisfying the axioms (i),(ii),(iii) above in the following way. One first *complexifies*, that is, one extends scalars from $\mathbb{R}^n$ to $V = \mathbb{C}^n$. Then one defines $\mathcal{P}$ to be the collection of (complexifications of) all affine subspaces spanned by *faces* of the polytope in the usual sense of convexity, that is the intersections of the convex polytope with the boundary hyperplanes of any of its supporting half-spaces.

9*.(*Regular real and complex polytopes*) Given an abstract polytope $\mathcal{P}$ in $V = \mathbb{C}^n$ in the sense of Problem 8, define its (linear) symmetry group

$$W = \mathrm{Aut}(P) := \{w \in \mathrm{Aut}_{\mathbb{C}}(V) = GL_n(\mathbb{C}) : w(F) \in \mathcal{P} \text{ for all } F \in \mathcal{P}\}.$$

Say that $\mathcal{P}$ is a *regular complex polytope*[4] if, in addition to the three axioms (i),(ii),(iii) above, it satisfies this axiom:

(iv) W acts transitively on the collection of maximal flags from $\varnothing$ to V in $\mathcal{P}$.

Our goal will be to show that this axiom (iv) not only shows that W is a reflection group, but also gives an adequate substitute for some of the properties enjoyed by real convex polytopes as a consequence of their *convexity*! An important role in this regard is played by the *vertices* of $\mathcal{P}$ which are defined to be the 0-dimensional faces of $\mathcal{P}$, and the notion of the *barycenter or centroid* b_F of a face F in $\mathcal{P}$

$$b_F := \frac{1}{t}\left(F_0^{(1)} + F_0^{(2)} + \cdots + F_0^{(t)}\right)$$

where $\{F_0^{(1)}, F_0^{(2)}, \ldots, F_0^{(t)}\}$ is the set of vertices lying on F.

(a) Explain why

- $b_\varnothing = 0$, the zero vector in V,
- $\{b_0\} = F_0$ for every vertex F_0,
- b_F lies on the face F when $\dim(F) \geq 0$, and
- any w in W that sends F to itself must fix b_F.

Now fix a base maximal flag $\mathcal{F}_0$ of faces from $\varnothing$ to V

$$\varnothing =: F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n := V$$

and name their barycenters $b_{-1}, b_0, b_1, b_2, \ldots, b_{n-1}$, so that $b_{-1} = 0$ and $b_0 = F_0$.

(b) For $j = 0, 1, 2, \ldots, n-1$, show both that

- $\{b_0, b_1, \ldots, b_j\}$ are $\mathbb{C}$-linearly independent, and
- therefore $\mathrm{Aff}\{b_0, b_1, \ldots, b_j\} = F_j$.

(*Hint:* Induct on j. In the base case $j = 0$, show that $b_0 = 0$ would force $F_0 = 0$ and then use Axiom (iv) to show that *every* vertex of $\mathcal{P}$ would be 0, contradicting Axiom (ii). In the inductive step, if b_j lies in $\mathrm{Lin}\{b_0, b_1, \ldots, b_{j-1}\}$ let m be the smallest index in $0 \leq m \leq j-1$ such that b_j lies in $\mathrm{Lin}\{b_0, b_1, \ldots, b_m\}$ but not in $\mathrm{Lin}\{b_0, b_1, \ldots, b_{m-1}\}$. Show this forces $F_m = \mathrm{Aff}\{b_0, b_1, \ldots, b_{m-1}, b_j\}$, and deduce

[4] For example, when $\mathcal{P}$ comes from the complexification of the affine spans of the faces of an n-dimensional convex polytope in $\mathbb{R}^n$, as discussed at the end of Problem 8, this extra axiom (iv) is the usual extra condition one requires for the convex polytope to be called a *regular polytope*.

from this that every w in W that preserves $F_0, F_1, \ldots, F_{m-1}, F_j$ must also preserve F_m. How does this contradict Axioms (ii) and (iv)?)

(c) Deduce W acts *simply* transitively on the maximal flags from $\varnothing$ to V, that is, there is a *unique* group element w taking the base flag $\mathcal{F}_0$ to any other such flag $\mathcal{F}$. In particular, there are $|W|$ such maximal flags.
(*Hint:* Part (b) showed $\{b_0, b_1, \ldots, b_{n-1}\}$ form a $\mathbb{C}$-linear basis for $V = \mathbb{C}^n$.)

