Reflection group counting and q-counting

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Outline

Shephard and Todd's proof that finite reflection subgroups W of $GL_n(\mathbb{C})$ have S^W polynomial relied on

- easy reduction to the case where W acts irreducibly, and
- classification of the irreducibles.

Theorem (Shephard and Todd 1955)

The finite subgroups W of $GL_n(\mathbb{C})$ generated by reflections that act irreducibly are among

- *one infinite family: the monomial groups $G(de, e, n)$,and*
- *a list of 34 exceptional groups*

Let's discuss who is there a bit...

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The wreath products $G(d, 1, n) \cong \mathbb{Z}_d \wr \mathfrak{S}_n$

Definition

For integers $d, n \geq 1$, the group $G(d, 1, n)$ is the group of $n \times n$ monomial matrices having exactly one nonzero entry, a d^{th} root-of-unity, in each row and column.

$G(d, 1, n)$ is isomorphic to the wreath product $\mathbb{Z}_d \wr \mathfrak{S}_n$ of a cyclic group $\mathbb{Z}_d$ of order d with the symmetric group $\mathfrak{S}_n$.

Example

$$w = \begin{bmatrix} +i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -i & 0 \end{bmatrix}$$

is an element of $G(4, 1, 3)$

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All the infinite family: $G(de, e, n)$ inside $G(de, 1, n)$

The map $G(d, 1, n) \xrightarrow{\varphi} \mathbb{C}^\times$ sending w to its product of nonzero entries is a homomorphism, with image the d^{th} roots-of-unity.

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For integers $d, e, n \geq 1$, the group $G(de, e, n)$ is the subgroup of $G(de, 1, n)$ consisting of those elements w for which $\varphi(w)$ is not just a $(de)^{th}$ root-of-unity, but actually a d^{th} root of unity.

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$$w = \begin{bmatrix} +i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -i & 0 \end{bmatrix}$$

has $\varphi(w) = (+i)(-i)(-1) = -1$, a 2^{nd}-root-of-unity, and hence lies inside the subgroup $G(4, 2, 3)$ of $G(4, 1, 3)$.

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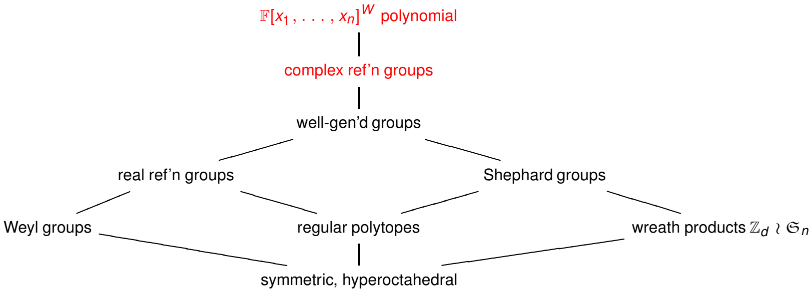
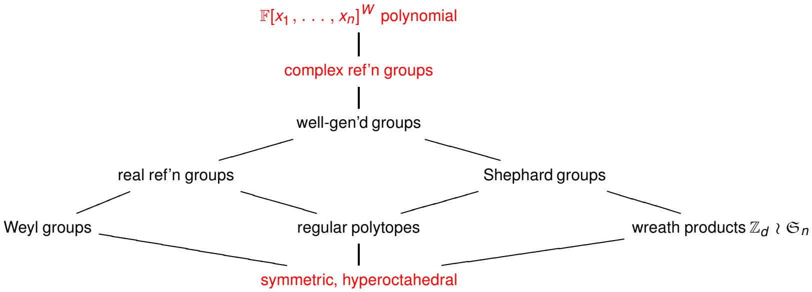
Taxonomy of reflection groups



Symmetric and hyperoctahedral groups



Symmetric and hyperoctahedral groups

We discussed them already, but

- the symmetric group $\mathfrak{S}_n = G(1,1,n)$,
- it is the Weyl group of type A_{n-1}, and
- the symmetry group of the regular $(n-1)$-simplex, while

- the hyperoctahedral group $\mathfrak{S}_n^{\pm} = G(2,1,n)$,
- is the Weyl group of type B_n or C_n, and
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  - the regular n-dimensional cross-polytope/hyperoctahedron,
  - and also of the regular n-cube.

