P-partitions revisited

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North Carolina State University
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Outline

1. (q-)counting linear extensions
2. Complete interesction posets
3. A product formula
4. Some context
5. Revisiting the ring of P-partitions

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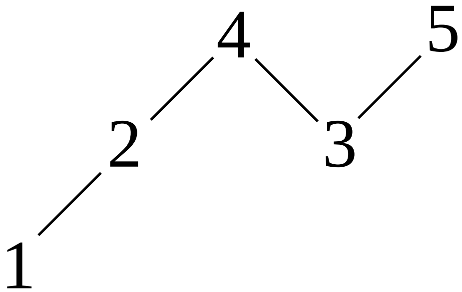
Posets

A poset (partially ordered set) P on labels $\{1, 2, \ldots, n\}$
is naturally labelled if $i <_P j$ implies $i <_{\mathbb{Z}} j$.

Example. For $n = 5$, our favorite poset P will be



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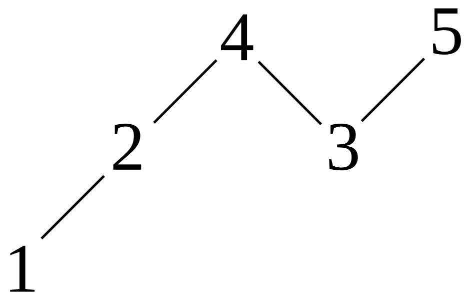
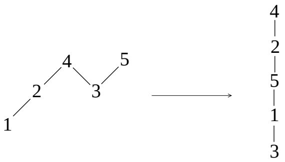


Linear extensions

A linear extension of P is a total order $w_1 <_w w_2 <_w \cdots <_w w_n$ that is stronger than P, that is, $i <_P j$ implies $i <_w j$.

The set of all linear extensions of P is denoted $\mathcal{L}(P)$.



Example. Our favorite P has

$$\mathcal{L}(P) = \left\{ \begin{matrix} 12345, & 13245, & 31245, \\ 12354, & 13254, & 31254, \\ 13524, & 31524, & 35124 \end{matrix} \right\}$$

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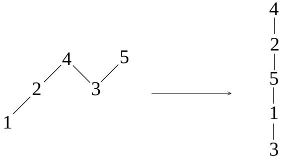

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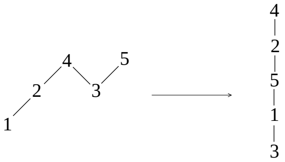

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$(q$-)counting

In general, $|\mathcal{L}(P)|$ is hard to count, or q-count by various statistics, such as

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)}$$

where the major index

$$\mathrm{maj}(w) := \sum_{i: w_i > w_{i+1}} i.$$

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An example q-count

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$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)} = q^0 + q^1 + 2q^2 + q^3 + 2q^4 + q^5 + q^6$$

$$= (1 + q + q^2)(1 + q^2 + q^4)$$

$$= [3]_q [3]_{q^2}$$

where $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$

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CI-posets

Such factorizations will occur for a class of posets that we call complete intersection (or CI) posets, defined here

- first in terms of their connected order ideals,
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Connected order ideals

An order ideal J in P is a down-set:
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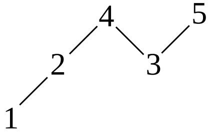
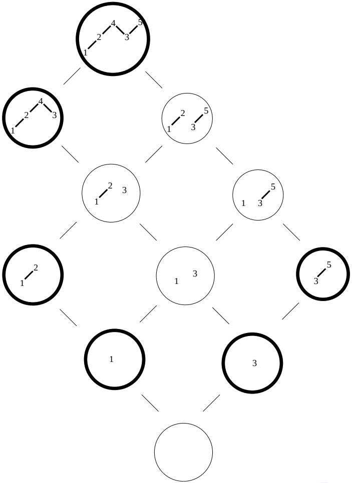
Example, with connected ideals darkly circled





Principal and nearly principal ideals

An obvious subclass of the connected order ideals are the principal ideals $P_{\leq x} = \{i \in P : i <_P x\}$.