(d) Show that for each $i = 0, 1, 2, \ldots, n-1$, the subgroup W_i of W preserving the flag $\mathcal{F}_0 \setminus \{F_i\}$

$$F_{-1} \subset F_0 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset F_{i+2} \subset \cdots \subset F_n$$

is a cyclic group $W_i = \langle r_i \rangle$ generated by some reflection r_i.
(*Hint:* W_i must fix pointwise the barycenters $b_0, b_1, \ldots, b_{i-1}, b_{i+1}, b_{i+2}, \ldots, b_{n-1}$, and hence also fixes pointwise the hyperplane H_i which they span $\mathbb{C}$-linearly. Use this together with finiteness of W to show the homomorphism $\det : W_i \to \mathbb{C}^\times$ is injective, and hence W_i is cyclic.)

(e) Explain why for any two maximal flags $w(\mathcal{F}_0)$ and $w'(\mathcal{F}_0)$ that are adjacent at dimension i, there is a power r_i^k in the cyclic group W_i such that $w' = wr_i^k$. Use this together with Problem 8(c) to show that $W = \langle r_0, r_1, \ldots, r_{n-1} \rangle$, so that W is a reflection group, called a *Shephard group*.

Our next goal is to say something about the nature of the relations among the generating reflections $S = \{r_0, r_1, \ldots, r_{n-1}\}$ for a Shephard group. Given two indices i, j satisfying $0 \leq i < j \leq n-1$, assume that

- r_i sends F_i to F_i', while preserving the other faces $\mathcal{F}_0 \setminus \{F_i\}$,
- r_j sends F_j to F_j', while preserving the other faces $\mathcal{F}_0 \setminus \{F_j\}$.

(f) Explain why if $|j - i| \geq 2$ then r_j preserves F_i', and r_i preserves F_j'. Why does this show that r_i, r_j commute?

(g) Show that r_i, r_{i+1} can *never* commute for $i = 0, 1, 2, \ldots, n-2$.
(*Hint:* Explain why $F_i' := r_i(F_i) \neq F_i$ and $F_{i+1}' := r_{i+1}(F_{i+1}) \neq F_{i+1}$. Then show that if r_i, r_{i+1} commute, one would have both of F_{i+1}, F_{i+1}' containing both of F_i, F_i', and why this is a contradiction.)

Remark 2.2. Part (g) explains why a regular (real) polytope has Coxeter system (W, S) which is *irreducible*, while part (f) explains why its Coxeter diagram is *unbranched*, that is, no node has degree 3 or higher.

Conversely, for any Coxeter system (W, S) with an irreducible unbranched Coxeter diagram and W finite, there is a construction due to Wythoff that allows one to produce a real regular convex polytope having W as its symmetry group.

3. Regular elements

Recall that an element c in a complex reflection subgroup W acting on $V = \mathbb{C}^n$ is a *regular* element if it has an eigenvector v in V that lie on none of the reflecting hyperplanes for reflections in W.

10^*.(*The regular elements in the symmetric group* $\mathfrak{S}_n$)
(a) Explain how you can decide whether a permutation w in $\mathfrak{S}_n$ is a power of an n-cycle based on its cycle type. Similarly explain how you can decide whether w is a power of an $(n-1)$-cycle.

(b) Prove that the only regular elements in $\mathfrak{S}_n$ are the powers of the n-cycles and the powers of the $(n-1)$-cycles.

11.(*The regular elements in hyperoctahedral group* $\mathfrak{S}_n^{\pm}$)
Recall that the *hyperoctahedral group* $W = \mathfrak{S}_n^{\pm} = G(2,1,n)$ is the (real) reflection group of all *signed permutation* matrices.

(a) Show the reflecting hyperplanes for W are

- $x_i = 0$ for $1 \leq i \leq n$
- $x_i = x_j$ for $1 \leq i < j \leq n$
- $x_i = -x_j$ for $1 \leq i < j \leq n$

One standard choice of *simple reflections* $S = \{s_1, \ldots, s_n\}$ making (W, S) a Coxeter system is to let s_i for $i = 1, 2, \ldots, n-1$ be the adjacent transposition $(i, i+1)$ as usual, and let s_n change the sign in the n^{th} coordinate, that is, reflecting through hyperplane $x_n = 0$. Their product is a Coxeter element

$$c = s_1 s_2 \cdots s_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & -1 \end{pmatrix}$$

in a two-line notation where $\begin{pmatrix} i \\ \pm j \end{pmatrix}$ means $w(e_i) = \pm e_j$.

