LECTURES ON MATROIDS AND ORIENTED MATROIDS

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Abstract. These lecture notes were prepared for the Algebraic Combinatorics in Europe (ACE) Summer School in Vienna, July 2005.

1. Lecture 1: Motivation, examples and axioms

Let's begin with a little "pep talk", some (very) brief history, and some of the motivating examples of matroids.

1.1. **Motivation.** Why learn about or study matroids/oriented matroids in geometric, topological, algebraic combinatorics? Here are a few of my personal reasons.

- They are general, so results about them are widely applicable.
- They have relatively few axioms and standard constructions/techniques, so they focus one's approach to solving a problem.
- They give examples of well-behaved objects: polytopes, cell/simplicial complexes, rings.
- They provide "duals" for non-planar graphs!

1.2. **Brief early history.** (in no way comprehensive ...)

1.2.1. *Matroids.*

- H. Whitney (1932, 1935) - graphs, duality, and matroids as *abstract linear independence*.
- G. Birkhoff (1935) - *geometric lattices* are simple matroids.
- S. Mac Lane (1938) - *abstract algebraic independence* give matroids.
- J. Edmonds and D.R. Fulkerson (1965) - *partial matchings* give matroids.

1.2.2. *Oriented matroids (OM's).*

- J. Folkman and J. Lawrence (1975) - *abstract hyperplane arrangements*
- R. Bland and M. Las Vergnas (1975) - *abstract linear programming*

1.3. **Motivating examples.** Let's start with the first few examples that originally motivated the definition of a matroid, before we actually give the definition.

Thanks to the Algebraic Combinatorics in Europe (ACE) training network for the opportunity to give these lectures, and to the students at the summer school who discovered numerous typos, incorrect exercises, and gave great suggestions for improvement! Particular thanks go to Andrew Berget for a later careful reading.

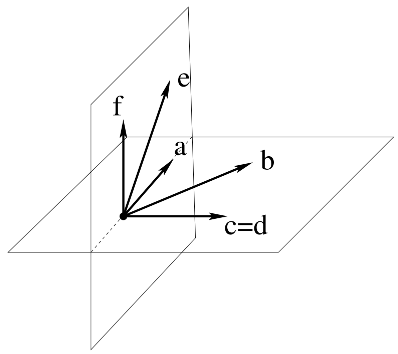

FIGURE 1.

1.3.1. *Motivating example: vector configurations.*

Let $\mathcal{V} = \{v_e\}_{e \in E}$ be a finite set of vectors in some vector space over a field $\mathbb{F}$.

Example 1.
Let $\mathcal{V} = \{a, b, c, d, e, f\}$ be the columns of

$$\begin{matrix} a & b & c & d & e & f \end{matrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Which subsets $I \subset E$ index *linearly independent* sets? Call them $\mathcal{I}$.

$$\begin{aligned} \mathcal{I} = \{&\varnothing, a, b, c, d, e, f, \\ &ab, ac, ad, ae, af, bc, bd, be, bf, ce, cf, de, df, ef, \\ &abe, abf, ace, acf, ade, adf, bce, bcf, bde, bdf, bef, cef, def\} \end{aligned}$$

Which subsets $I \subset E$ index *bases* for the span of $\mathcal{V}$? Call them $\mathcal{B}$.

$$\mathcal{B} = \{abe, abf, ace, acf, ade, adf, bce, bcf, bde, bdf, bef, cef, def\}$$

Some properties of the collection of *independent sets* $\mathcal{I} \subset 2^E$:

I1. $\varnothing \in \mathcal{I}$.
I2. $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$ implies $I_2 \in \mathcal{I}$.

(I1 and I2 together say that $\mathcal{I}$ is an *abstract simplicial complex* on E).

I3. (*Exchange axiom*) $I_1, I_2 \in \mathcal{I}$ and $|I_2| > |I_1|$ implies there exists $e \in I_2 - I_1$ with $I_1 \cup \{e\} \in \mathcal{I}$.

Some properties of the collection of *bases* $\mathcal{B} \subset 2^E$:

B1. $\mathcal{B} \neq \varnothing$.