An important disjoint subclass for us are the nearly principal ideals J, defined by

- J is connected, and
- $J = J_1 \cup J_2$ with J_1, J_2 connected ideals having $J_i \subsetneq J$, and
- this expression $J = J_1 \cup J_2$ is unique

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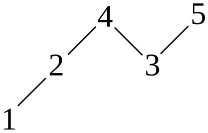
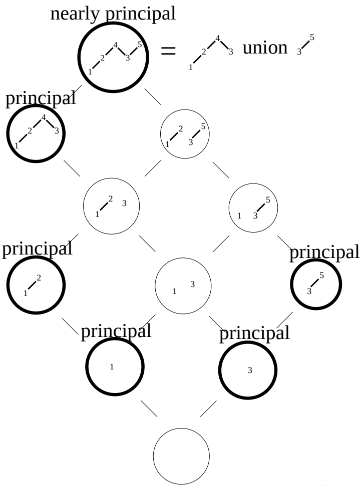
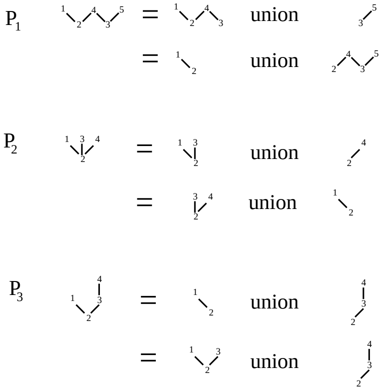
Our favorite example is CI





The three minimal non-CI examples

These P_1, P_2, P_3 are not CI, and are the minimal obstructions to being CI, as induced subposets.



Factorization theorem

Theorem. (Féray-R.)
Naturally labelled CI-posets P on $\{1, 2, \ldots, n\}$ have

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)} = [n]!_q \cdot \frac{\prod_{\{J_1, J_2\}} [\ |J_1| + |J_2|\ ]_q}{\prod_J [\ |J|\ ]_q}$$

where

- $[n]!_q := [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q$,
- the denominator runs over connected order ideals J, while
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Our favorite example...

... has these connected ideals

ideal	$\{1\}$	$\{3\}$	$\{1,2\}$	$\{3,5\}$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$
size	1	1	2	2	4	5

and only one (unordered) pair intersecting nontrivially, namely

$$\{J_1 = \{3,5\} \quad , \quad J_2 = \{1,2,3,4\}\}$$

$$|J_1| + |J_2| = 2 + 4 = 6.$$

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The theorem therefore asserts that

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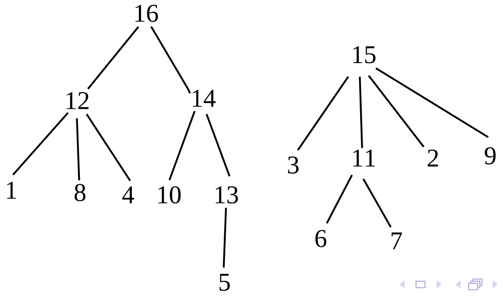
Special case: forest posets

A special case of the factorization theorem occurs when the poset CI-poset P has every connected ideal principal, so none are nearly principal.

Then P is a forest poset in the sense that every element is covered by at most one other element.



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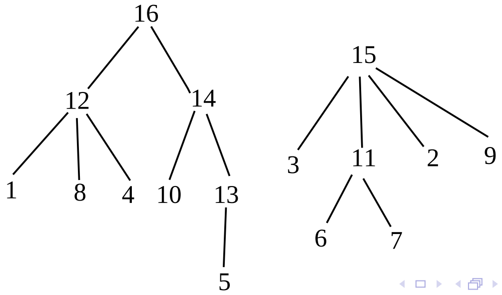


