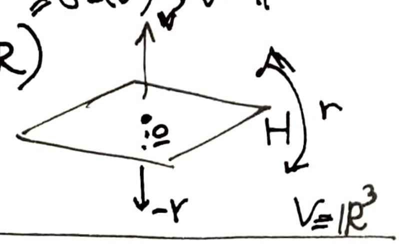
Reflection groups, combinatorics, and invariant theory

AMS Intro to Research Seminar Oct. 22, 2025

OUTLINE: ① Ref'n groups
② Combinatorics
③ Invariant theory

$\boxed{\text{Ref: See my Math 8680 Fall 2022 syllabus \& notes}}$

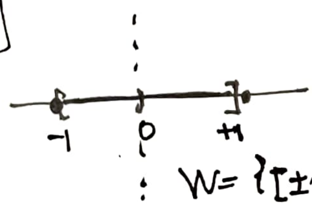
① Ref'n groups $:\overset{\text{DEF}}{=}$ finite subgroups $W \subset GL_n(\mathbb{R})$ $= GL(V)$, $V=\mathbb{R}^n$
generated by reflections r



(MAIN) EXAMPLES:

$W = S_n^{\pm} = \{$ signed $n\times n$ permutation matrices $\} =$ linear symmetries of n-cube $[-1,+1]^n \subset \mathbb{R}^n =: V$

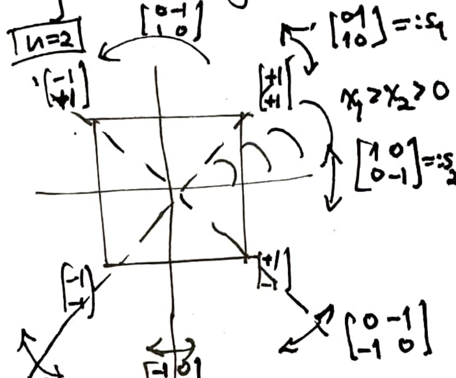
$\boxed{n=1}$



$W = \{[\pm 1]\} \subset GL_1(\mathbb{R})$

$|W| = 2$
$= 2^1 \cdot 1!$

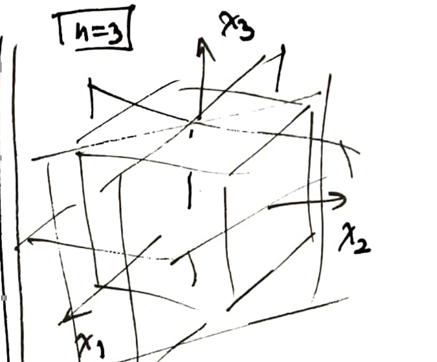
$\boxed{n=2}$



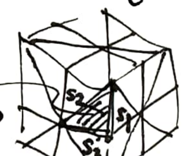
$W = \left\{ \begin{bmatrix} \pm1 & 0 \\ 0 & \pm1 \end{bmatrix}, \begin{bmatrix} 0 & \pm1 \\ \pm1 & 0 \end{bmatrix} \right\}$

$|W| = 8$
$= 2^2 \cdot 2!$

$\boxed{n=3}$



$W = \left\{ \begin{bmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & \pm1 \\ 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \end{bmatrix} \right\}$



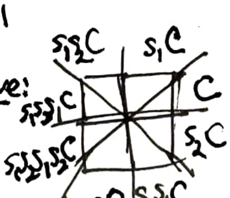
$|W| = 3! \cdot 2^3$
$= n!\,2^n$ in general

FACTS: ① One obtains a nice (Coxeter) presentation for W from the reflections in the walls of a fundamental chamber C := a connected component of $\mathbb{R}^n - \{$all ref'n hyperplanes$\}$
$s_1, s_2, \ldots, s_n$

$$W = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle \quad \text{some } m_{ij} \in \{2,3,\ldots\}$$

e.g. $W = S_n^{\pm} = \langle \underset{(1,2)}{s_1}, \underset{(2,3)}{s_2}, \ldots, \underset{(n-1,n)}{s_{n-1}}, \underset{\binom{n}{-n}}{s_n} \mid s_i^2 = 1,\ (s_i s_j)^2 = 1 \text{ if } |i-j| \geq 2,\ (s_i s_{i+1})^3 = 1 \text{ for } i = 1, \ldots, n-2,\ (s_{n-1} s_n)^4 = 1 \rangle$

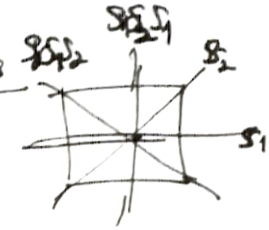
② The map $W \longrightarrow \{$all chambers $C'\}$ is bijective:
$w \longmapsto w(C)$



② Combinatorics

Let's compute the generating functions counting $w \in W$ by 2 interesting statistics: (1st) $\mathrm{codim}_{\mathbb{R}}(\underbrace{V^w}_{\text{fixed space of } w}) := n - \dim_{\mathbb{R}}\{v \in V = \mathbb{R}^n : w(v) = v\}$

