# q-counting and invariant theory

Vic Reiner
University of Minnesota

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Lecture

1: Monday — Invitation to q-counts & representation theory
- quotients of Boolean algebras

2: Tuesday — Representation theory review & reflection groups

TODAY

3: Thursday — Molien's Theorem & coinvariant algebras

4: Thursday — Cyclic Sieving Phenomena (CSP) & Springer's Theorem

see ECCO 2018 lecture notes

5: Friday — More CSP's & the deformation idea

Recall

**DEFINITION**: A representation of a group $G$ on a vector space $V \cong \mathbb{C}^n$ is a homomorphism

$$G \xrightarrow{\rho} GL(V) \cong GL_n(\mathbb{C})$$

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Combinatorics provides many...

**EXAMPLES**

1. Permutation representations :=
   those that factor

$$G \hookrightarrow \mathfrak{S}_n \xrightarrow{\rho_{perm}} GL_n(\mathbb{C})$$

$$\sigma \longmapsto n \times n \text{ permutation matrix}$$

e.g. $\sigma = (245)(13) \in \mathfrak{S}_5 \longmapsto$

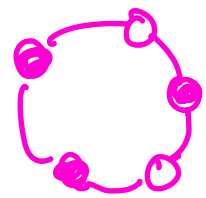
$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Some permutation representations:

- $G = C_n = \langle (1\,2\,\cdots\,n) \rangle \hookrightarrow \mathfrak{S}_n$

  and also $G \hookrightarrow \mathfrak{S}_{2^n}$.

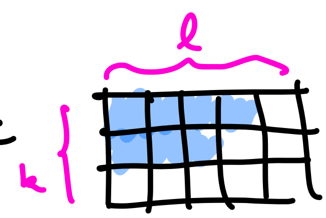
  The latter's $G$-orbits were black/white necklaces



- $G = \mathfrak{S}_k[\mathfrak{S}_\ell] \hookrightarrow \mathfrak{S}_{k\ell}$

  and also $G \hookrightarrow \mathfrak{S}_{2^{k\ell}}$

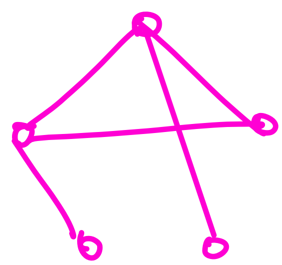
  The latter's $G$-orbits were Ferrers diagrams $\lambda \subset$




- $G = \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{\binom{n}{2}}$

  and also $G \hookrightarrow \mathfrak{S}_{2^{\binom{n}{2}}}$

  The latter's $G$-orbits were unlabeled graphs



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- The regular representation $\rho_{reg}$

$$G \hookrightarrow \mathfrak{S}_{|G|} \xrightarrow{\rho_{reg}} GL(\mathbb{C}G)$$

where $\rho_{reg}(g)(h) := gh \quad \forall h \in G, \; g \in G$

## EXAMPLES (of representations, continued)

2. 1-dimensional representations

$$G \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$

such as the trivial representation

$$\mathbb{1} = \mathbb{1}_G : \quad G \to \mathbb{C}^\times$$
$$g \mapsto 1 \quad \forall g \in G$$

or the determinant representation

$$\det : \; GL(V) \longrightarrow \mathbb{C}^\times$$
$$g \longmapsto \det(g)$$

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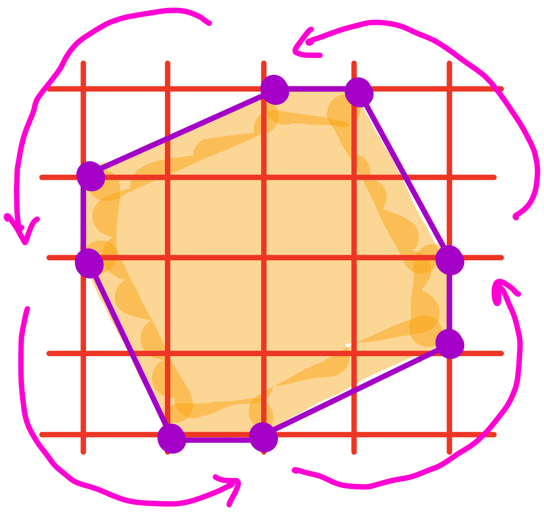
3. (Linear) symmetry groups of geometric objects $P \subset \mathbb{R}^d$

$$G = \mathrm{Aut}_{\text{linear}}(P) := \{ g \in GL_n(\mathbb{R}) : g(P) = P \}$$