Special case: The maj q-hook-formula for forests

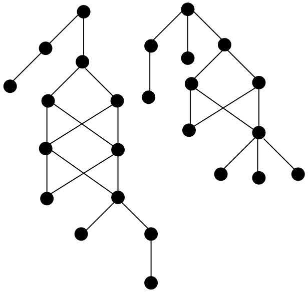
Theorem. (Knuth 1973 for $q = 1$, Björner and Wachs 1989)

Naturally labelled forest posets P on $\{1, 2, \ldots, n\}$ have

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)} = \frac{[n]!_q}{\prod_{i=1}^{n} \left[\, |P_{\leq i}| \,\right]_q}.$$

A typical CI poset

Still, one might ask “How special are CI-posets?”
Here’s a typical-looking one:



Characterizations of CI posets

Theorem. T.F.A.E. for a poset P:

- P is CI, that is, every connected order ideal is either principal or nearly principal.
- P avoids P_1, P_2, P_3 as induced subposets.
- P is the smallest class of posets containing the one-element poset and closed under 3 operations: *disjoint union*, *hanging*, and *twinning*.
- The P-partition affine semigroup ring has a complete intersection presentation ...

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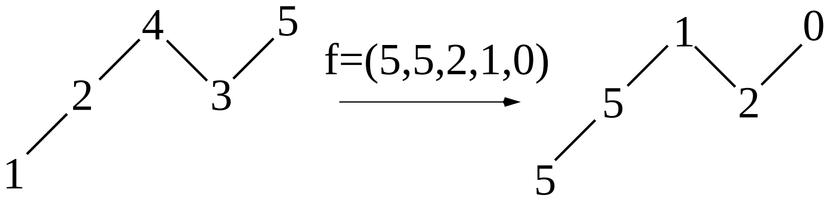
P-partition review

It is time to revisit the rings behind Richard Stanley's (1971) concept of P-partitions for a naturally-labelled poset P on $\{1, 2, \ldots, n\}$.

These are functions $f : P \to \mathbf{N}$ which are (weakly) order-reversing: if $i <_P j$ then $f(i) \geq_{\mathbf{N}} f(j)$.



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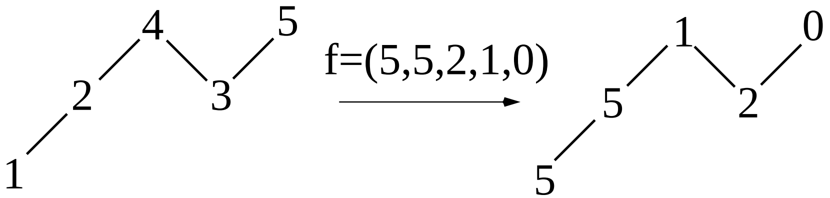

An affine semigroup ring

These P-partitions are the lattice points in a convex polyhedral cone of dimension n:

- $f(i) \geq 0$ for $i = 1, 2, \ldots, n$, and
- $f(i) \geq f(j)$ for $i <_P j$.

Thus the sum $f_1 + f_2$ of two P-partitions f_1, f_2 is another; they are a (finitely generated, cancellative) semigroup under addition.

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Making P-partitions f correspond to monomials x^f

$$f = (5,5,2,1,0) \quad \leftrightarrow \quad x^f = x_1^5 x_2^5 x_3^2 x_4^1 x_5^0$$

they form a k-basis for an affine semigroup ring

$$\begin{aligned} R_P &:= k - \text{span of} \{x^f : f \text{ a } P - \text{partition}\} \\ &\subseteq k[x_1, \ldots, x_n]. \end{aligned}$$

This ring was studied a bit by Adriano Garsia around 1980.

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Why major index ?

Introduce a standard grading on R_P where $\deg(x_i) = 1$.

