q-Ehrhart Theory

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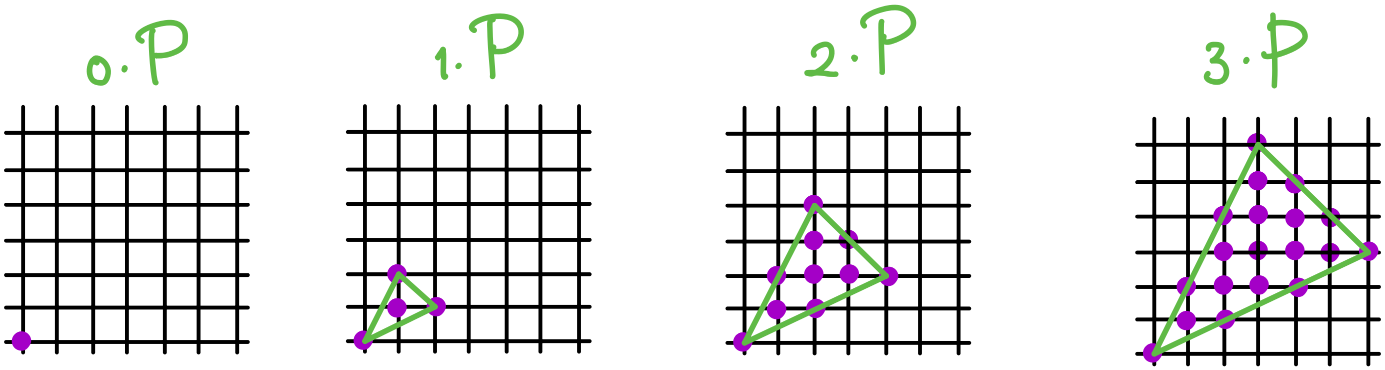


$$1 + 4t^1 + 10t^2 + 19t^3 + \dots = E_P(t)$$

$$:= \sum_{m=0}^{\infty} t^m \cdot \#\left(\mathbb{Z}^n \cap mP\right)$$

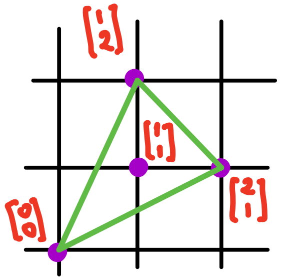
$$= \frac{1+t+t^2}{(1-t)^3}$$

q-analogue

$$1 + (1+2q+q^2)t^1 + \begin{pmatrix} 1+2q+3q^2 \\ +3q^3+q^4 \end{pmatrix} t^2 + \begin{pmatrix} 1+2q+3q^2+4q^3 \\ +5q^4+3q^5+q^6 \end{pmatrix} t^3 + \dots = E_P(q,t)$$

$$= \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

q-analogue of $\#\mathbb{Z}^n \cap mP$ comes from deformation ...



$Z := \mathbb{Z}^n \cap mP$

coordinate ring

$$\rightsquigarrow \quad \mathbb{R}[Z] = \mathbb{R}[x,y] \big/ I(Z)$$

$$I(Z) = \underset{\begin{bmatrix}0\\0\end{bmatrix}}{(x-0,\, y-0)} \cap \underset{\begin{bmatrix}1\\1\end{bmatrix}}{(x-1,\, y-1)} \cap \underset{\begin{bmatrix}1\\2\end{bmatrix}}{(x-1,\, y-2)} \cap \underset{\begin{bmatrix}2\\1\end{bmatrix}}{(x-2,\, y-1)}$$

$$= (2xy - y^2 - 2x + y,\; x^2 - y^2 - 3x + y,\; y^3 - 3y^2 + 2y)$$

deform

(= take top degree forms)

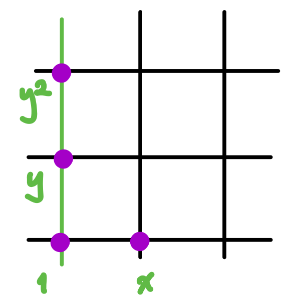
associated graded ring

$$\operatorname{gr} \mathbb{R}[Z] = \mathbb{R}[x,y] \big/ \operatorname{gr} I(Z)$$

$$\operatorname{gr} I(Z) = (2xy - y^2,\; x^2 - y^2,\; y^3)$$

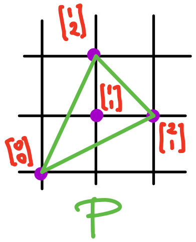
$$= \operatorname{span}_{\mathbb{R}} \{1, x, y, y^2\}$$