(b) Show that c is a regular element directly, by exhibiting a c-eigenvector v that avoids all the reflecting hyperplanes in (a).

We wish to define the *cycle type* of an element w in $W = \mathfrak{S}_n^{\pm}$, which parametrizes its W-conjugacy class. First decompose $\{1, 2, \ldots, n\}$ into blocks called *cycles for* w by saying i, j lie in the same cycle of w if some power of w sends e_i to $\pm e_j$.

Then say a cycle of w is *even-sign* (resp. *odd-sign*) if the number of negative signs in the two-line notation for the cycle (i.e. the number of ordered pairs (i, j) in the cycle for which $w(+e_i) = -e_j$) is even (resp. odd). Lastly say that w has *cycle type* (λ_+, λ_-) where λ_+, λ_- are number partitions whose parts are the sizes of the even-, odd-sign cycles of w. In particular, $|\lambda_+| + |\lambda_-| = n$

(c) Show that cycle type exactly parametrizes W-conjugacy class.

(d) How do you recognize whether w in W is conjugate to a power of the Coxeter element c?

(e) Show the regular elements in W are exactly the W-conjugates of powers of c.

4. Noncrossing and nonnesting partitions

12*. (*Noncrossing partitions in type A_{n-1}* Let $W = \mathfrak{S}_n$ denote the symmetric group on n letters acting on $V = \mathbb{R}^n$ by permuting coordinates, generated by its set T of reflections, that is the set of all *transpositions* $t = (i\,j)$ for $1 \leq i < j \leq n$.

(a) For w a permutation in W, let $c(w)$ denote the number of cycles in the cycle decomposition of w, counting fixed points each as one cycle. Show that if $t = (i\,j)$ then

$$c(wt) = \begin{cases} c(w) - 1 & \text{if } i, j \text{ lie in different cycles of } w, \\ c(w) + 1 & \text{if } i, j \text{ lie in the same cycle of } w. \end{cases}$$

(b) Show that the *(absolute) length function*

$$\ell_T(w) := \min\{\ell : w = t_1 t_2 \cdots t_\ell \text{ for some } t_i \in T\}$$

has these equivalent reformulations:

$$\ell_T(w) = n - c(w) = \operatorname{codim}_{\mathbb{R}}(V^w)$$

where $V^w = \{v \in V = \mathbb{R}^n : w(v) = v\}$.

(c) Define a binary relation $<$ on W by taking the reflexive, transitive closure of the relation $w < wt$ for $w \in W, t \in T$ with $\ell_T(w) < \ell_T(wt)$. Show that $\leq$ is a partial order on W having the identity element e as its unique minimum element.

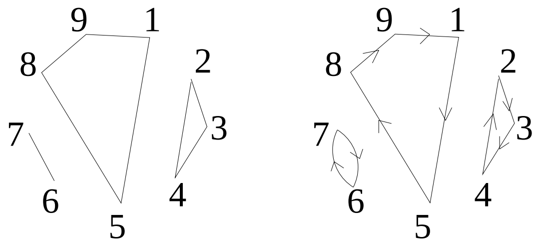
(d) Show that $u \leq v$ if and only if

$$\ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v).$$

Show that ℓ_T gives a *rank function* for $\leq$ on W in the sense that every maximal totally ordered subset (*chain*) in the interval from e to w has length $\ell_T(w)$.

Recall the definition of the poset of noncrossing partitions $NC(n)$. It is a subposet of the refinement order Π_n on all partitions of the n-element set $[n] := \{1, 2, \ldots, n\}$, and consists of those partitions whose blocks have disjoint convex hulls when $[n]$ labels (in clockwise order) the vertices of a convex n-gon.

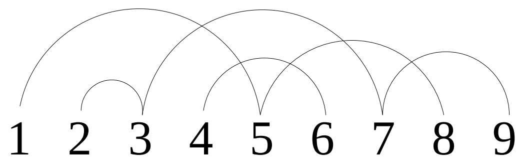
(e) Let c be the n-cycle $(123 \cdots n-1\,n)$ in W. Show that $w \leq c$ in the partial order on W if and only if the cycles of w form a noncrossing partition of $[n]$ in which each cycle is directed clockwise around the n-gon.

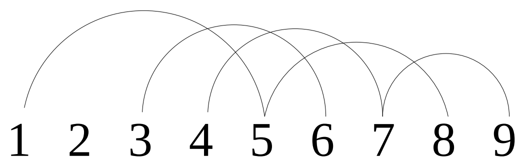


(f) Show that $NC(n) \cong [e, c]_<$ where $[e, c]_<$ denotes the interval from e to w in the partial order on W described above.