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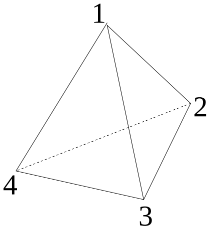
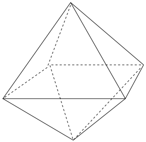
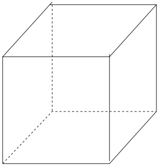
Simplices, cross-polytopes, cubes







Why are they reflection groups?

The transpositions (i,j) swapping coordinates x_i and x_j in $\mathbb{R}^n$ are orthogonal reflections through the hyperplane $x_i = x_j$.
These are the reflections in $\mathfrak{S}_n$, and they generate it.

The sign change $x_i \mapsto -x_i$ in coordinate i is an orthogonal reflection through the hyperplane $x_i = 0$.
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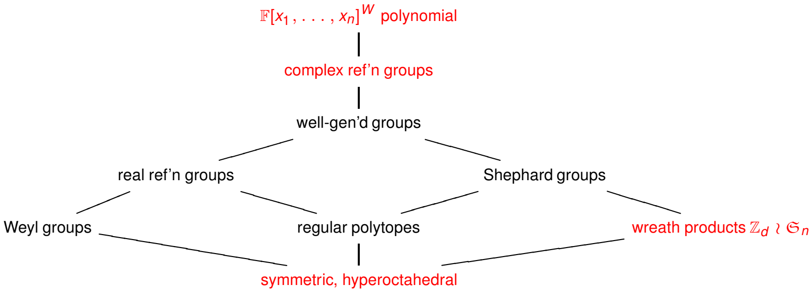
$\mathbb{F}[x_1, \ldots, x_n]^W$ polynomial
complex ref'n groups
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real ref'n groups
Shephard groups
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wreath products $\mathbb{Z}_d \wr \mathfrak{S}_n$
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Wreath products $\mathbb{Z}_d \wr \mathfrak{S}_n$

Although we already discussed them a bit,

- $\mathbb{Z}_d \wr \mathfrak{S}_n = G(d, 1, n)$,
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- it is the symmetry group of the n-dimensional regular complex polytopes(!) generalizing
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Complex polytopes!?

Think of an n-dimensional convex polytope P in $V = \mathbb{R}^n$ as the arrangement $\mathcal{P}$ of affine subspaces spanned by its *faces*.

- vertices $\leftrightarrow$ 0-dimensional subspaces
- edges $\leftrightarrow$ 1-dimensional subspaces
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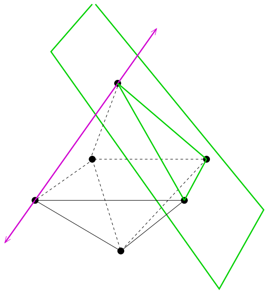

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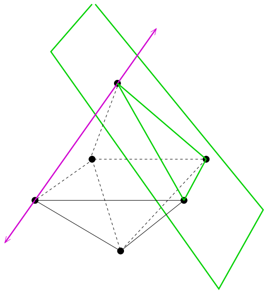

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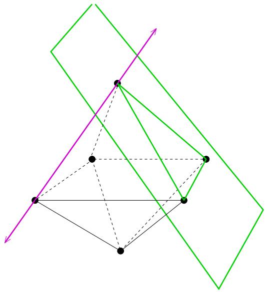

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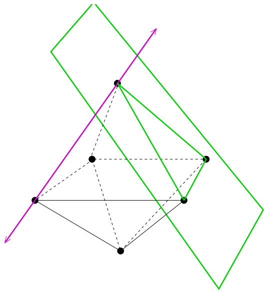

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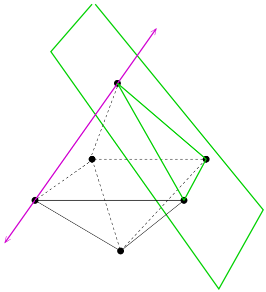

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A few of the properties of the arrangement $\mathcal{P}$ of face subspaces of a convex polytope P are captured in these polytope axioms:

1. The empty affine subspace $\varnothing$, and $V = \mathbb{R}^n$ are both in $\mathcal{P}$.
2. Nested subspaces $F \subset F''$ with $\dim F'' - \dim F \geq 2$ have the open interval
$$(F, F'') := \{F' \in \mathcal{P} : F \subsetneq F' \subsetneq F''\}$$
containing at least two intermediate subspaces.
3. Nested subspaces $F \subset F''$ of $\mathcal{P}$ with $\dim F'' - \dim F \geq 3$ have connected open interval (F, F''), considered as a poset under inclusion.

