Reflection group counting and q-counting

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Outline

1. Lecture 1
   - Things we count
   - What is a finite reflection group?
   - Taxonomy of reflection groups
2. Lecture 2
   - Back to the Twelvefold Way
   - Transitive actions and CSPs
3. Lecture 3
   - Multinomials, flags, and parabolic subgroups
   - Fake degrees
4. Lecture 4
   - The Catalan and parking function family
5. Bibliography

Multinomials and flags of subsets

The examples of

- k-subsets counted by $\binom{n}{k}$, and
- ordered k-subsets counted by $n(n-1)\cdots(n-(k-1))$,

are special cases of objects parametrized by a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n.

Definition

An α-flag (of subsets) is a chain of nested subsets of $\{1, 2, \ldots, n\}$

$$\varnothing \subset S_{\alpha_1} \subset S_{\alpha_1+\alpha_2} \subset \cdots \subset S_{\alpha_1+\cdots+\alpha_{\ell-1}} \subset \{1, 2, \ldots, n\}$$

in which each subset S_j has cardinality given by its subscript j.

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$\mathfrak{S}_n$ acts transitively, so here we go ...

Proposition

The α-flags are counted by the multinomial coefficient

$$\binom{n}{\alpha} = \binom{n}{\alpha_1, \ldots, \alpha_\ell} = \frac{n!}{\alpha_1! \cdots \alpha_\ell!}.$$

Proof.

They carry a transitive $\mathfrak{S}_n$-action, with the stabilizer of one particular flag conjugate to the Young subgroup

$$\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_\ell}.$$

□

Their invariant ring, and q-analogue

Here one has

$$S^{\mathfrak{S}_\alpha} = \mathbb{C}\left[e_1(\mathbf{x}^{(1)}), \ldots, e_{\alpha_1}(\mathbf{x}^{(1)}), \quad \cdots, \quad e_1(\mathbf{x}^{(\ell)}), \ldots, e_{\alpha_\ell}(\mathbf{x}^{(\ell)})\right]$$

where $\mathbf{x}^{(i)}$ is the variable set

$$\{x_{\alpha_1+\cdots+\alpha_{i-1}+1}, \; x_{\alpha_1+\cdots+\alpha_{i-1}+2}, \; \ldots \; x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\}$$

and hence

$$\begin{aligned} X(q) &= \frac{\mathrm{Hilb}(S^{\mathfrak{S}_\alpha}, q)}{\mathrm{Hilb}(S^{\mathfrak{S}_n}, q)} \\ &= \frac{1/((1-q)\cdots(1-q^{\alpha_1})\cdots(1-q)\cdots(1-q^{\alpha_\ell}))}{1/((1-q)\cdots(1-q^n))} \\ &= \frac{[n]!_q}{[\alpha_1]!_q \cdots [\alpha_\ell]!_q} =: \begin{bmatrix} n \\ \alpha \end{bmatrix}_q \end{aligned}$$

the traditional q-multinomial.

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The usual CSP

Corollary

One has a triple $(X, X(q), C)$ *giving a CSP where*

$$X = \alpha\text{-flags of subsets of } \{1, 2, \ldots, n\}$$
$$X(q) := \begin{bmatrix} n \\ \alpha \end{bmatrix}_q$$
$$C := \langle c \rangle$$

where c *is an* n*-cycle or* $(n-1)$*-cycle in* $\mathfrak{S}_n$*.*

But q-multinomials have further meanings

$W = \mathfrak{S}_n$ as a Coxeter group, has usual length function $\ell(w) = \ell_S(w)$ with respect to Coxeter generators S, and

$$\sum_{w \in W} q^{\ell(w)} = [n]!_q.$$

Similarly $W_J = \mathfrak{S}_\alpha$ is a Coxeter group in its own right, a parabolic subgroup, inheriting the same length function, with

$$\sum_{w \in W_J} q^{\ell(w)} = [\alpha_1]!_q \cdots [\alpha_\ell]!_q.$$

The theory says the minimum length coset representatives W^J for W/W_J will have

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Geometric meaning

Definition

For any field $\mathbb{F}$ and composition α of n, one can consider the partial flag variety of all α-flags of $\mathbb{F}$-subspaces in $\mathbb{F}^n$

$$\{0\} \subset V_{\alpha_1} \subset V_{\alpha_1+\alpha_2} \subset \cdots \subset V_{\alpha_1+\cdots+\alpha_{\ell-1}} \subset \mathbb{F}^n$$

Alternatively, it is the homogeneous space G/P_α where $G = GL_n(\mathbb{F})$ and P_α is the block-triangular matrix subgroup fixing a standard α-flag where $V_i = \mathbb{F}e_1 + \mathbb{F}e_2 + \cdots + \mathbb{F}e_i$.