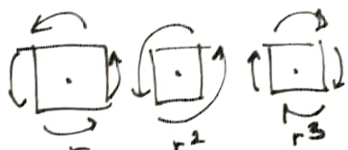
EXAMPLE: ~~length~~ $W = S_2^{\pm}$ has 4 reflections all with $\mathrm{codim}(V^w) = 1$



and 4 rotations, through angles of $0, \pi/4, \pi, 3\pi/4$

$e =$

$\mathrm{codim}(V^e) = 0$



$\mathrm{codim}(V^r) = \mathrm{codim}(V^{r^2}) = \mathrm{codim}(V^{r^3}) = 2$

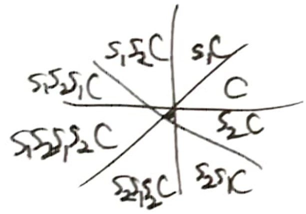
$$\rightsquigarrow \sum_{w \in S_2^{\pm}} t^{\mathrm{codim}(V^w)} = 4t^1 + t^0 + 3t^2 = 1 + 4t + 3t^2 = (1+t)(1+3t)$$

(2nd) (Coxeter) length $\ell(w) := \min\{\ell : w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with s_i from Coxeter generators$\}$

$\underset{\text{FACT}}{=} \#\{$refln hyperplanes separating C from $wC\}$

EXAMPLE:

$W = S_2^{\pm}$



$$\sum_{w \in W} q^{\ell(w)} = q^0 + 2q^1 + 2q^2 + 2q^3 + q^4$$
$$= 1 + 2q + 2q^2 + 2q^3 + q^4$$
$$= (1+q)(1+q+q^2+q^3) = [2]_q[4]_q$$

where $[m]_q = 1+q+q^2+\cdots+q^{m-1}$

THEOREMS (Shephard–Todd 1955) (a) For a reflection group $W \subset GL_n(\mathbb{R})$ $\exists$ magical integers $d_1, d_2, \ldots, d_n$ s.t.

$$\sum_{w \in W} t^{\mathrm{codim}(V^w)} = (1+(d_1-1)t)(1+(d_2-1)t)\cdots(1+(d_n-1)t)$$

(Solomon 1966) (b) $$\sum_{w \in W} q^{\ell(w)} = [d_1]_q [d_2]_q \cdots [d_n]_q$$

EXAMPLE: For $W = S_n^{\pm}$, the $(d_1, d_2, \ldots, d_n) = (2, 4, 6, \ldots, 2n)$

e.g. $W = S_2^{\pm}$ has $(d_1, d_2) = (2, 4)$ $\begin{cases} \rightsquigarrow (1+t)(1+3t) \\ \rightsquigarrow [2]_q[4]_q \end{cases}$

③ Invariant theory - gives the $(d_1, d_2, \ldots, d_n)$

W acts on $V = \mathbb{R}^n$ and acts by linear substitutions of variables on polynomials $\mathbb{R}[x_1, \ldots, x_n] \longrightarrow \mathbb{R}[x_1, \ldots, x_n]$

$$f(\underline{x}) \stackrel{w}{\longmapsto} f(w\underline{x})$$

THEOREM: The W-invariant polynomials $\mathbb{R}[x_1, \ldots, x_n]^W := \{ f(\underline{x}) \in \mathbb{R}[\underline{x}] : w(f) = f \ \forall w \in W \}$

(Shephard-Todd, Chevalley 1955) for finite $W \subset GL_n(\mathbb{R})$ will again be a polynomial subalgebra

$$\mathbb{R}[f_1, f_2, \ldots, f_n]$$

$\Longleftrightarrow$ W is a reflection group

[and then $(d_1, d_2, \ldots, d_n) = (\deg(f_1), \deg(f_2), \ldots, \deg(f_n))$ are the magic numbers]

EXAMPLE: $W = S_n^{\pm}$ acts $\mathbb{R}[x_1, \ldots, x_n]$ by $w(x_i) = \pm x_{w(i)}$

with $\mathbb{R}[x_1, \ldots, x_n]^{S_n^{\pm}} = \mathbb{R}[\underbrace{x_1^2 + \ldots + x_n^2}_{f_1}, \underbrace{x_1^2x_2^2 + x_1^2x_3^2 + \ldots + x_{n-1}^2x_n^2}_{f_2}, \underbrace{x_1^2x_2^2x_3^2 + \ldots}_{f_3}, \ldots, \underbrace{x_1^2x_2^2 \cdots x_n^2}_{f_n}]$

degrees: $(2, 4, 6, \ldots, 2n)$

$= (d_1, d_2, d_3, \ldots, d_n)$.

REMARKS: ① Both $\sum_{w \in W} t^{\mathrm{codim}(V^w)}$, $\sum_{w \in W} q^{\ell(w)}$ have topological interpretations as Poincaré polynomials for interesting spaces X

$X = \mathbb{C}^n - \{\text{complex refln hyperplanes}\}$ | X = flag manifold G/B for associated Lie group G

② Much of the story generalizes to refln groups $W \subset GL_n(\mathbb{C})$

where "reflection" r means $\mathrm{codim}_{\mathbb{C}}((\mathbb{C}^n)^r) = 1$

i.e. r fixes a hyperplane H