Koszulity and Stirling representations
- a preliminary report

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Various Guises of Reflection Arrangements
ICMS Edinburgh March 2023
1. Stirling numbers $c(n,k)$, $S(n,k)$

2. (Familiar) algebras of the 1st kind

3. Koszulity review

4. Supersolvability

5. (Koszul dual) algebras of the 2nd kind

6. Properties and Questions
1. Stirling numbers $c(n,k), S(n,k)$

(Signless) Stirling number of the 1st kind

$$c(n,k) := \text{# permutations in } S_n \text{ with } k \text{ cycles}$$

for $1 \leq k \leq n$

<table>
<thead>
<tr>
<th>$c(4,4)$</th>
<th>$c(4,3)$</th>
<th>$c(4,2)$</th>
<th>$c(4,1)$</th>
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Generating function definition:

\[ \sum_{k=1}^{n} \binom{n}{k} t^{n-k} = (1+t)(1+2t)(1+3t) \cdots (1+(n-1)t) \]

\[ n=4: \]

\[ (1+t)(1+2t)(1+3t) = 1 + 6t + 11t^2 + 6t^3 \]

\[ c(4,4) \quad c(4,3) \quad c(4,2) \quad c(4,1) \]
(Signless) **Stirling number of the 2nd kind**

\[ S(n, k) := \text{# partitions of } \{1, 2, \ldots, n\} \text{ with } k \text{ blocks} \]

for \( 1 \leq k \leq n \)

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<thead>
<tr>
<th>( S(4,4) )</th>
<th>( S(4,3) )</th>
<th>( S(4,2) )</th>
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Generating function definition:

\[ \sum_{n=k}^{\infty} S(n,k) t^n = \frac{t^k}{(1-t)(1-2t)(1-3t) \cdots (1-k \cdot t)} \]

Rewritten for later purposes:

\[ \sum_{i=0}^{\infty} S(n-1+i, n-1) t^i = \frac{1}{(1-t)(1-2t) \cdots (1-(n-1)t)} \]

(cf. \[ \sum_{k=1}^{n} c(n,k) t^{n-k} = (1+t)(1+2t)(1+3t) \cdots (1+(n-1)t) \) )
**Triangles and recursions**

<table>
<thead>
<tr>
<th></th>
<th>k=1</th>
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<td>n=1</td>
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<td>n=4</td>
<td>6</td>
<td>11</td>
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<td>n=5</td>
<td>24</td>
<td>35</td>
<td>10</td>
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For \( c(n,k) \):

\[
c(n,k) = (n-1) \cdot c(n-1,k) + c(n-1,k-1)
\]

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<td>n=4</td>
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<td>15</td>
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For \( s(n,k) \):

\[
s(n,k) = k \cdot s(n-1,k) + s(n-1,k-1)
\]
2. (Familiar) algebras of the 1st kind

\[ A(n) := \bigwedge^* \text{Conf}(n, \mathbb{R}^d) \]

(skew-commutative exterior algebra)

\[ A_{os}(n) = \bigwedge \{ e_{ij} \} / (e_{ij} e_{ik} - e_{ij} e_{jk} + e_{ik} e_{jk}) \quad \text{for } 1 \leq i < j < k \leq n \]

\[ A_{VG}(n) = \mathbb{K}[x_{ij}] / (x_{ij}^2 \text{ and } x_{ij} x_{ik} - x_{ij} x_{jk} + x_{ik} x_{jk}) \quad \text{for odd } d \]

(ordered) configuration space of \( n \) distinct \( d \)-dimensional points \( (p_1, p_2, \ldots, p_n) \) in \( \mathbb{R}^d \)

\( p_i \neq p_j \)

Arnold 1968

F. Cohen 1972

rescale cohomological grading by \( d-1 \)

commutative polynomial ring
Both $A(n) = A_{os}(n)$, $A_{vg}(n)$ have

$$\text{Hilb}(A(n), t) = (1+t)(1+xt)(1+xt^2) \cdots (1+(n-1)t)$$

since those were Gröbner bases for the ideals, with initial terms $e_{ik}e_{jk}$ and $x_{ik}x_{jk}, x_i^2$, leading to standard monomial NBC bases =

"at most one finger from each hand"  

Barcelo 1988

$n=6$: 

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</table>

NBC monomial of deg 3

$w \in \mathcal{G}_6$ counted by $c(6,6,3) = c(63)$
As $G_n$-representations, both $A(n) = A_{os}(n), A_{vc}(n)$ are well-studied, but not completely understood.