G/P_α turns out to be a

- smooth projective variety, with
- Schubert cell decomposition X_w, indexed by w in W^J:

$$G/P_\alpha = \bigsqcup_{w \in W^J} X_w$$

- and the Schubert cell $X_w = BwP_\alpha$ isomorphic to $\mathbb{F}^{\ell(w)}$

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This lets one prove these classical facts about $X(q) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q$.

Theorem

- *When* $\mathbb{F} = \mathbb{F}_q$, *one has* $X(q) = |G/P_\alpha|$.
- *When* $\mathbb{F} = \mathbb{R}$, *one has*

$$X(q) = \mathrm{Poin}_{\mathbb{Z}_2}(G/P_\alpha, q) := \sum_{i \geq 0} q^i \cdot \dim_{\mathbb{Z}_2} H_i(G/P_\alpha; \mathbb{Z}_2).$$

- *When* $\mathbb{F} = \mathbb{C}$, *one has*

$$X(q) = \mathrm{Poin}_{\mathbb{Z}}(G/P_\alpha, q^{\frac{1}{2}}) := \sum_{i \geq 0} q^i \cdot \mathrm{rank}_{\mathbb{Z}} H_{2i}(G/P_\alpha; \mathbb{Z}).$$

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Compositions and flags for reflection groups

We've seen the Boolean algebra 2^{n-1}
of all compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n generalizes from
$W = \mathfrak{S}_n$ to real reflection groups W with simple generators S:

2^{n-1} generalizes to the Boolean algebra 2^S,
with α corresponding to the subset $J \subseteq S$ generating $W_J = \mathfrak{S}_\alpha$

Example

For $W = \mathfrak{S}_9$ with $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, the composition $\alpha = (2, 4, 3)$ corresponds to the subset $J = \{s_1, \ s_3, s_4, s_5, \ s_7, s_8\}$ generating $W_J = \mathfrak{S}_2 \times \mathfrak{S}_4 \times \mathfrak{S}_3$.

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The real reflection group W has degrees $(d_1, \ldots, d_n)$.
The parabolic subgroup W_J has its own degrees $(d_1^J, \ldots, d_n^J)$.

One has q-analogues of $|W|, |W^J|, [W : W_J] = |W^J|$ as before:

- $\sum_{w \in W} q^{\ell(w)} = [d_1]_q [d_2]_q \cdots [d_n]_q$
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Parabolic subgroups and quotients

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Of course, the same CSP

Corollary

In the above setting, one has a CSP triple $(X, X(q), C)$ *with*

$$X = W/W_J$$

$$X(q) = \frac{\mathrm{Hilb}(S^{W_J}, q)}{\mathrm{Hilb}(S^W, q)} = \sum_{w \in W^J} q^{\ell(w)} = \prod_{i=1}^{n} \frac{[d_i]_q}{[d_i^J]_q}$$

$$C = \langle c \rangle$$

where c is any regular element of W,
and C *acts on* W/W_J *via left-translation of cosets* wW_J.

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$$X(q) = \frac{\mathrm{Hilb}(S^{W_J}, q)}{\mathrm{Hilb}(S^{W}, q)} = \sum_{w \in W^J} q^{\ell(w)} = \prod_{i=1}^{n} \frac{[d_i]_q}{[d_i^J]_q}$$

$$C = \langle c \rangle$$

where c is any regular element of W,
and C acts on W/W_J via left-translation of cosets wW_J.

Generalized partial flag varieties for Weyl groups

Furthermore, if W is also a Weyl group, then there is an associated semisimple algebraic group G over any field $\mathbb{F}$.