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Regular polytopes

One needs only one more axiom for such configurations $\mathcal{P}$ to capture the nature of regular polytopes:

- The linear symmetry group

$$W = \{w \in GL_n(\mathbb{R}) : w(\mathcal{P}) = \mathcal{P}\}$$

acts transitively on the collection of maximal flags

$$\varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq \mathbb{R}^n.$$

Theorem

An arrangement of affine subspaces $\mathcal{P}$ in $\mathbb{R}^n$ satisfying these four axioms has symmetry group W generated by reflections.

Proof.

See the exercises! □

In fact, such arrangements in $\mathbb{R}^n$ all come from real regular polytopes, and ...

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Regular complex polytopes

Theorem (Shephard 1952)

Any arrangement of affine subspaces $\mathcal{P}$ in $\mathbb{C}^n$ satisfying the same four axioms has symmetry group W generated by (unitary) reflections.

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Such $\mathcal{P}$ are called regular complex polytyopes,
their associated symmetry groups called Shephard groups.

They have a nice classification, and cover more than half of the exceptional complex reflection groups.

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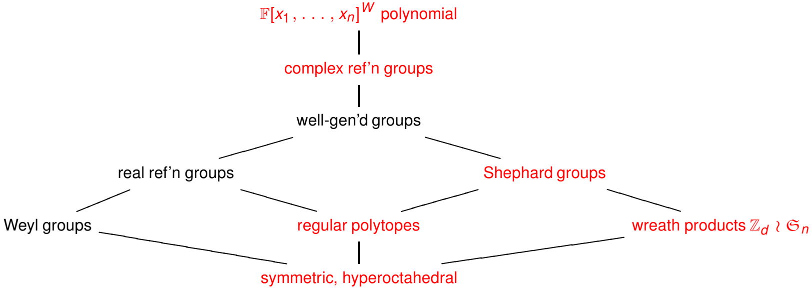

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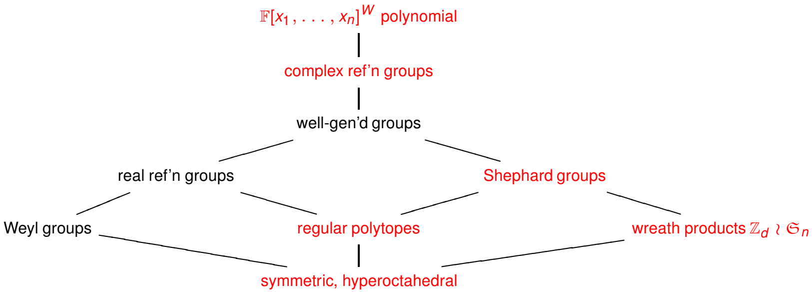

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Real reflection groups

Theorem (Coxeter 1934)

W a finite subgroup of $GL_n(\mathbb{R})$ is generated by (orthogonal) reflections if and only if it has a Coxeter presentation (W,S):

$$W = \langle S : (s_i s_j)^{m_{i,j}} = e \rangle$$

with $m_{i,j} \in \{2,3,\ldots\}$ and $m_{i,i} = 2$.

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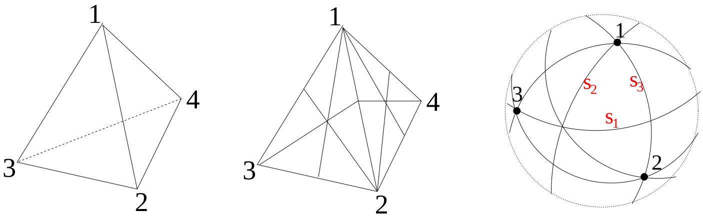
Example

For the symmetric group $W = \mathfrak{S}_n$,
acting (irreducibly) on $\mathbb{R}^{n-1} \cong \mathbb{R}^n / \mathbb{R}[1, 1, \ldots, 1]$,
one can take as simple reflections

$$S = \{s_1, \ldots, s_{n-1}\}$$

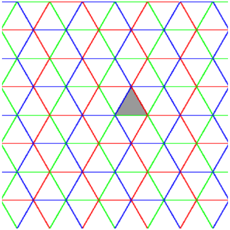
the adjacent transpositions s_i swapping $x_i \leftrightarrow x_{i+1}$.

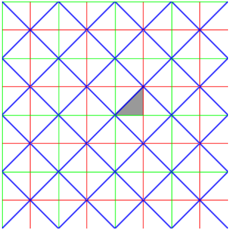


Weyl groups

Definition

The Weyl groups W are the finite crystallographic reflection groups in $GL_n(\mathbb{R})$: those that preserve a lattice, like $\mathbb{Z}^n$, inside $V = \mathbb{R}^n$.





