Reflection group counting and q-counting

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Outline

The middle column of the 12-fold way

balls N	boxes X	any f	injective f	surjective f
dist.	dist.	x^n	$(x)(x-1)(x-2)\cdots(x-(n-1))$	x! S(n,x)
indist.	dist.	$\binom{x+n-1}{n}$	$\binom{x}{n}$	$\binom{n-1}{n-x}$
dist.	indist.	$S(n,1)$ $+S(n,2)$ $+\cdots$ $+S(n,x)$	1 if $n \leq x$ 0 else	S(n,x)
indist.	indist.	$p_1(n)$ $+p_2(n)$ $+\cdots$ $+p_x(n)$	1 if $n \leq x$ 0 else	$p_x(n)$

The nontrivial entries both count sets with transitive $\mathfrak{S}_n$-action:

- $\binom{n}{k}$ counts k-subsets, and
- $n(n-1)(n-2)\cdots(n-(k-1))$ counts ordered k-subsets

taken from the n-set $\{1, 2, \ldots, n\}$.

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Their traditional q-analogues

Both have some traditional q-analogues,
part of the usual list, whose $q \to 1$ limits recover certain counts:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} \quad \stackrel{q\to 1}{\to} \quad n$$

$$[n]!_q := [n]_q[n-1]_q \cdots [2]_q[1]_q \quad \stackrel{q\to 1}{\to} \quad n!$$

$$[n]_q[n-1]_q \cdots [n-(k-1)]_q \quad \stackrel{q\to 1}{\to} \quad n(n-1)\cdots(n-(k-1))$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q} \quad \stackrel{q\to 1}{\to} \quad \binom{n}{k}$$

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They have more pleasant properties

All of these q-analogues

- are polynomials in q, lying in $\mathbb{Z}[q]$, and
- even have nonnegative coefficients, lying in $\mathbb{N}[q]$.

Example

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]!_q}{[2]!_q [2]!_q} = \frac{[4]_q [3]_q}{[2]_q [1]_q}$$
$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{1+q}$$
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And a cyclic sieving phenomenon

Consider a finite set X permuted by a cyclic group $C = \langle c \rangle = \{e, c, c^2, \ldots, c^{m-1}\} \cong \mathbb{Z}_m$, and a generating function $X(q)$ lying in $\mathbb{Z}[q]$.

Definition

Say $(X, X(q), C)$ exhibits the cyclic sieving phenomenon (CSP) if each c^d in C, has the cardinality of its fixed point set $X^{c^d} = \{x \in X : c^d(x) = x\}$ predicted by

$$|X^{c^d}| = [X(q)]_{q=\zeta^d}$$

where $\zeta = e^{\frac{2\pi i}{m}}$.

In other words, the m^{th} root-of-unity evaluations of $X(q)$ encode all the information about the C-orbit sizes on X

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The prototype CSP

Theorem (R.-Stanton-White 2004)

This triple $(X, X(q), C)$ exhibits the CSP:

$$X := k\text{-subsets of } \{1, 2, \ldots, n\}$$

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Proof.

Exercises do a brute force proof; we'll discuss a better one. □

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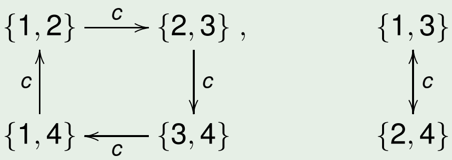

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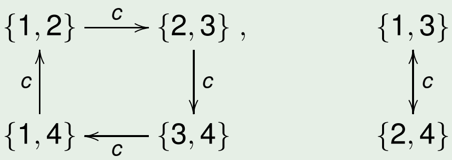

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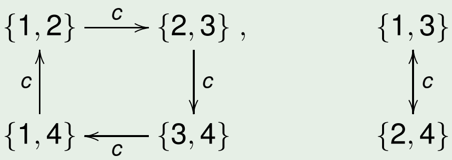

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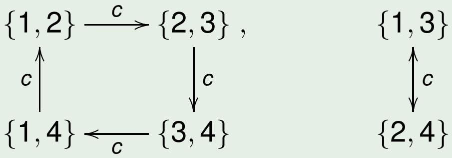

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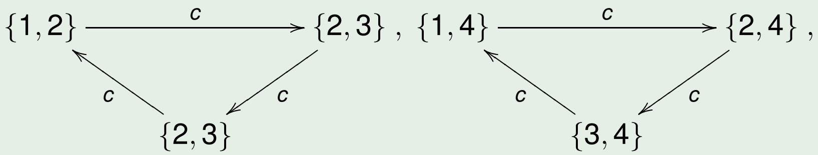

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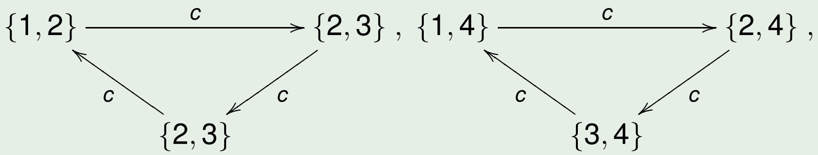