Stanley's Basic lemma on P-partitions gives
a unimodular triangulation of the polyhedral cone,
with maximal cones indexed by $\mathcal{L}(P)$,
and the following easy Hilbert series computation:

$$\mathrm{Hilb}(R_P, q) = \sum_f q^{f(1)+\cdots+f(n)} = \frac{\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

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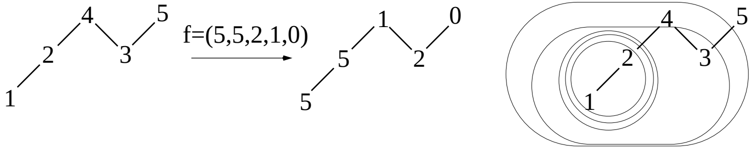
Generators for R_P

It's easy to see that R_P is generated by the monomials $x_J := \prod_{i \in J} x_i$ as one runs through the order ideals J of P, e.g.

$$x^f = x_1^5 x_2^5 x_3^2 x_4^1 x_5^0$$
$$= (x_1x_2x_3x_4)(x_1x_2x_3)(x_1x_2)^3.$$



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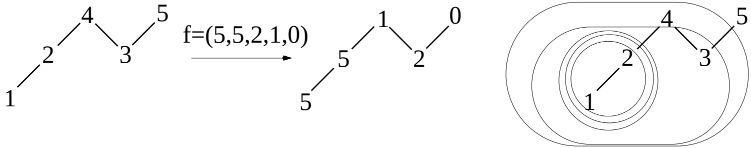

Why connected order ideals?

It's also easy to see that monomials x_J for disconnected ideals J give redundant generators, e.g.

$$x_1 x_2 x_3 = x_1 x_2 \cdot x_3$$

Proposition. (Boussicault-Féray-Lascoux-R.)

- Extreme rays of the P-partition cone are the connected ordered ideals J of P, and
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Minimal presentation for R_P

Introducing indeterminates U_J for the connected ideals J, one has a surjection $k[U_J] \to R_P$ sending $U_J \mapsto x_J$.
Its kernel is often called the toric ideal I_P.

Theorem. (Féray-R.)
The presentation $R_P \cong k[U_J]/I_P$, has the toric ideal I_P minimally generated by the binomials

$$U_{J_1} U_{J_2} - U_{J_1 \cup J_2} \cdot \prod_i U_{J^{(i)}}$$

where

- J_1, J_2 are connected order ideals that intersect nontrivially: $\varnothing \subsetneq J_1 \cap J_2 \subsetneq J_1, J_2$, and
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The running example

Example. Our favorite example has

$$\begin{array}{rllllll} R_P & = k[x_1, & x_3, & x_1x_2, & x_3x_5, & x_1x_2x_3x_4, & x_1x_2x_3x_4x_5] \\ & \cong k[U_1, & U_3, & U_{12}, & U_{35}, & U_{1234}, & U_{12345}] \quad / \; I_P \end{array}$$

where I_P is the (principal) ideal generated by the element

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in degree $2 + 4 = 6$.
Consequently,

$$\mathrm{Hilb}(R_P, q) = \frac{1 - q^6}{(1-q)(1-q)(1-q^2)(1-q^2)(1-q^4)(1-q^5)}$$

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Complete intersections

The same trick works just as well whenever $R_P \cong k[U_J]/I_P$ is a complete intersection presentation,
that is,

- the Krull dimension n for R_P, and
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These generators form a Gröbner basis for the toric ideal with respect to certain term orders.

- This corresponds to a new (non-unimodular) triangulation of the P-partition cone.
- It shows that a certain associated graded ring is Koszul.

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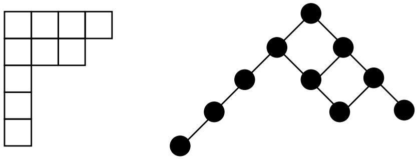

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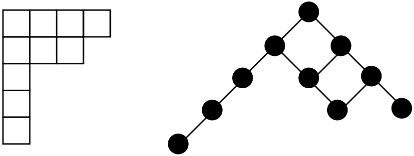

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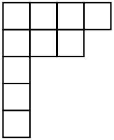

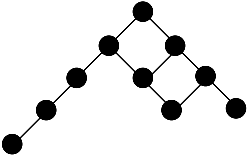

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Thanks for listening!