13^*.(*Nonnesting partitions in type* A_{n-1})
Recall the definition of the poset of nonnesting partitions $NN(n)$. Given a partition π of the set $\{1,2,\ldots,n\}$, call a *bump* of π a pair (i,j) in the same block of π with no integers k having $i<k<j$ in the same block. One can picture π by drawing $1,2,\ldots,n$ along a horizontal line, and drawing the bumps (i,j) of π as semicircular arcs above i,j. For example, the figure below depicts in this way two set partitions of $\{1,2,3,4,5,6,7,8,9\}$, namely

$$\pi_1 = \{\{1,5,8\},\{2,3,7,9\},\{4,6\}\} \text{ and,}$$
$$\pi_2 = \{\{1,5,8\},\{2\},\{3,6\},\{4,7,9\}\}.$$





Then $NN(n)$ is a subposet of the refinement order Π_n on all partitions of $[n]$, and consists of those partitions having no pair of bumps $(i,j)\neq(i',j')$ which are *nested*: $i\leq i'\leq j'\leq j$. For example, π_1 above is nesting, because its bumps $\{2,3\},\{1,5\}$ are nested, or because its bumps $\{4,6\},\{3,7\}$ are nested. However, π_2 is nonnesting, so an element of $NN(n)$.

Consider the usual crystallographic root system Φ of type A_{n-1}, along with one of its usual choices of positive roots Φ^+:

$$\Phi := \{e_i - e_j : 1\leq i\neq j\leq n\}$$
$$\Phi^+ := \{e_i - e_j : 1\leq i< j\leq n\}$$

Let $\mathbb{N}\Phi^+$ denote the set of all nonnegative integral combinations of the positive roots.

(a) Show that a partition π of $[n]$ is the transitive closure of the relation determined by its collection of bumps, and hence that π is uniquely determined by its bumps. Show that the collection of bumps (i,j) of a partition π always corresponds to a linearly independent set of roots e_i-e_j in Φ^+.

(b) Show that the relation on Φ^+ defined by

$$\alpha'\leq\alpha \text{ if } \alpha-\alpha'\in\mathbb{N}\Phi^+$$

defines a partial order on Φ^+. We will call this the *(positive) root poset*. Draw this poset for $n=2,3,4,5$.

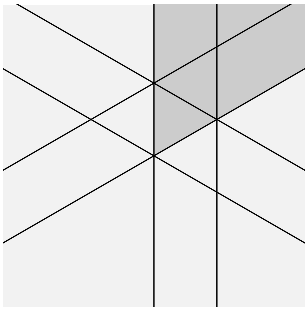


FIGURE 1. T
he Shi arrangement for $n = 3$ (or type A_2), with the positive cone shaded. Figure taken from [1].

(c) Show that two bumps $(i, j), (i', j')$ are nested if and only if their corresponding roots $\alpha = e_i - e_j, \alpha' = e_{i'} - e_{j'}$ satisfy $\alpha' < \alpha$.

Consequently, the map that sends a nonnesting partition π to the roots corresponding to its bumps gives a bijection between $NN(n)$ and the set of all *antichains* (= collections of pairwise incomparable elements) in the positive root poset.

(d) Biject $NN(n)$ to Dyck paths with number of bumps going to number of peaks? Corollary: Narayana numbers count $NN(n)$ by numbers of blocks or by bumps.

Recall that an *order filter* in a poset P is a subset $F \subseteq P$ with the property that $x \in F$ and $y > x$ in P implies $y \in F$.

(e) For any poset P show that the map sending an antichain A to the set $F := \{x \in P : x \geq a \text{ for some } a \in A\}$ gives a bijection between the antichains in P and the filters in P.

Consider the *Shi arrangement* of hyperplanes in $V = \mathbb{R}^n$, namely the hyperplanes of the form $\langle x, \alpha \rangle = 0, 1$ as α ranges through Φ^+. Removing these hyperplanes from V leaves open connected components called *regions*. This arrangement is depicted for $n = 3$ in Figure ??, after modding out by the 1-dimensional subspace $x_1 = \cdots x_n$ which is parallel to all of the hyperplanes. Here the *positive cone* containing the regions where $\langle x, \alpha \rangle > 0$ is shown shaded.