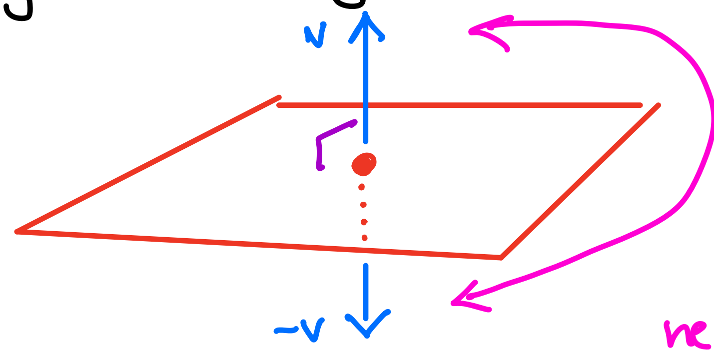
e.g.



$$G = \mathrm{Aut}_{\text{linear}}(P) \cong C_4$$
$$\hookrightarrow O_2(\mathbb{R}) \text{ orthogonal group}$$
$$\hookrightarrow GL_2(\mathbb{R})$$
$$\hookrightarrow GL_2(\mathbb{C})$$

## 4. (Real) reflection groups :=

finite subgroups $G \hookrightarrow O_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{C})$
generated by Euclidean reflections $t$




$t$ fixes $H$ = reflecting hyperplane,
negates normal vector $v$

Good examples of reflection groups are
are $G = \mathrm{Aut}_{\text{linear}}(P)$ for regular polytopes $P$

$G$ is transitive on maximal flags of faces
$\{\text{vertex} \subset \text{edge} \subset \text{polygon} \subset \cdots \subset \text{facet}\}$

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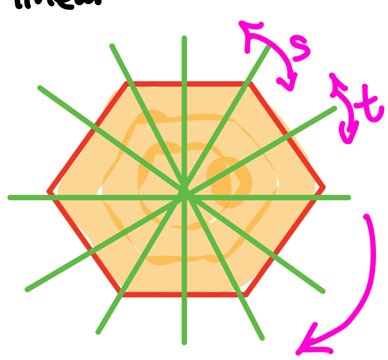
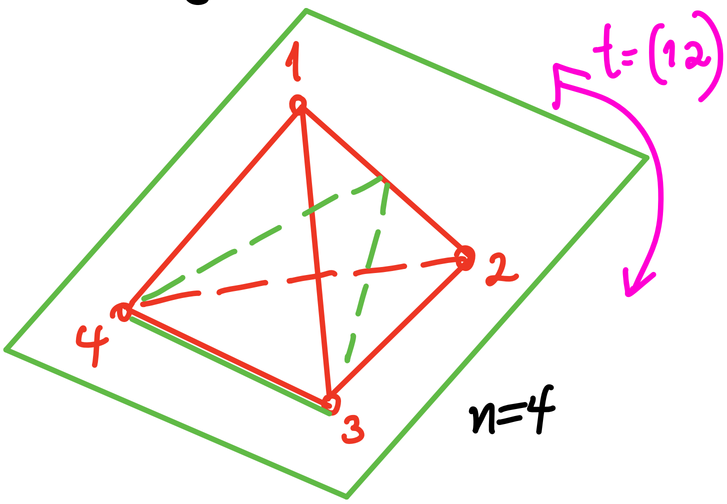
$G = I_2(m)$ = dihedral group of order $2m$
$= \mathrm{Aut}_{\text{linear}}(\text{regular } m\text{-gon}) = \langle s,t \mid s^2 = t^2 = 1, (st)^m = 1 \rangle$

