Reflection group counting and q-counting

Vic Reiner
Univ. of Minnesota
reiner@math.umn.edu

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Outline

Twelve-fold way

Question

How many ways to place a set N of n balls (distinguishable or indistinguishable) into a set X of x boxes (distinguishable or indistinguishable)?

Equivalently, how many functions $N \xrightarrow{f} X$ with labelled/unlabelled source N and target X?

What if we insist that f be injective or surjective?

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Twelve-fold way

$$N \xrightarrow{f} X$$

balls N	boxes X	any f	injective f	surjective f
dist.	dist.	x^n	$(x)(x-1)(x-2)\cdots(x-(n-1))$	x! S(n,x)
indist.	dist.	$\binom{x+n-1}{n}$	$\binom{x}{n}$	$\binom{n-1}{n-x}$
dist.	indist.	$S(n,1)$ $+S(n,2)$ $+\cdots$ $+S(n,x)$	1 if $n \leq x$ 0 else	S(n,x)
indist.	indist.	$p_1(n)$ $+p_2(n)$ $+\cdots$ $+p_x(n)$	1 if $n \leq x$ 0 else	$p_x(n)$

Some other popular counts

- composition numbers 2^{n-1},
- multinomials $\binom{n}{k_1,k_2,\ldots,k_\ell}$,
- triangular numbers $\binom{n}{2}$,
- Stirling numbers of the 1st kind $s(n,k)$,
- tableaux numbers f^λ,
- parking functions $(n+1)^{n-1}$,
- Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$,
  Kirkman-Cayley numbers $\frac{1}{k+1}\binom{n+k+1}{k}\binom{n-1}{k}$,
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Motivation and goals

These numbers count various objects associated to the symmetric group $\mathfrak{S}_n$ on n letters, often carrying natural $\mathfrak{S}_n$-actions.

Each generalizes naturally to other finite reflection groups, in particular, to Coxeter groups/systems (W, S) with W finite.

This often leads to q-analogues with good properties.

Moral

Pay attention to Prof. Brenti's lectures!

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