<table>
<thead>
<tr>
<th>$n=4$</th>
<th>1 + 6$t$ + 11$t^2$ + 6$t^3$</th>
<th>[H_k[G_4]] (ungraded)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{vc}(4)$</td>
<td>[\text{regular rep.}]</td>
<td>2 copies of $[H_k[G_4]/{G_2} \times G_1 G_1]_k$</td>
</tr>
<tr>
<td>$A_{os}(4)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
As $G_n$-representations,

$$
\sum_{n=0}^{\infty} \sum_{k=1}^{n} \text{ch } A(n)_{n-k} \ t^k =
$$

\[
\sum_{\lambda=1^{m_1}2^{m_2}} t^{l(\lambda)} \prod_{j=1}^{\infty} h_{m_j}[\text{Lie}_j] = \prod_{m=1}^{\infty} (1-p_m) \quad \text{for VG}
\]

\[
\sum_{\lambda=1^{m_1}2^{m_2}} t^{l(\lambda)} \prod_{j=\text{odd}} h_{m_j}[\pi_j] \cdot \prod_{j=\text{even}} e_{m_j}[\pi_j] = \prod_{m=1}^{\infty} (1+(-1)^m p_m) \quad \text{for OS}
\]

where $a_m(t) = \frac{1}{m} \sum_{d \mid m} \mu(d) t^d$

Many results by Lehrer-Solomon, Whitehouse, Douglass-Pfeiffer-Rörle, ...
Branching for $A(n)$

The recurrence

$$C(n,k) = (n-1) \cdot C(n-1,k) + C(n-1, k-1)$$

lifts easily (via the generating functions):

**PROPOSITION:** Both $A(n) = A_{OS}(n), A_{VG}(n)$ have these branching rules for restriction $G_n$ to $G_{n-1}$:

$$A(n)_i \downarrow G_n \cong \chi_{\text{def}}^{(n-1)} \otimes A(n-1)_{i-1} \oplus A(n-1)_i$$

defining rep. of $G_{n-1}$ as permutation matrices

<table>
<thead>
<tr>
<th>$k$</th>
<th>$C(n,k)$</th>
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<tbody>
<tr>
<td></td>
<td>1 2 3 4 5</td>
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<tr>
<td>1</td>
<td>1 1</td>
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<tr>
<td>2</td>
<td>1 1</td>
</tr>
<tr>
<td>3</td>
<td>2 3 1</td>
</tr>
<tr>
<td>4</td>
<td>6 11 6 1</td>
</tr>
<tr>
<td>5</td>
<td>24 50 25 10 1</td>
</tr>
</tbody>
</table>
Better phrasing:

**PROPOSITION:** $A(n)=\left\{ \frac{A_{os}(n)}{A_{VG}(n)} \right\}$ have $G_{n-1}$-equivariant s.e.s.

$0 \rightarrow A(n-1) \rightarrow A(n) \left[ \begin{array}{c} \rightarrow \end{array} \right] \left[ \begin{array}{c} \frac{G_{n}}{G_{n-1}} \end{array} \right] \rightarrow \left[ \begin{array}{c} \chi_{\text{def}}^{(n-1)} \otimes A(n-1) \end{array} \right] (-1) \rightarrow 0$

This generalizes to \{Orlik-Solomon, Varchenko-Gelfand\} algebras

$A_{os}(X) := \Lambda \left\{ e_{H} \right\}_{H \in A} \left/ \langle \partial e_{C} : \text{circuits } C \rangle \right.$

$A_{VG}(X) := \Lambda \left\{ x_{H} \right\}_{H \in A} \left/ \langle \partial x_{C} : \text{circuits } C \rangle \right.$

where for a circuit $C=\left\{ H_{1}, \ldots, H_{p} \right\}$ with $\sum_{j=1}^{p} \alpha_{j} = 0$ if $H_{j} = \ker(\alpha_{j})$