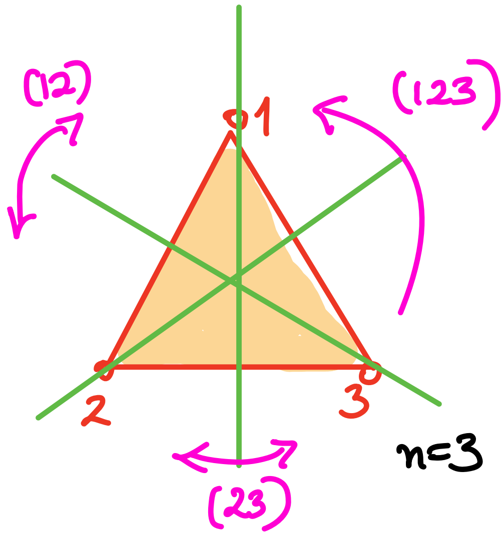
$m = 6$




$r = st$ = rotation

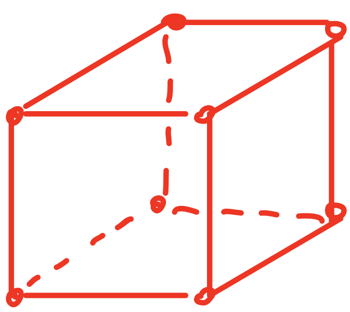
$G =$ symmetries of regular $(n-1)$-simplex $\cong \mathfrak{S}_n$

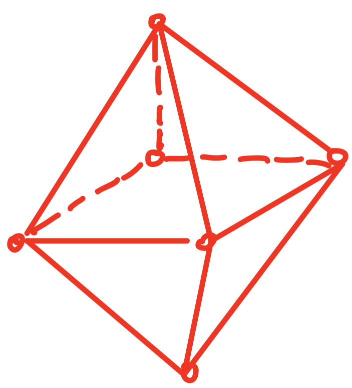



$$G \xrightarrow{\rho_{ref}} O_{n-1}(\mathbb{R})$$

$G =$ symmetries of $n$-dimensional cube
(= symmetries of $n$-dimensional cross-polytope)
= hyperoctahedral group $B_n$
$\cong$ $n \times n$ signed permutation matrices $\begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$





**QUESTION:** Can one classify all representations of a group $G$ up to equivalence,

meaning $G \xrightarrow{\rho} GL(V)$
$G \xrightarrow{\rho'} GL(V')$

have a $\mathbb{C}$-linear isomorphism $V \xrightarrow[\sim]{\varphi} V'$ which is G-equivariant:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\varphi"'] & V' \end{tikzcd}$$

commutes $\forall g \in G$

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**ANSWER:** Yes, when $G$ is finite and $V \cong \mathbb{C}^n$

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A key tool are traces:

**DEF'N:** The character $\chi_\rho$ of $G \xrightarrow{\rho} GL_n(\mathbb{C})$ is the (conjugacy) class function

$$G \xrightarrow{\chi_\rho} \mathbb{C}$$
$$g \longmapsto \chi_\rho(g) := \mathrm{Trace}(\rho(g))$$

meaning
$\chi_\rho(ghg^{-1}) = \chi_\rho(h)$

# Finite group representation theory "review"

## 1. Maschke's Theorem

One can always decompose $\rho = \bigoplus_{i=1}^{t} \rho_i$,

meaning

$$\rho(g) = \begin{array}{c|c} & \begin{array}{cccc} V_1 & V_2 & \cdots & V_t \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ \vdots \\ V_t \end{array} & \left[\begin{array}{c|c|c|c} \rho_1(g) & 0 & 0 & 0 \\ \hline 0 & \rho_2(g) & \cdots & 0 \\ \hline 0 & 0 & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \rho_t(g) \end{array}\right] \end{array}$$

where $V = \bigoplus_{i=1}^{t} V_i$ and each $G$-representation $V_i$ is simple/irreducible

i.e., no $G$-stable subspaces $U$ with $\{0\} \subsetneq U \subsetneq V_i$

2. The list of inequivalent irreducible representations $\{\rho_1, \rho_2, \dots, \rho_r\}$ has size $r = \#$ G-conjugacy classes.