degrees: $\quad 0 \quad 1 \quad 1 \quad 2$



$$4 \quad \rightsquigarrow \quad 1 + 2q + q^2$$

$$= \#Z \qquad\qquad = \operatorname{Hilb}(\operatorname{gr} \mathbb{R}[Z], q)$$

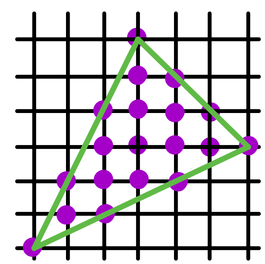


$$E_P(t) := \sum_{m=0}^{\infty} t^m \cdot \#\left(\mathbb{Z}^n \cap mP\right) = \frac{1+t+t^2}{(1-t)^3}$$

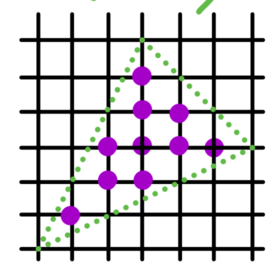
CLASSICAL Ehrhart Theorems: (Ehrhart, Macdonald, Stanley)
1928-2023

- $E_P(t) = \dfrac{1+h_1^* t + h_2^* t^2 + \dots + h^*_{\dim P}\, t^{\dim P}}{(1-t)^{\dim P+1}}$

3 · P



int(3 · P)

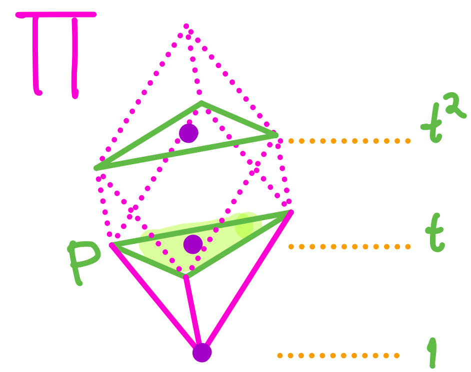


- $E_P\left(\frac{1}{t}\right) = (-1)^{\dim P+1} E_{\mathrm{int}(P)}(t)$

  RECIPROCITY

- $h_i^* \geq 0$ for $i=1,2,\dots,\dim P$

- For lattice simplices with vertices $\{\underline{v}^{(j)}\}$,



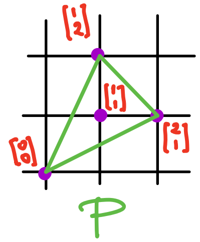
$$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$$

semi-open parallelepiped $\Pi := \sum_j [0,1) \cdot \begin{bmatrix} \underline{v}^{(j)} \\ 1 \end{bmatrix}$ in $\mathbb{R}^{n+1}$

$$E_P(t) := \sum_{m=0}^{\infty} t^m \cdot \#\left(\mathbb{Z}^n \cap mP\right) = \frac{1+t+t^2}{(1-t)^3}$$

$\xrightarrow{q\text{-analogue}}$

$$E_P(q,t) := \sum_{m=0}^{\infty} t^m \cdot \mathrm{Hilb}\left(\mathrm{gr}\,\mathbb{R}[\mathbb{Z}^n \cap mP], q\right) = \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$



CLASSICAL Ehrhart Theorems: $\rightsquigarrow$ CONJECTURES (Rhoades-R. 2024)

Classical Ehrhart Theorems:

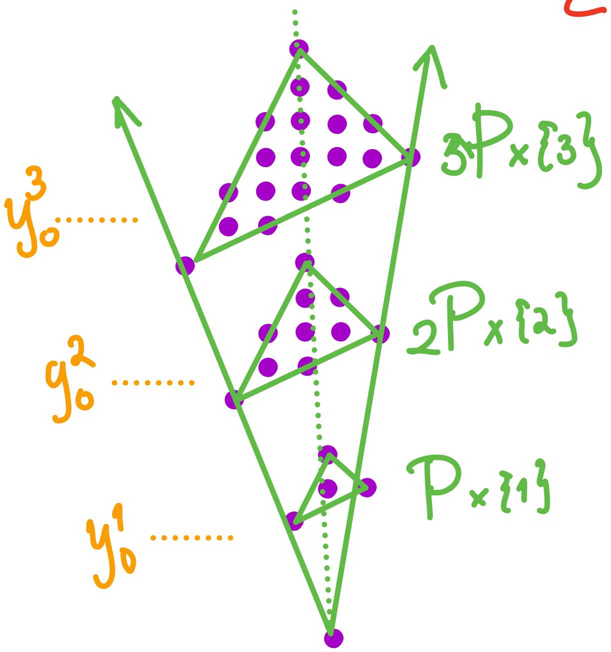
- $$E_P(t) = \frac{1+h_1^* t + h_2^* t^2 + \ldots + h_{\dim P}^* t^{\dim P}}{(1-t)^{\dim P+1}}$$
- $$E_P\left(\tfrac{1}{t}\right) = (-1)^{\dim P+1} E_{\mathrm{int}(P)}(t)$$
  RECIPROCITY
- $h_i^* \geq 0$ for $i = 1, 2, \ldots, \dim P$
- For lattice simplices,
  $$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$$