(f) Given a region R of the Shi arrangement lying in the positive cone, let F be the collection of positive roots $\alpha \in \Phi^+$ satisfying $\langle x, \alpha \rangle > 1$ for every $x \in R$. Show that F is a filter in the positive root order. Explain how this gives an injective map from the set of such regions to antichains in Φ^+ and nonnesting partitions. (This map turns out to be bijective, but this is not obvious.)

5. q-ANALOGUES AND CYCLIC SIEVING PHENOMENA

14*. (*q-binomial coefficient warm-up*) Recall the q-analogues

$$\begin{aligned} [n]_q &:= 1 + q + q^2 + \cdots + q^{n-1} \\ [n]!_q &:= [n]_q [n-1]_q \cdots [2]_q [1]_q \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{[n]!_q}{[k]!_q} [n-k]!_q \end{aligned}$$

(a) Prove the q-Pascal recurrences

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q . \end{aligned}$$

(b) Prove that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda} q^{|\lambda|}$$

as $\lambda = (n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 0)$ runs through all number partitions having at most k nonzero parts with each part of size at most $n-k$, or equivalently, the *Ferrers diagram* for λ fits inside a $k \times (n-k)$ rectangle.

(c) Prove that when q is a power of a prime, so the cardinality of a finite field $\mathbb{F}_q$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = |\{k\text{-dimensional } \mathbb{F}_q\text{-subspaces of } V = (\mathbb{F}_q)^n\}|.$$

(*Hint*: this can be done either using part (a) or part (b)).

15*.(*What happens to q-binomials when one plugs in roots of unity for q?*)
Let ζ be a primitive d^{th} root of unity in $\mathbb{C}$, such as $\zeta = e^{\frac{2\pi i}{d}}$.

(a) Show that when a, b are positive integers having $a \equiv b \bmod d$, one has

$$\lim_{q \to \zeta} \frac{[a]_q}{[b]_q} = \begin{cases} \frac{a}{b} & \text{if } a \equiv b \equiv 0 \bmod d \\ 1 & \text{if } a \equiv b \not\equiv 0 \bmod d. \end{cases}$$

(b) Prove that if n, k have quotients n', k' and remainders n'', k'' upon division by d, meaning that

$$\begin{aligned} n &= n'd + n'', \\ k &= k'd + k'', \end{aligned}$$

where $0 \leq n'', k'' \leq d-1$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\zeta} = \binom{n'}{k'} \cdot \begin{bmatrix} n'' \\ k'' \end{bmatrix}_{q=\zeta} .$$

16*. (*Cyclic sieving phenomena for subsets and multisets*)
Recall that

- a finite set X with
- an action of a cyclic group $C = \langle c \rangle \cong \mathbb{Z}/d\mathbb{Z}$ permuting X, and
- a polynomial $X(q)$ in $\mathbb{Z}[q]$

give a triple $(X, X(q), C)$ exhibiting the *cyclic sieving phenomenon (or CSP)* if all elements c^k in C have the size of their fixed point set $X^{c^k} = \{x \in X : c^k(x) = x\}$ predicted by a evaluating $X(q)$ at powers of a primitive d^{th} root-of-unity ζ:

$$|X^{c^k}| = [X(q)]_{q=\zeta^k}\,.$$

(a) Check directly that one has a CSP for the triple

$$\begin{aligned} X &= \{k\text{-element subsets of } \{1,2,\ldots,n\}\} \\ X(q) &= \begin{bmatrix} n \\ k \end{bmatrix}_q \\ C &= \langle c \rangle \cong \mathbb{Z}/n\mathbb{Z} \end{aligned}$$

where c is the n-cycle $c = (1, 2, 3, \ldots, n)$ inside $\mathfrak{S}_n$, acting via

$$c\{i_1, \ldots, i_k\} = \{i_1 + 1, \ldots, i_k + 1\} \bmod n.$$

(*Hint:* Problem 13 makes the root-of-unity evaluations easy, and it's not hard to analyze how many subsets are fixed by a power of c.)

(b) Similarly check directly that one has a CSP for the triple

$$\begin{aligned} X &= \{k\text{-element multisubsets of } \{1,2,\ldots,n\}\} \\ X(q) &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \\ C &= \langle c \rangle \cong \mathbb{Z}/n\mathbb{Z} \end{aligned}$$

where the n-cycle $c = (1, 2, 3, \ldots, n)$ acts similarly.

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[3] L. Solomon, Invariants of finite reflection groups. *Nagoya Math. J.* **22** (1963), 57–64.