# Reflection group counting and $q$-counting 

Vic Reiner<br>Univ. of Minnesota reiner@math.umn.edu

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## Outline

(1) Lecture 1

- Things we count
- What is a finite reflection group?
- Taxonomy of reflection groups
(2) Lecture 2
- Back to the Twelvefold Way
- Transitive actions and CSPs
(3) Lecture 3
- Multinomials, flags, and parabolic subgroups
- Fake degrees
(4) Lecture 4
- The Catalan and parking function family
(5) Bibliography


## Twelve-fold way

## Question

How many ways to place a set $N$ of $n$ balls (distinguishable or indistinguishable) into a set $X$ of $x$ boxes (distinguishable or indistinguishable)?

Equivalently, how many functions $N \xrightarrow{ } X$
with labelled/unlabelled source $N$ and target $X$ ?
What if we insist that $f$ be injective or surjective?

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$$
N \xrightarrow{f} X
$$

| balls N | boxes X | any f | injective f | surjective f |
| :---: | :---: | :---: | :---: | :---: |
| dist. | dist. | $x^{n}$ | $(x)(x-1)(x-2) \cdots(x-(n-1))$ | x ! S(n,x) |
| indist. | dist. | $\binom{x+n-1}{n}$ | $\binom{x}{n}$ | $\binom{n-1}{n-x}$ |
| dist. | indist. | $S(n, 1)$ <br> $+S(n, 2)$ <br> $+\cdots$ <br> $+S(n, x)$ | 1 if $n \leq x$ <br> 0 else | $\mathrm{S}(\mathrm{n}, \mathrm{x})$ |
| indist. | indist. | $p_{1}(n)$ <br> $+p_{2}(n)$ <br> $+\ldots$ <br> $+p_{x}(n)$ | 1 if $n \leq x$ <br> 0 else | $p_{x}(n)$ |

## Some other popular counts

- composition numbers $2^{n-1}$,
- multinomials $\binom{n}{k_{1}, k_{2}, \ldots, k_{n}}$,
- triangular numbers $\binom{n}{2}$,
- Stirling numbers of the 1 st kind $s(n, k)$,
- tableaux numbers $f^{\lambda}$,
- parking functions $(n+1)^{n-1}$,

Catalan numbers $\frac{1}{n+1}\binom{2 n}{n}$,
Kirkman-Cayley numbers $\frac{1}{k+1}\binom{n+k+1}{k}\binom{n-1}{k}$,
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## Motivation and goals

These numbers count various objects associated to the symmetric group $\mathfrak{S}_{n}$ on $n$ letters, often carrying natural $\mathfrak{S}_{n}$-actions.

Each generalizes naturally to other finite reflection groups, in particular, to Coxeter groups/systems $(W, S)$ with $W$ finite.

This often leads to $q$-analogues with good properties.

## Moral

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