$\partial e_{C} := \sum_{j=1}^{p} (-1)^{j} e_{H_{1}} \wedge \cdots \wedge \hat{e}_{H_{j}} \wedge \cdots e_{H_{p}}$

$\partial x_{C} := \sum_{j=1}^{p} \text{sgn}(c_{j}) x_{H_{1}} \cdots \hat{x}_{H_{j}} \cdots x_{H_{p}}$
PROPOSITION: Assume the arrangement $\mathcal{A}$ has symmetries $W$ and $X$ a modular coatom (= line) with $W$-stabilizer $N_X$.

Then both algebras $A(\mathcal{A})=A_{\text{o.s.}}(\mathcal{A}), A_{\text{v.g.}}(\mathcal{A})$ have an $N_X$-equivariant s.e.s.

$$0 \rightarrow A(\mathcal{A}_X) \rightarrow A(\mathcal{A}) \rightarrow \bigoplus_{H: H \not\approx X}^W \mathcal{J}_H \rightarrow \left[ \bigoplus_{H: H \not\approx X} A(\mathcal{A}^H) \right](-1) \rightarrow 0$$

The maps $j_H$ come from addition-deletion sequences:

$$0 \rightarrow A(\mathcal{A}-\{H\}) \rightarrow A(\mathcal{A}) \rightarrow A(\mathcal{A}^H) \rightarrow 0$$

- $e_H \wedge \ldots \wedge e_H$ if $H \not\in \{H_1, \ldots, H_p\}$
- $e_H \wedge e_{H_1} \wedge \ldots \wedge e_{H_p}$
Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a standard graded associative $k$-algebra, generated by $A_1 = V$

$$= k \bigoplus_{d=0}^{\infty} A_1 \oplus A_2 \oplus ..., \quad \text{where } k \text{-basis } x_1, ..., x_n \text{ for } V = A_1$$

$$\cong \frac{k\langle x_1, ..., x_n \rangle}{T(V)}$$

where the 2-sided ideal $I$ is homogeneous:

$I = \bigoplus_{d=0}^{\infty} I_d$ where $I_d = I \cap T^d(V) = k\langle x_1, ..., x_n \rangle_d$
**Definition:** A is Koszul if \( \exists \) an \( A \)-free resolution of \( \mathfrak{k} = A / A_+ \) which is linear:

\[
0 \leftarrow \mathfrak{k} \leftarrow A \leftarrow A(-1) \leftarrow A(-2) \leftarrow A(-3) \leftarrow \ldots
\]

\[
\begin{bmatrix}
0 \\
1 \\
\vdots \\
1
\end{bmatrix} \leftarrow \begin{bmatrix}
[x_1] \\
\vdots \\
[x_n]
\end{bmatrix}
\]

\[
\begin{bmatrix}
\ldots \\
\ldots \\
\ldots
\end{bmatrix} \leftarrow \begin{bmatrix}
\ldots \\
\ldots \\
\ldots
\end{bmatrix}
\]

Equivalently, \( \text{Tor}_i^A(\mathfrak{k}, \mathfrak{k})_j = 0 \) unless \( i = j \)

( or same for \( \text{Ext}_i^A(\mathfrak{k}, \mathfrak{k})_j \) )
When $A$ is Koszul, one can write down a beautiful explicit resolution of $k$ called the Priddy complex ...