Corresponding to the subset $J \subseteq S$, one has a parabolic subgroup P_J of G, playing the role of P_α.

One again has the generalized partial flag variety G/P_J, which is smooth, projective, with a Schubert cell decomposition into cells $X_w \cong \mathbb{F}^{\ell(w)}$ indexed by w in W^J.

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The same geometric interpretations

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Theorem

- *When* $\mathbb{F} = \mathbb{F}_q$, *one has* $X(q) = |G/P_J|$.
- *When* $\mathbb{F} = \mathbb{R}$, *one has* $X(q) = \mathrm{Poin}_{\mathbb{Z}_2}(G/P_J, q) := \sum_{i \geq 0} q^i \cdot \dim_{\mathbb{Z}_2} H_i(G/P_J; \mathbb{Z}_2)$.
- *When* $\mathbb{F} = \mathbb{C}$, *one has* $X(q) = \mathrm{Poin}_{\mathbb{Z}}(G/P_J, q^{\frac{1}{2}}) := \sum_{i \geq 0} q^i \cdot \mathrm{rank}_{\mathbb{Z}} H_{2i}(G/P_J; \mathbb{Z})$.

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CSPs for involutions are "$q = -1$ phenomena"

Definition (Stembridge 1994)

Suppose one has a CSP triple $(X, X(q), C)$ has the cyclic group $C = \mathbb{Z}_2 = \langle \tau \rangle$ of order two.

In other words, one has τ is an involution on X, and

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A $q = -1$ phenomenon involving partitions

Definition

Say that a number partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_\ell > 0)$ fits in a $k \times (n-k)$ rectangle if $\lambda_1 \leq n-k$ and $\ell \leq k$.

Example

$\lambda = 553 = 5530$ fits in a 4×5 rectangle:

5	□	□	□	□	□
5	□	□	□	□	□
3	□	□	□	·	·
0	·	·	·	·	·

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Theorem

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\lambda \text{ fitting in a} \\ k\times(n-k) \text{ rectangle}}} q^{|\lambda|}.$$

Proof.

See the exercises. ☐

Example

$$q^4 + q^3 + 2q^2 + q^1 + q^0 = 1 + q + 2q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$$

The q-binomial q-counts partitions in a rectangle

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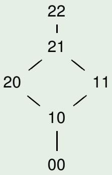

q^4

$+q^3$

$+2q^2$

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An involution on partitions in a rectangle

Having fixed the dimensions $k \times (n-k)$ of the rectangle, define an involution τ on all such λ fitting in it, by rotating the picture 180°, and taking the complementary boxes.

Example

Having fixed $k \times (n-k)$ as 4×5, one has $\tau(553) = 52$:

5	□	□	□	□	□		5	□	□	□	□	□
5	□	□	□	□	□	$\longleftrightarrow$	2	□	□	.	.	.
3	□	□	□	.	.			.	.	.	.	.
	.	.	.	.	.			.	.	.	.	.

How many such λ are fixed by the involution τ?

The phenomenon

Theorem (Stembridge 1994)

The involution τ on the set X of λ in a $k \times (n-k)$ rectangle, with $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$, exhibits a $q = -1$ phenomenon.

(He really proved something more general for plane partitions.)

Example

E.g. for $n = 4, k = 2$,

$X = \left\{ 22 \overset{\tau}{\leftrightarrow} 00, \quad 21 \overset{\tau}{\leftrightarrow} 10, \quad \circlearrowleft^{\tau} 20 \quad \circlearrowleft^{\tau} 11 \right\}$ has

$X^\tau = \{20, 11\}$,

$$\text{and } X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

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Eng's generalization

Stembridge's result is a special case, for $W = \mathfrak{S}_n$ and $W_J = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$, of the following.

Let W be a finite real reflection group W, simple reflections S, longest element w_0, and pick any subset $J \subseteq S$.

Theorem (O. Eng 2001)

The involution τ on $X = W/W_J$ defined by $\tau(wW_J) := w_0 wW_J$, with $X(q) = \sum_{w \in W^J} q^{\ell(w)}$, gives a $q = -1$ phenomenon.

Proof.

(Not Eng's) It follows from our general CSP: $X(q)$ is our usual for $X = W/W_J$, and the longest element w_0 is regular. □