Conjectures (Rhoades-R. 2024):

- $$E_P(q,t) = \frac{N_P(q,t)}{\prod_i (1-q^{a_i} t^{b_i})}$$ with $N_P(q,t)$ in $\mathbb{Z}[q,t]$
- $$E_P\left(\tfrac{1}{q}, \tfrac{1}{t}\right) = (-1)^{\dim P+1} q^{\dim P} E_{\mathrm{int}(P)}(q,t)$$
  q-RECIPROCITY
- ??
- For lattice simplices, $N_P(q,t)$ lies in $\mathbb{N}[q,t]$

One of Stanley's celebrated techniques ...

Affine semigroup ring $\mathbb{R}[\mathrm{cone}\,P]$
$\subset \mathbb{R}[y_0, y_1, \ldots, y_n]$



$\mathrm{cone}(P) \subset \mathbb{R}^{n+1}$

- finitely generated $\mathbb{R}$-algebra (Gordan)
- Cohen-Macaulay (Hochster)
- Canonical module (Stanley, Danilov)
  $\Omega\, \mathbb{R}[\mathrm{cone}\,P] \cong \mathbb{R}[\mathrm{int}(\mathrm{cone}\,P)]$

q-analogue
$\rightsquigarrow$

Harmonic algebra

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m\, V_{\mathbb{Z}^n \cap mP}$$
$$\subset \mathbb{R}[y_0, y_1, \ldots, y_n]$$

where V_Z = harmonic space for $\mathrm{gr}\, I(Z)$
$\subset \mathbb{R}[y_1, \ldots, y_n]$

$$:= \bigcap_{f(\underline{x}) \in \mathrm{gr}\, I(Z)} \ker f\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)\Bigg|_{\mathbb{R}[\underline{y}]}$$

CONJECTURE (Rhoades-R. 2024):

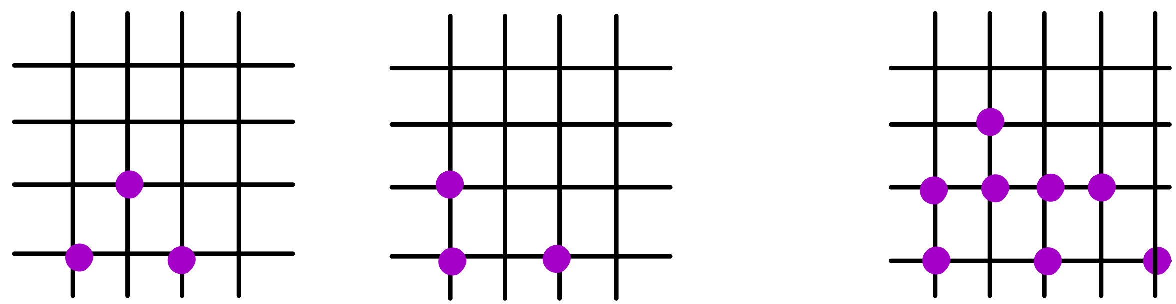
- $\mathcal{H}_P$ is finitely generated
- $\mathcal{H}_P$ is Cohen-Macaulay
- $\Omega\, \mathcal{H}_P \cong \mathcal{H}_{\mathrm{int}\,P}$

Existence of the algebra $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m V_{\mathbb{Z}^n \cap mP}$ is not obvious:

$V_{\mathbb{Z}^n\cap m_1P} \cdot V_{\mathbb{Z}^n\cap m_2P} \subseteq V_{\mathbb{Z}^n \cap (m_1+m_2)P}$ follows from a surprising ...

THEOREM (Rhoades-R. 2024) For finite point sets Z_1 and $Z_2 \subset \mathbb{R}^n$,

$$V_{Z_1} \cdot V_{Z_2} \subseteq V_{Z_1+Z_2}$$



Z_1 $\qquad$ Z_2 $\qquad$ Z_1+Z_2 = Minkowski sum

Thank you organizers, and

HAPPY 80th BIRTHDAY RICHARD!