In fact,

- the character $\chi_\rho$ determines $\rho$ upto equivalence
- because the irreducible characters $\{\chi_{\rho_1}, \chi_{\rho_2}, \dots, \chi_{\rho_r}\}$ give a $\mathbb{C}$-basis for the $\mathbb{C}$-vector space of class functions $G \to \mathbb{C}$
- and this basis is orthonormal with respect to the Hermitian inner product

$$\langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \cdot \chi_2(g)$$

3. Orthonormality $\Longrightarrow$ to uniquely decompose

$$\rho = \bigoplus_{i=1}^{r} \rho_i^{\oplus m_i}$$

into irreducibles $\rho_1, \rho_2, \ldots, \rho_r$,

since $\quad \chi_\rho = m_1 \chi_{\rho_1} + \ldots + m_r \chi_{\rho_r}$

one can compute the multiplicities $m_i$ from

$$\langle \chi_\rho, \chi_{\rho_i} \rangle_G = \Big\langle \sum_{j=1}^{r} m_j \chi_{\rho_j}, \ \chi_{\rho_i} \Big\rangle_G = m_i$$

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4. Similarly, it implies

$$\langle \chi_\rho, \chi_\rho \rangle_G = \Big\langle \sum_{j=1}^{r} m_j \chi_{\rho_j}, \sum_{i=1}^{r} m_i \chi_{\rho_i} \Big\rangle_G$$

$$= \sum_{j=1}^{r} m_j^2 = 1$$

$$\Updownarrow$$

$\rho = \rho_i$ is irreducible

# (Standard) EXAMPLES

1. 1-dimensional representations $G \xrightarrow{\rho} \mathbb{C}^\times$ are the same as their character: $\rho = \chi_\rho : G \to \mathbb{C}$

2. Permutation representations

$$\begin{tikzcd} G \arrow[r, hook] \arrow[rr, bend right, "\rho"'] & \mathfrak{S}_n \arrow[r, "\rho_{perm}"] & GL_n(\mathbb{C}) \end{tikzcd}$$

have $\chi_\rho(\sigma) = \mathrm{Trace}(\rho_{perm}(\sigma)) = \#$ of fixed points (=1-cycles) of $\sigma$

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In particular,

multiplicity of $\mathbb{1}_G$ in $\rho$ $= \langle \chi_\rho, \chi_{\mathbb{1}} \rangle_G$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \overline{\chi_\rho(\sigma)}$$

$$= \frac{1}{|G|} \sum_{\sigma \in G} (\#\text{ of fixed points of } \sigma)$$

Burnside's Lemma (and see EXER. 1.1.2(e))

$= \#G$-orbits on $[n]$

3. The regular representation

group algebra

$$G \xhookrightarrow{\rho_{reg}} \mathfrak{S}_{|G|} \hookrightarrow GL(\mathbb{C}G)$$

has $\rho_{reg}(g)(h) = gh \neq h$ if $g \neq e$

so $\chi_{\rho_{reg}}(g) = \text{Trace}\ \rho_{reg}(g) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e. \end{cases}$

Hence $\langle \chi_{\rho_{reg}}, \chi_{\rho_i} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_{reg}}(g)} \cdot \chi_{\rho_i}(g)$

$$= \frac{1}{|G|} \cdot |G| \cdot \chi_{\rho_i}(e)$$

$$= \dim_{\mathbb{C}}(V_i) \quad \text{if } G \xrightarrow{\rho_i} GL(V_i)$$

COROLLARY The regular representation contains every irreducible $\rho_i$ with multiplicity $\dim_{\mathbb{C}}(V_i)$:

$$\rho_{reg} = \bigoplus_{i=1}^{r} \rho_i^{\oplus \dim_{\mathbb{C}}(V_i)}$$

take dimensions

$$|G| = \sum_{i=1}^{r} \dim_{\mathbb{C}}(V_i)^2$$

4. Irreducible representations of

$G = \mathfrak{S}_3 = \{ \underbrace{e}, \mid \underbrace{(12), (13), (23)} \mid, \underbrace{(123) \quad (132)} \}$ ?