$A$ is necessarily a quadratic algebra

i.e. $A = k\langle x_1, \ldots, x_n \rangle / I$ with $I = (\begin{vmatrix} \mathbf{I}_1 \end{vmatrix})_{\mathbf{V} \otimes \mathbf{V}}$

so one can define its quadratic dual algebra

$A^! := k\langle y_1, \ldots, y_n \rangle / J$ where $J := (\begin{vmatrix} \mathbf{I}_1 \end{vmatrix})^\perp$

$V^*$ basis dual to $x_1, \ldots, x_n$

$\langle x \otimes x', y \otimes y' \rangle := (x,y) \cdot (x',y')$

The Priddy complex is $A \otimes (A^!)^*$, linearly resolving $k$

$0 \leftarrow k \leftarrow A \otimes (A^!)^* \leftarrow A \otimes (A^!)^* \leftarrow A \otimes (A^!)^* \leftarrow \cdots$

with $d_i = \text{mult by } \sum_{j=1}^{n} x_j \otimes (y_j)^*$
Exactness of the Priddy complex shows

\[ \text{Hilb}(A'^*_t, -t) \cdot \text{Hilb}(A_t) = 1 \quad \left[= \text{Hilb}(\mathbb{A}^1_k, t) \right] \]

so

\[ \text{Hilb}(A'^*_t) = \frac{1}{\text{Hilb}(A_t)} \]

When \( W \) acts on \( A \) via graded \( k \)-algebra automorphisms, exactness in degree \( d \) shows virtual character identities

\[ (A'^*_d)^* - A_1 \otimes (A'^*_d - 1)^* + A_2 \otimes (A'^*_d - 2)^* - \ldots \pm (A'^*_d) \otimes A_{d-1} \pm A_d = 0 \]

expressing \( (A'^*_d)^* \) (and hence also \( A'^*_d \)) recursively in terms of \( A_0, A_1, \ldots, A_d \).
A useful sufficient condition for Koszulity of $A$:

**PROPOSITION:**

When $A = \Lambda(e_1, \ldots, e_n)/I$

or

$K[x_1, \ldots, x_n]/I$

and $\prec$ a monomial order $\prec$ on $\Lambda(e_1, \ldots, e_n)$

$K[x_1, \ldots, x_n]$

for which $I$ has a quadratic Gröbner basis

i.e. $\text{in}_\prec(I)$ is a quadratic (monomial) ideal,

then $A$ is Koszul.
4. Supersolvability

**DEFINITION:** A central arrangement $\mathcal{A}$ is supersolvable if it has an $M$-chain := maximal flag of modular flats $\emptyset = X_1 \supset X_2 \supset \ldots \supset X_{r-1} \supset \{0\}$

It partitions $\mathcal{A} = A_1 \cup A_2 \cup \ldots \cup A_r$

where $A_i := A_{X_i} - A_{X_{i-1}} = \{ H \in \mathcal{A} : H \supset X_i \}$

and defines exponents $e_1, e_2, \ldots, e_r$ by

$$e_i := |A_i|$$
THEOREM

Either of the rings $A(t) = A_{os}(t), A_{rg}(t)$ has a term order $<$ on $N(e_1, \ldots, e_n)$ or $I_k(x_1, \ldots, x_n)$ for which its defining ideal has a quadratic Gröbner basis

$$\iff A \text{ is supersolvable}$$

In this case, the term orders $<$ which work are those having $x_{h_i} < x_{h_j}$ if $\{ h_i \in A_i \}$ with $i < j$.

Furthermore, the initial terms look like $x_{h_i}x_{h_j}$ for $h_i, h_j \in A_i$ so the standard monomial NBC-basis is the set of monomials with at most one “finger” $x_{h_i}$ from each “hand” $A_i$:

$$\implies \text{Hillb}(A(A), t) = (1 + e_1 t)(1 + e_2 t) \ldots (1 + e_r t)$$
**Corollary:** When $A$ is supersolvable, both $A(A) = A_{os}(A), A_{vc}(A)$ are Koszul, with $\text{Hilb}(A^i, t) = \frac{1}{\text{Hilb}(A_{ij}, t) = \frac{1}{(1-e_1 t)(1-e_2 t) \ldots (1-e_r t)}}$

**Corollary:** For $A$ supersolvable, $A(A) = A_{os}(A)$ have (noncommutative) quadratic Grobner-basis with initial terms $Y_{H_j} Y_{H_i}$ for $\{H_i \in A_i \text{ with } i < j\}$ and standard monomial basis

\[ \{ m^{(1)}, m^{(2)}, \ldots, m^{(r)} : m^{(i)} \text{ noncommutative monomial in } \{y_{H_j} \}_{H \in A_i} \} \]

i.e. "no revisiting an earlier hand".