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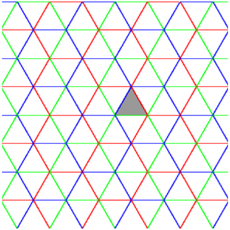

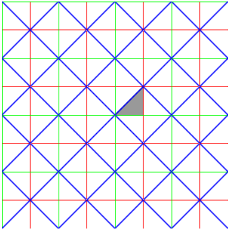

Weyl groups

Weyl groups are special because they come equipped with semisimple Lie groups/algebras, giving extra connections with

- representation theory of the Lie group/algebra,
- flag manifold, Grassmannian geometry/topology,
- affine reflection groups and Coxeter systems, and
- representation theory of
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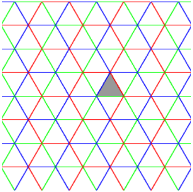

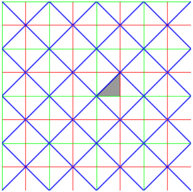

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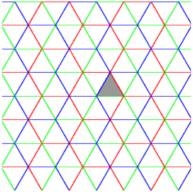

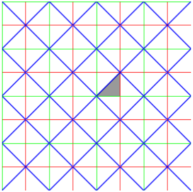

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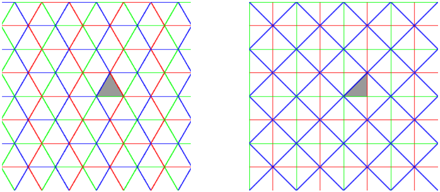

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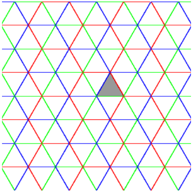

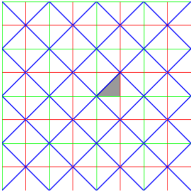

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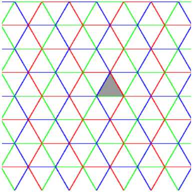
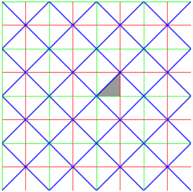
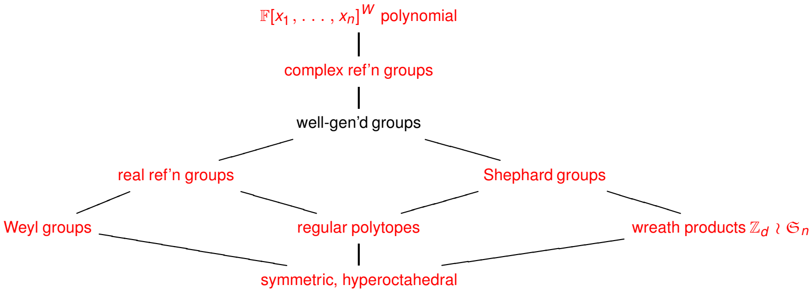




Only one left ...



Well-generated complex reflection groups

Definition

A complex reflection group W in $GL_n(\mathbb{C})$ is well-generated if it can be generated by n reflections.

Example

Real ref'n groups are well-gen'd, by their simple reflections S.

Example

Shephard groups are well-gen'd; see exercises.

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Who is well-generated among $G(de, e, n)$?

Example

$G(d, 1, n)$ is Shephard, hence well-gen'd.

Example

$G(e, e, n)$ is well-gen'd by $\{s_1, \ldots, s_{n-1}\}$ together with one extra reflection sending

$$\begin{array}{ccc} e_i & \longmapsto & \zeta e_j \\ e_j & \longmapsto & \zeta^{-1} e_i, \end{array}$$

that is, in the plane spanned by basis $\{e_1, e_2\}$, acting by

$$\begin{bmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{bmatrix}.$$

Who is not well-gen'd among $G(de, e, n)$?

Proposition

For $d, e, n \geq 2$, the group $G(de, e, n)$ is not well-generated.

Example

$G(4, 2, 2)$ acts on $V = \mathbb{C}^2$, but requires at least 3 elements to generate it, e.g. these 3:

$$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} +i & 0 \\ 0 & -i \end{bmatrix} \right\}.$$

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Well-generated groups are mysterious

Well-generated complex reflection groups seem very well-behaved from several viewpoints, for example, with regard to the Catalan combinatorics to be discussed later.

We do not really understand this!

Many beautiful facts with uniform statements have been checked to hold for the non-real well-generated groups only in a case-by-case fashion, using the Shephard-Todd classification.

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Review: the taxonomy