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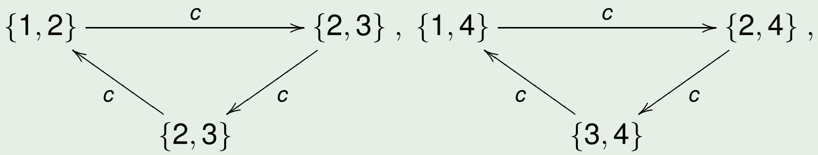

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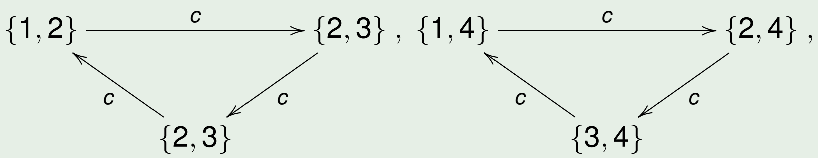

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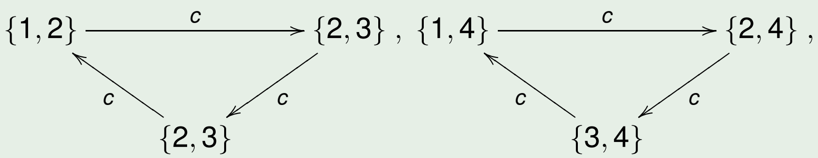

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Note that this CSP can fail for $C = \langle c \rangle$ with c in $\mathfrak{S}_n$ that are neither n-cycles nor $(n-1)$-cycles.

Example

$n = 4, k = 1$, and $C = \langle c \rangle = \mathbb{Z}_2$ with $c = (1,2)(3)(4)$:

$\{1\} \overset{c}{\longleftrightarrow} \{2\} \qquad \overset{c}{\circlearrowleft} \{3\} \qquad \overset{c}{\circlearrowleft} \{4\}$

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The same would work also for ordered k-subsets:

Theorem

This triple $(X, X(q), C)$ exhibits the CSP:

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But this one is not as interesting, and there is something much more general, due to the transitive $\mathfrak{S}_n$-action.

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Transitive actions are coset actions

Recall that for a group G permuting a set X transitively, the elements x_0 in X all have G-conjugate stabilizer subgroups

$$G_{x_0} := \{g \in G : g(x_0) = x_0\}.$$

Fixing some x_0 in X and defining $H := G_{x_0}$, the map

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g(x) \end{aligned}$$

induces a G-equivariant bijection

$$G/H \to X$$

Corollary

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A q-analogue of $[G : H]$

In particular, when G acts transitively on X, with H the stabilizer of x_0 in X, one has

$$|X| = [G : H].$$

But when G is a finite subgroup of $GL_n(\mathbb{F})$, we claim that there is also an appropriate q-analogue.

Recall that G acts via linear substitutions on

$$S = \mathbb{F}[x_1, \ldots, x_n]$$

with graded G-invariant subring S^G, having Hilbert series

$$\mathrm{Hilb}(S^G, q) := \sum_{d \geq 0} q^d \cdot \dim_{\mathbb{F}}(S^G)_d.$$

A q-analogue of $[G : H]$

In particular, when G acts transitively on X, with H the stabilizer of x_0 in X, one has

$$|X| = [G : H].$$

But when G is a finite subgroup of $GL_n(\mathbb{F})$, we claim that there is also an appropriate q-analogue.

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Definition

For a finite subgroup G of $GL_n(\mathbb{F})$ acting transitively on X, with H the stabilizer of x_0, define

$$X(q) := \frac{\mathrm{Hilb}(S^H, q)}{\mathrm{Hilb}(S^G, q)}.$$

Theorem

$X(q)$ is a rational function in q with no pole at $q = 1$, and

$$X(1) = [G : H].$$

Proof.

See exercises! □

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Unhappy that $X(q)$ is rational, not polynomial?

Theorem

When $S^G = \mathbb{F}[f_1, \ldots, f_n]$ is a polynomial algebra, then any subgroup H of G has $X(q) := \frac{\mathrm{Hilb}(S^H,q)}{\mathrm{Hilb}(S^G,q)}$

1. *a polynomial in q, that is, lying in $\mathbb{Z}[q]$, and*
2. *if furthermore $|G|$ is in $\mathbb{F}^\times$, then $X(q)$ lies in $\mathbb{N}[q]$.*

In particular, both hold when G is a complex reflection group.

Proof.

(Sketch)
The first assertion comes from Hilbert's syzygy theorem, saying S^H will have a finite free S^G-module resolution.

In the second case, S^H will be Cohen-Macaulay, and hence actually a free S^G-module. □

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And these $X(q)$ have a general CSP!

Definition

For G a complex reflection group in $GL_n(\mathbb{C})$, say that c in G is a regular element if there is some c-eigenvector v in $V = \mathbb{C}^n$ that avoids all reflecting hyperplanes for G.
(Equivalently, G permutes the eigenvector v freely.)

