q-counting and invariant theory

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Lecture

1: Monday
Invitation to q-counts
& representation theory
- quotients of Boolean algebras

2: Tuesday
Representation theory review
& reflection groups

3: Thursday
Molien's Theorem
& coinvariant algebras

TODAY

4: Thursday
Cyclic Sieving Phenomena (CSP)
& Springer's Theorem

see ECCO 2018 lecture notes

5: Friday
More CSP's
& the deformation idea

GOAL: Enhance the structure of the coinvariant algebra (Springer's Theorem) to explain some interesting counting formulas:

CSP's = cyclic sieving phenomena

(S. Pfannerer's talk has more to say on CSP's)

We've already seen an instance of a CSP...

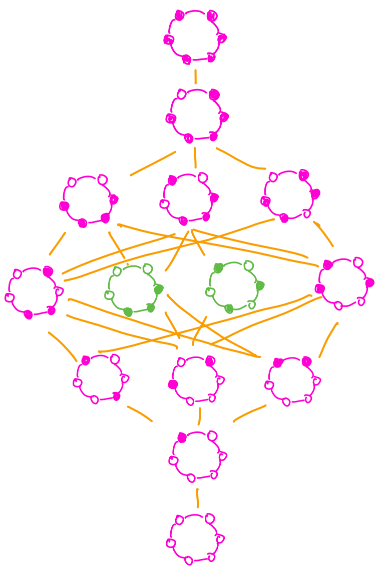
deBruijn 1959

THEOREM For any subgroup G of $\mathfrak{S}_n$, consider

- the set $X := 2^{[n]}/G$ = G-orbits $\mathcal{O}$ of subsets $A \subseteq \{1,2,\ldots,n\} = [n]$
- the generating function
$$X(q) := r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n$$
where $r_k = \#$ G-orbits $\mathcal{O}$ on $\binom{[n]}{k}$
- and the $\mathbb{Z}/2\mathbb{Z}$-action on X induced by complementation $A \mapsto [n] \setminus A$ sending an orbit $\mathcal{O} \xmapsto{c} \mathcal{O}^c := \{[n] \setminus A : A \in \mathcal{O}\}$.

Then
$$\underbrace{\#\{x \in X : c(x) = x\}}_{\substack{\#\text{ of}\\ \text{self-complementary}\\ G\text{-orbits } \mathcal{O}}} = \underbrace{\left[X(q)\right]_{q=-1}}_{r_0 - r_1 + r_2 - \cdots \pm r_n}$$

EXAMPLE

$X=$ 

$$X(q)=1+q+3q^2+4q^3+3q^4+q^5+q^6$$

$$\Big\downarrow q=-1$$

$$[X(q)]_{q=-1} = 2$$

This is an example of what Stembridge (1994) called a "$q=-1$ phenomenon":

A set X with an action of $\mathbb{Z}/2\mathbb{Z}=\{1,c\}$ and a polynomial $X(q)$ such that

$$X(+1) = \#X$$

$$X(-1) = \#\{x\in X : c(x)=x\}$$

He found several interesting examples.

More generally ...

DEFINITION: Say that a
R.-Stanton-White
2004

- finite set X
- with the action of a cyclic group $C = \{1, c, c^2, \ldots, c^{m-1}\} \cong \mathbb{Z}/m\mathbb{Z}$
- and a polynomial $X(q)$

exhibit a cyclic sieving phenomenon (CSP)

if for every $c^d \in C$ one has

$$\#\{x \in X : c^d(x) = x\} = \left[X(q)\right]_{q=\zeta^d}$$

where $\zeta = e^{2\pi i/m}$ = primitive m^{th} root of 1 in $\mathbb{C}^\times$

(Proto-) EXAMPLE

- set $X = \binom{[n]}{k}$ = k-element subsets of $[n]$
- action $\circlearrowleft$ $C \cong \mathbb{Z}/n\mathbb{Z} = \langle \underbrace{\overset{n\text{-cycle}}{(1,2,\ldots,n)}}_{c:=} \rangle$
- polynomial $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ q-binomial coefficient

$$:= \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

where $\underset{\text{q-factorial}}{[n]!_q} := [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q$

$\underset{\text{q-number}}{[n]_q} := 1+q+q^2+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$

THEOREM RSW 2004

EXER. 1.4.3 gives a 'brute force' proof

This
- X $\circlearrowleft$
- C
- $X(q)$

exhibit a CSP.