(or "all-you-can-eat salad bar, but keep it moving!"")
5. (Koszul dual) algebras of the 2nd kind

**EXAMPLE** Type A reflection arrangement in $V = \mathbb{R}^n/\mathbb{R}[1]$ for $W = \mathbb{G}_n$ has hyperplanes $\{ H_{ij} = \{ x_i = x_j \} : 1 \leq i < j \leq n \}$, supersolvable, M-chain:

$$V \supset \{ x_1 = x_2 \} \supset \{ x_1 = x_2 = x_3 \} \supset \ldots \supset \{ x_1 = x_2 = \ldots = x_{n-1} \} \supset \{ x_1 = x_2 = \ldots = x_n \} = \{ 0 \}$$

![Diagram](image)

**COROLLARY**: 

$$\text{Hilb}(A(n)!, t) = \frac{1}{(-t)(1-2t) \ldots (1-(n-1)t)} = \sum_{i=0}^{8} S(n-1+i, n-1) t^i$$
**Example**

$A(C_3)$ has standard monomials $m^{(1)}$, $m^{(2)}$, $m^{(3)}$, $m^{(4)}$, where $m^{(i)}$ picks noncommutative monomials in the hand $A_i$:

\[
\begin{array}{cccc}
1 & 1 & 2 & 1 \\
2 & 3 & 4 & 1 \\
3 & 4 & 5 & 1 \\
4 & 5 & 6 & 1 \\
5 & 6 & 7 & 1 \\
6 & 7 & 8 & 1 \\
7 & 8 & 9 & 1 \\
8 & 9 & 10 & 1 \\
9 & 10 & 11 & 1 \\
10 & 11 & 12 & 1 \\
11 & 12 & 13 & 1 \\
12 & 13 & 14 & 1 \\
\end{array}
\]

- e.g. $m^{(1)} = m^{(2)} \cdot m^{(3)} \cdot m^{(4)} = Y_{12}Y_{12} \cdot Y_{13}Y_{23}Y_{23}Y_{13} \cdot Y_{34}Y_{14} \cdot Y_{45}Y_{25} \in A(C_3)$

To get a set partition, insert spacers and record 1st indices $i$ in $Y_{ij}$ to get restricted growth function.

\[
\begin{align*}
1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 3 & 3 & 1 & 4 & 4 & 2 & 5 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\end{align*}
\]

$m \rightarrow$ block 1 {1, 2, 3, 5, 8, 11} 2 {4, 6, 7, 14} 3 {9, 10} 4 {12, 13}$
6. Properties and Questions

- Branching rule

\[ S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1) \]

generalizes to...

**PROPOSITION:** Both \( A(n)' = A_{os}(n)', A_{vg}(n)' \) have these branching rules for restriction \( G_n \) to \( G_{n-1} \):

\[ A(n)'_i \mid G_n \cong \chi_{\text{def}}^{(n-1)} \otimes A(n)'_{i-1} \mid G_{n-1} \oplus A(n-1)'_i \]
Better phrasing:

**PROPOSITION:** For a supersolvable, $\mathbf{X}$ a modular $\mathbf{X}$, one has an $\mathbf{N}_{\mathbf{X}}$-equivariant s.e.s.
Representation stability

**DEFINITION (Church--Farb 2005)** A sequence \( \{V_n\}_{n \geq 1, 2, \ldots} \) of \( G_n \)-reps is representation-stable if there exist\( x^{(1)}, \ldots, x^{(t)} \) and constants \( c_1, \ldots, c_t \in \mathbb{N} \) such that if \( n \gg 0 \), the \( G_n \)-irreducible decomposition looks like

\[
V_n \cong \bigoplus_{i=1}^{t} \left[ \chi^{(i)} \right] \oplus C_i
\]
THEOREM: For fixed $i = 0, 1, 2, \ldots$