Theorem (R.-Stanton-White 2004)

In the above setting and for any subgroup H of G, the triple $(X, X(q), C)$ exhibits the CSP, where

- $X = G/H$
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Regular elements in $\mathfrak{S}_n$

How does this generalize the CSP's that we saw?
Firstly, who are the regular elements in $\mathfrak{S}_n$?

Let $\zeta_n := e^{\frac{2\pi i}{n}}$.

Example

An n-cycle $c = (1, 2, 3, \ldots, n)$ is a regular element, since it has an eigenvector $v = (1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})$ avoiding all reflection hyperplanes $x_i = x_j$.

Example

An $(n-1)$-cycle $c = (1, 2, 3, \ldots, n-1)(n)$ is a regular element, since it has an eigenvector $v = (1, \zeta_{n-1}, \zeta_{n-1}^2, \ldots, \zeta_{n-1}^{n-2}, 0)$. avoiding all $x_i = x_j$.

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Powers of regular elements are always regular.

Example

In $\mathfrak{S}_n$, the longest permutation $w_0 = (1, n)(2, n-1) \cdots$ is a power of an n-cycle (n even) or of an $(n-1)$-cycle (n is odd), hence always a regular element.

(In finite real reflection groups, the longest element w_0 is always a regular element.)

Theorem

All regular elements in $\mathfrak{S}_n$ are powers of n- and $(n-1)$-cycles.

Proof.

See the exercises! □

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k-subsets X of $\{1, 2, \ldots, n\}$, have transitive action of $G = \mathfrak{S}_n$, and $x_0 = \{1, 2, \ldots, k\}$ has $H = G_{x_0} = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$, the Young subgroup permuting $\{1, 2, \ldots, k\}, \{k+1, k+2, \ldots, n\}$.

$$S^G = \mathbb{C}[e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)]$$
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What is magical about regular elements?

So what made this general CSP work?

Theorem

For any subgroup H of G a complex reflection group, and any regular element c of G, the triple $(X, X(q), C)$ exhibits the CSP, where

- $X = G/H$
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It is a shadow of Springer's theory of regular elements, generalizing work of Shephard-Todd and Chevalley on the coinvariant algebra for G.

The coinvariant algebra

Definition

The coinvariant algebra is the quotient $S/(S_+^G)$ of S by the ideal generated by G-invariant elements S_+^G of positive degree.

Example

When $S^G = \mathbb{F}[f_1, \ldots, f_n]$, then $S_+^G = (f_1, \ldots, f_n)$ and $S/(S_+^G) = \mathbb{F}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

Example

In particular, the symmetric group $\mathfrak{S}_n$ has $S/(S_+^{\mathfrak{S}_n}) = \mathbb{F}[x_1, \ldots, x_n]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$.

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The coinvariant algebra

Theorem (Shephard-Todd 1955, Chevalley 1955)

For G a complex reflection group inside $GL_n(\mathbb{C})$, one has an isomorphism of G-representations

$$S/(S_+^G) \cong \mathbb{C}[G]$$

where G acts via

- *linear substitutions on $S/(S_+^G)$, and*
- *via the (left-)regular representation on $\mathbb{C}[G]$.*

Thus $S/(S_+^G)$ is a natural graded version of $\mathbb{C}[G]$.

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Springer enhanced this with a commuting cyclic action.
Given a regular element c in G, with eigenvector v avoiding the reflecting hyperplanes, let $c(v) = \zeta \cdot v$ for some ζ in $\mathbb{C}^\times$.

Theorem (Springer 1974)

For G any complex reflection group and letting C be the cyclic group generated by any regular element c, one has an isomorphism of $G \times C$-representations

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Taking H-fixed subspaces in Springer's $G \times C$-isomorphism

$$S/(S^G_+) \cong \mathbb{C}[G]$$

it becomes an isomorphism of C-representations:

$$\begin{array}{ccc} (S/(S^G_+))^H & \cong & (\mathbb{C}[G])^H \\ \| & & \| \\ S^H/(S^G_+) & & \mathbb{C}[G/H] \end{array}$$

Any c^d in C acts with same trace on the two extreme ends:

- One can show the trace on the left side $S^H/(S^G_+)$ is $X(\zeta^d)$ where $X(q) = \frac{\mathrm{Hilb}(S^H,q)}{\mathrm{Hilb}(S^G,q)}$, and
- the trace on the right size is $|X^{c^d}|$ where $X = G/H$.

This proves that general CSP theorem.

How does this help?

Taking H-fixed subspaces in Springer's $G \times C$-isomorphism

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- One can show the trace on the left side $S^H/(S^G_+)$ is $X(\zeta^d)$ where $X(q) = \frac{\mathrm{Hilb}(S^H,q)}{\mathrm{Hilb}(S^G,q)}$, and
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This proves that general CSP theorem.

How does this help?

Taking H-fixed subspaces in Springer's $G \times C$-isomorphism

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