$r = 3$ conjugacy classes

$\Rightarrow$ 3 irreducible representations

Start with its 1-dimensional characters:

since $\mathfrak{S}_3 = \langle \underset{s}{\overset{}{(12)}}, \underset{t}{(23)} \rangle$, with $s = (12)$, $t = (23)$

and $s, t$ are conjugate, any 1-dimensional character $\chi$ has $\chi(s) = \chi(t)$.

Since $s^2 = t^2 = 1$, also $\chi(s) = \chi(t) \in \{\pm 1\}$.

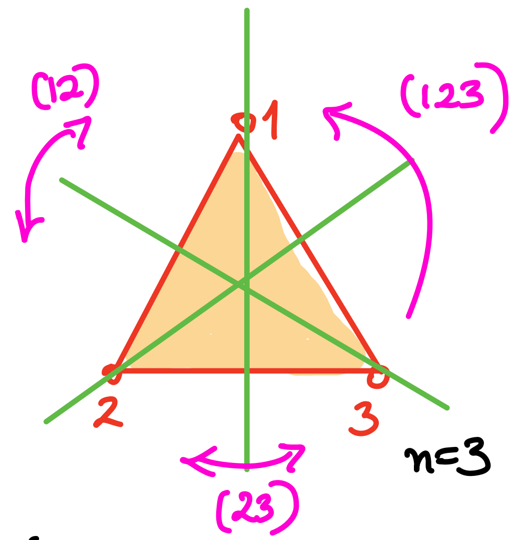
Two possibilities:

$$\begin{aligned} \mathfrak{S}_3 &\xrightarrow{\mathbb{1}} \mathbb{C}^\times \\ s, t &\longmapsto +1 \end{aligned}$$

$$\begin{aligned} \mathfrak{S}_3 &\xrightarrow{\operatorname{sgn}} \mathbb{C}^\times \\ s, t &\longmapsto -1 \end{aligned}$$

Need one more irreducible representation,
and we claim the reflection representation

$$\mathfrak{S}_3 \xrightarrow{\rho_{ref}} O_2(\mathbb{R}) \hookrightarrow GL_2(\mathbb{C})$$




is irreducible,
e.g via computing its character:

$$\chi_{ref}(e) = \text{Trace}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 2$$

$$\chi_{ref}((i\,j)) = \text{Trace}\begin{pmatrix}\text{reflection}\\ \text{in } \mathbb{R}^2\end{pmatrix} = \text{Trace}\begin{bmatrix}+1 & 0\\ 0 & -1\end{bmatrix} = 0$$

$$\chi_{ref}((i\,j\,k)) = \text{Trace}\begin{pmatrix}120^\circ\\ \text{rotation}\end{pmatrix} = \text{Trace}\begin{bmatrix}e^{2\pi i/3} & 0\\ 0 & e^{-2\pi i/3}\end{bmatrix} = -1$$

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Hence $\langle \chi_{ref}, \chi_{ref} \rangle_{\mathfrak{S}_3} = \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} \overline{\chi_{ref}(\sigma)} \cdot \chi_{ref}(\sigma)$

$$= \frac{1}{3!}\Big(\overset{e}{1 \cdot 2 \cdot 2} + \overset{\substack{(12)\\(13)\\(23)}}{3 \cdot 0 \cdot 0} + \overset{\substack{(123)\\(132)}}{2 \cdot (-1)(-1)}\Big) = \frac{1}{3!}(6) = 1$$

irreducible ✓

## CONCLUSION:

$\mathfrak{S}_3$ has irreducible character table

| | $e$ | $(12), (13), (23)$ | $(123), (132)$ |
|---|---|---|---|
| $\mathbb{1}$ | 1 | 1 | 1 |
| Sgn | 1 | $-1$ | 1 |
| $\rho_{ref}$ | 2 | 0 | $-1$ |

**REMARK:** In general, $\mathfrak{S}_n$ has irreducible representations $\{\rho_1, \rho_2, \dots, \rho_r\} = \{\rho_\lambda : \text{partitions } \lambda \text{ of } n\}$ with $\rho_\lambda(e) = \dim_{\mathbb{C}} V_\lambda =: \dim(\lambda) = \dfrac{n!}{\prod_{\square \in \lambda} h_\square}$

in V. Feray's talks