EXAMPLE $n=4 \quad k=2$

$$X = \binom{[4]}{2}$$

$$\circlearrowright$$

$$C = \langle \overset{c=}{(1\,2\,3\,4)} \rangle$$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 4 \arrow[ur] & & 2 \arrow[dl] \\ & 3 \arrow[ul] & \end{tikzcd}$$

$$= \{e, c, c^2, c^3\}$$

$$\begin{tikzcd} & \{1,2\} \arrow[dr, "c"] & \\ \{1,4\} \arrow[ur, "c"] & & \{2,3\} \arrow[dl, "c"] \\ & \{3,4\} \arrow[ul, "c"] & \end{tikzcd} \qquad \begin{tikzcd} \{1,3\} \arrow[d, bend left, "c"] \\ \{2,4\} \arrow[u, bend left, "c"] \end{tikzcd}$$

$$X$$

$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]!_q}{[2]!_q\,[2]!_q} = \frac{[4]_q[3]_q[2]_q[1]_q}{[2]_q[1]_q \cdot [2]_q[1]_q}$$

$$= \frac{[4]_q[3]_q}{[2]_q} = \frac{(1+q+q^2+q^3)(1+q+q^2)}{1+q}$$

$$= 1 + q + 2q^2 + q^3 + q^4$$

$$\boxed{\begin{array}{c} m=4 \\ \Downarrow \\ \zeta = e^{2\pi i/4} = i \end{array}}$$

$q = \zeta^0 = 1$

$1+1+2+1+1$

$= 6 = |X|$

$= |X^{c^0}|$

$q = \zeta^2 = -1$

$1-1+2-1+1$

$= 2$

$= |\underbrace{\{13, 24\}}_{X^{c^2}}|$

$q = \zeta^1 = i$

$1+i-2-i+1$

$= 0$

$= |X^{c^1}|$

That Proto-EXAMPLE has many proofs,
one of which generalizes to reflection groups, via...

THEOREM (Springer 1972) In a finite reflection group
$$G \subset GL_n(\mathbb{C}) = GL(V),$$
say $c \in G$ is a regular element if it has an eigenvector $v \in V$, say $c(v) = \zeta \cdot v$, lying on none of the reflecting hyperplanes.

Then its cyclic subgroup $C = \{e, c, c^2, \ldots, c^{m-1}\} \cong \mathbb{Z}/m\mathbb{Z}$ gives us an isomorphism of $G \times C$-representations:

coinvariant algebra $\qquad\qquad$ regular representation $\mathbb{C}G$

$$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \cong \rho_{reg}$$

$\circlearrowright$ $\qquad\qquad$ $\circlearrowright$

- G acts as before by linear substitutions
- C acts by scalar substitutions $c(x_i) = \zeta x_i$

- G left-translates $h \xrightarrow{g} gh$
- C right-translates $h \xrightarrow{c^d} hc^d$

NEW!

Here is a general CSP corollary:

THEOREM When a finite reflection group
RSW 2004
$G \subset GL_n(\mathbb{C})$ acts transitively on a set X,
every regular element $c \in G$ gives a CSP:

- X $\quad (\cong G/H$ for some subgroup $H)$
  $\circlearrowleft$
- $C = \{e, c, c^2, \ldots, c^{m-1}\} \cong \mathbb{Z}/m\mathbb{Z}$
- $X(q) := \dfrac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q)} = \prod_{i=1}^{n}(1-q^{d_i}) \cdot \mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)$

In other words,

$$\#\{x \in X : c^d(x) = x\} = \left[X(q)\right]_{q=\zeta^d}$$

$$\|$$

$$\#\{\text{cosets } gH : c^d gH = gH\}$$

where $\zeta = e^{2\pi i/m}$

How does this generalize the Proto-EXAMPLE?

- $X = \binom{[n]}{k} = \underbrace{\mathfrak{S}_n}_{G} \Big/ \underbrace{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}_{H}$

  $G = \mathfrak{S}_n$ acts transitively on k-subsets of $[n]$

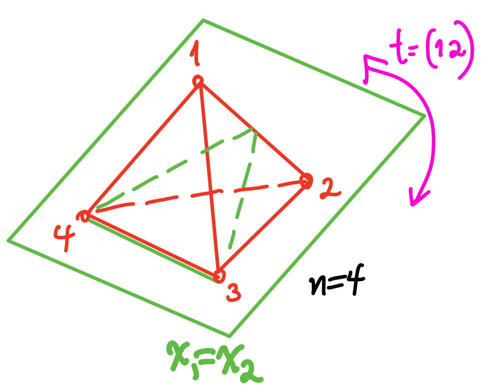
  $H = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ is the stabilizer of $\{1,2,\ldots,k\} \subset [n]$.