(Church-Farb)

\[ A(n) = \begin{cases} \mathcal{AOS}(n) \\ \mathcal{AVG}(n) \end{cases} \]

both \( A(n) \) have \( \{ A(n) \} \) representation - stable

\[ H^i \text{Conf}(n, X) \] is rep-stable

for certain kinds of manifolds \( X \).
COROLLARY (ARS 2023) For fixed $i=0,1,2,\ldots$

$\{A(n)_i\}$ are also representation-stable

---

proof: Induct on $i$. Virtually we have

$$A(n)_i = \sum_{j=1}^{i} A(n)_j \otimes A(n)_{i-j}$$

rep-stable by Church-Farb

rep-stable by induction

rep-stable by Murnaghan's Stability Thm:

$\chi \otimes \chi$ stabilizes for large $n$. 
QUESTION:
Can we use more FI-module theory to bound the stable range for $n>>0$?

This could help approach ... 

QUESTION:
Can we find exact formulas for $A(n)_i$? 

e.g., generating functions for $\sum_{n,i} \text{ch} A(n)_i \cdot t^i$

involving plethysm or infinite products?
### Boundary cases

**OS**

- $A_{os}(1)$
- $A_{os}(2)$
- $A_{os}(3)$
- $A_{os}(4)$
- $A_{os}(5)$

**VG**

- $A_{vg}(1)$
- $A_{vg}(2)$
- $A_{vg}(3)$
- $A_{vg}(4)$
- $A_{vg}(5)$

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$S(n-1, n-1) = 1 \implies \text{ch } A(n)_{n-1} = s$

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$S(m, 1) = 1 \implies \text{ch } A_{os}(2)_i = s_{\text{even}}$

$\text{ch } A_{vg}(2)_i = \begin{cases} s_{\text{even}} & i \text{ even} \\ s_{\text{odd}} & i \text{ odd} \end{cases}$
\[ S(n, n-1) = \binom{n}{2} \implies \chi \lambda(n)_{n-1} = \begin{cases} \text{for OS} & S_{\frac{n-2}{2}} \quad \text{for VG} \\
-2 & \text{for OS} \\
\text{for VG} \end{cases} \]

\[ S(n, 2) = 2^{n-1} - 1 \implies A_{OS}(3)_{n-1} = 1 + \chi^{P} + \chi^{P} \otimes \chi^{P} + \chi^{P} \otimes \chi^{P} \otimes \chi^{P} + \ldots + \left(\chi^{P}\right)^{\otimes i} \]
Most mysterious ...

**CONJECTURE:** $A_{os}^i(n)!$ is always an $G_n$-permutation representation (!)

Verified for $n = 2, 3, 4$

$i = 0, 1$

(The $G_n$-orbit stabilizers are not all parabolic subgroups $G_{α_1} \times ... \times G_{α_e}$)
Homotopy Lie algebras  (see A. Suciu's talk ?)

\[ \exists \text{ a graded Lie (super)algebra } L = \bigoplus_{d=0}^{\infty} L_d \text{ with } \]

\[ A(n)^\perp = \text{Ext}^{A(n)}(k, k) = U(L) \quad \text{universal enveloping algebra} \]

\[ \cong \ 
\text{PBW Thm.} \ 
\text{Sym}^+ \mathfrak{sl}(L) \quad \text{graded polynomial algebra} \]

\[ \begin{cases} 
\bigoplus_{\lambda=1^{m_1} 2^{m_2} \ldots} \otimes \text{Sym}^{m_i}(L_i) \\
\bigoplus_{\lambda=1^{m_1} 2^{m_2} \ldots} \otimes \bigwedge^{m_i}(L_i) \otimes \bigwedge^{m_i}(L_i) 
\end{cases} \quad \text{for } \mathfrak{A}_{os}(n) \]

\[ \bigotimes_{i=1}^{\infty} \text{Sym}^{m_i}(L_i) \quad \text{for } \mathfrak{A}_{os}(n) \]
So if we understood the $G_n$-representations on $L = \bigoplus_{d=0}^{\infty} L_d$ it would help us understand those on $A(n)$. "

Sadly, calculations for $A_{os}(n) = \text{Sym}(L)$ show that these $L_d$ are not $G_n$-permutation reps?"
Thanks ICMS,
and thank you for your attention!