B2. (*Exchange axiom*) Given $B_1, B_2 \in \mathcal{I}$ and $x \in B_1 - B_2$, there exists $y \in B_2 - B_1$ with
$$(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}.$$

1.3.2. *Motivating example: algebraic independence and transcendence bases (Mac Lane 1938).*

Let $\mathcal{V} = \{f_e\}_{e \in E}$ be a finite subset of vectors of an extension field of a field $\mathbb{F}$.

Example 2.
In the rational function field $\mathbb{F}(x, y, z) \supset \mathbb{F}$, let
$$\begin{aligned} \mathcal{V} = \{a &= x, \\ b &= xy, \\ c &= y, \\ d &= y^2 - 1, \\ e &= \frac{x}{z}, \\ f &= z^2 + 2\} \end{aligned}$$

Which subsets $I \subset E$ index *algebraically independent* subsets?

Which subsets $B \subset E$ index *transcendence bases* for the subfield that $\mathcal{V}$ generates[1] ?

Some of the *minimal algebraic dependences*:
$$\begin{aligned} 0 &= ac - b, \\ 0 &= d - (c^2 - 1), \\ 0 &= a^2 + 2e^2 - fe^2. \end{aligned}$$

Note that we have cooked up this example so that these minimal algebraic dependences involve the same sets of elements as the *minimal linear dependences* in our vector configuration example from before Figure 1.

1.3.3. *Motivating example: forests and spanning trees (Whitney 1932).*

Let $G = (V, E)$ be a finite, connected graph, such as the one in Figure 2.

Which subsets $I \subset E$ index *forests* of edges?

Which subsets $B \subset E$ index *spanning trees* for G?

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[1]Recall from field theory that a transcendence basis for a field extension of $\mathbb{F}' \supseteq \mathbb{F}$ is a set of elements $\{\alpha_i\}$ which are algebraically independent and have the property that the extension $\mathbb{F}' \supseteq \mathbb{F}(\{\alpha_i\})$ is algebraic, i.e. every β in $\mathbb{F}'$ satisfies an algebraic equation with coefficients in $\mathbb{F}(\{\alpha_i\})$.

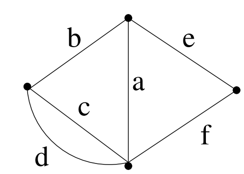


FIGURE 2.

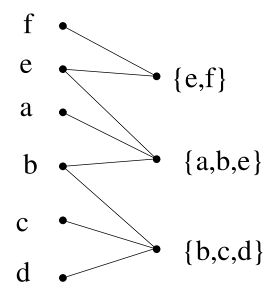


FIGURE 3.

1.3.4. *Historically later example: partial matchings in bipartite graphs (Edmonds and Fulkerson 1965).*

Consider a bipartite graph with vertex bipartition $E \sqcup F$, such as the one in Figure 3.

Which subsets $I \subset E$ can be *matched* along edges into F?

Which subsets $B \subset E$ are the left endpoints of *maximum-size matchings*?

The *exchange* axioms I3 for $\mathcal{I}$, and B2 for $\mathcal{B}$ do hold in this situation, but this is not at all obvious!.

1.4. **Definition.** Finally, the definition of a matroid (Whitney 1935)...

Definition 3.
Say $\mathcal{I} \subset 2^E$ forms the *independent sets* of a *matroid* M on E (and write $\mathcal{I} = \mathcal{I}(M)$) if $\mathcal{I}$ satisfies properties $I1, I2, I3$ from before.

Alternatively, ...

Definition 4.
Say $\mathcal{B} \subset 2^E$ forms the *bases* of a *matroid* M on E (and write $\mathcal{B} = \mathcal{B}(M)$) if $\mathcal{B}$ satisfies properties $B1, B2$ from before.

Example 5.