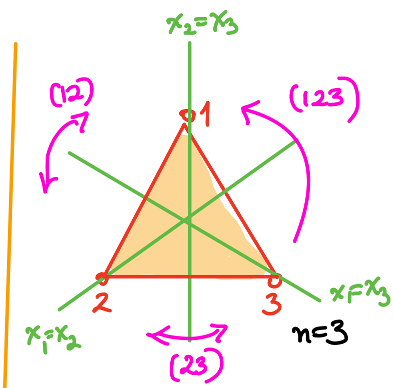
- The n-cycle $c = (1\,2 \cdots n)$ inside $\mathfrak{S}_n$ is a regular element: acting on $V = \mathbb{C}^n$:

  it has an eigenvector $v = \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$, where $\zeta = e^{2\pi i/n}$

  with $c(v) = \zeta \cdot v$

  lying on no reflecting hyperplanes $x_i = x_j$ since its coordinates are distinct.





What about $X(q)$? We claim...

- $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]!_q}{[k]!_q\,[n-k]!_q}$

$$= \frac{1}{(1-q^1)\cdots(1-q^k)\cdot(1-q^1)\cdots(1-q^{n-k})} \Big/ \frac{1}{(1-q^1)(1-q^2)\cdots(1-q^n)}$$

$$= \mathrm{Hilb}\big(\mathbb{C}[\underline{x}]^{\overbrace{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}^{H}}, q\big) \Big/ \mathrm{Hilb}\big(\mathbb{C}[\underline{x}]^{\underbrace{\mathfrak{S}_n}_{G}}, q\big)$$

since just as

$$\mathbb{C}[\underline{x}]^{\mathfrak{S}_n} = \mathbb{C}[\overset{1}{e_1}, \overset{2}{e_2}, \overset{\cdots}{\ldots}, \overset{n}{e_n}] \leftarrow \text{degrees}$$

one also has

$$\mathbb{C}[\underline{x}]^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}} = \mathbb{C}[\overset{1}{e_1(x_1,\ldots,x_k)}, \overset{2\,\cdots}{\ldots}, \overset{k}{e_k(x_1,\ldots,x_k)},$$

degrees

$$\overset{1}{e_1(x_{k+1},\ldots,x_n)}, \overset{2\,\cdots}{\ldots}, \overset{n-k}{e_{n-k}(x_{k+1},\ldots,x_n)}]$$

RECAP:

THEOREM When a finite reflection group $G \subset GL_n(\mathbb{C})$ acts transitively on a set X, every regular element $c \in G$ gives a CSP:

- $X \quad (\cong G/H$ for some subgroup $H)$

$\circlearrowleft$

- $C = \{e, c, c^2, \ldots, c^{m-1}\} \cong \mathbb{Z}/m\mathbb{Z}$
- $$X(q) := \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q)} = \prod_{i=1}^{n}(1-q^{d_i}) \cdot \mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)$$

Proto-EXAMPLE

- $X = \binom{[n]}{k} = \mathfrak{S}_n / \mathfrak{S}_k \times \mathfrak{S}_{n-k}$

$\circlearrowleft$

- $C = \langle \overset{n\text{-cycle}}{(1\,2\,\cdots\,n)} \rangle$ inside $\mathfrak{S}_n$
- $$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{\mathfrak{S}_n}, q)}$$

Q: How to get the CSP THEOREM from Springer's ?

A: Our favorite technique: comparison of traces !

Sketch proof:

Springer gave us a $G \times C$-rep isomorphism

$$\underbrace{\mathbb{C}[\underline{x}]/(f_1,\ldots,f_n)}_{\text{coinvariant algebra}} \cong \underbrace{\rho_{reg}}_{\text{regular rep } \mathbb{C}G}$$

Taking H-fixed subspaces leaves a C-rep isomorphism where we can compare the trace of c^d on both sides.

$$\left(\mathbb{C}[\underline{x}]/(\underline{f})\right)^H \cong (\mathbb{C}G)^H \text{ as } C\text{-reps}$$

Compare the trace of c^d on both sides:

LEFT:

$\left(\mathbb{C}[\underline{x}]/(\underline{f})\right)^H$ is a graded $\mathbb{C}$-vector space, and c^d acts via scalar $(\zeta^d)^k$ in the k^{th} homogeneous piece.

Also, can show

$$X(q) = \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q)}$$

$$= \mathrm{Hilb}\left(\left(\mathbb{C}[\underline{x}]/(\underline{f})\right)^H, q\right)$$

Hence c^d acts on left with trace $[X(q)]_{q=\zeta^d}$

RIGHT:

One can identify

$$(\mathbb{C}G)^H \cong \mathbb{C}[H\backslash G]$$

permutation representation of C on $X = G/H$, where $c^d(gH) = c^d gH$

Hence c^d acts on right with trace

$$\#\{x \in X : c^d(x) = x\}$$

$\blacksquare$