(1) **Linear matroids**: represented over a field $\mathbb{F}$ by vectors $\{v_e\}_{e\in E}$
$\mathcal{I}$ = linearly independent subsets
$\mathcal{B}$ = bases for their span

(2) **Algebraic matroids**: represented over a field $\mathbb{F}$ by elements of $\{f_e\}_{e\in E}$ of an extension field
$\mathcal{I}$ = algebraically independent subsets
$\mathcal{B}$ = transcendence bases for the subfield they generate

(3) **Graphic matroids**: represented by a (connected) graph $G = (V, E)$
$\mathcal{I}$ = forests of edges
$\mathcal{B}$ = spanning trees

(4) **Transversal matroids:** represented by a bipartite graph on vertex set $E \sqcup F$
$\mathcal{I}$ = endpoints in E of partial matchings
$\mathcal{B}$ = endpoints in E of maximum size matchings

In Exercise 1, you are shown how to prove these impliciations:

Graphic $\searrow$

Linear $\rightarrow$ **Algebraic**

Transversal $\nearrow$

1.5. **Other axiom systems.** One of the features of matroids that makes them flexible (and occasionally frustrating to the novice) is that they have several equivalent axiom systems. Here are a few notable ones...

1.5.1. *Circuits.*

The *circuits* $\mathcal{C}$ of a matroid are the inclusion-minimal *dependent sets* (= sets not in $\mathcal{I}$).

Example 6.
The circuits $\mathcal{C}$ consists of $\{abc, abd, cd, aef, bcef, bdef\}$ in the example we have been using so far.

Definition 7.
Say $\mathcal{C} \subset 2^E$ forms the *circuits* of a *matroid* M on E (and write $\mathcal{C} = \mathcal{C}(M)$) if $\mathcal{I}$ satisifies these three axioms:

C1. $\varnothing \notin \mathcal{C}$.
C2. $C_1, C_2 \in \mathcal{C}$ and $C_1 \subset C_2$ implies $C_1 = C_2$.
C3. (*Circuit elimination*) $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ implies there exists $C_3 \in \mathcal{C}$ with $C_3 \subset (C_1 \cup C_2) - \{e\}$.

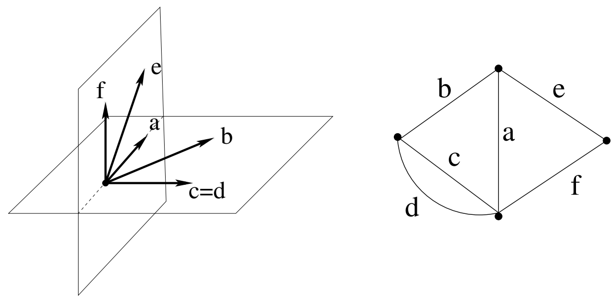


FIGURE 4

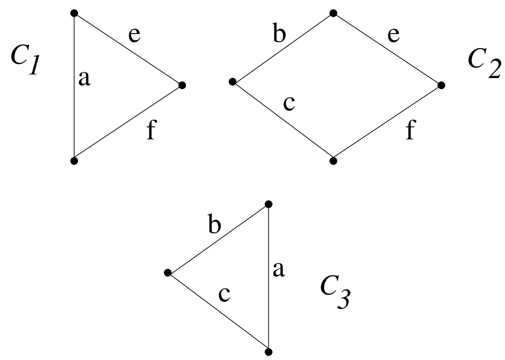


FIGURE 5

Example 8.
Here is an example of circuit elimination for vector configurations:

$$\begin{gathered} C_1 : 0 = a - e + f \\ C_2 : 0 = -b + c + e - f \\ \text{implies} \\ C_3 : 0 = a - b + c \end{gathered}$$

Exercise 4 asks you to prove the equivalence between the circuit axioms and the independent set axioms.

1.5.2. *The semimodular rank function.*

Given a matroid M on E, for $A \subset E$ define the *rank function*

$$r(A) := \max\{|I| : I \in \mathcal{I}(M) \text{ and } I \subseteq A\}.$$

It satisfies these *rank axioms*:

R1. $0 \leq r(A) \leq |A|$.
R2. $A_1 \subset A_2$ implies $r(A_1) \leq r(A_2)$.
R3. *(Semimodularity)*

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

One can recover the independent sets from these as follows:

$$\mathcal{I} = \{I \subset E : r(I) = |I|\}.$$

1.5.3. *The exchange closure operation.*

The *closure* operation

$$\overline{A} := \{e \in E : r(A \cup \{e\}) = r(A)\}$$

satisfies these *matroid/exchange closure axioms*:

CL1. $A \subseteq \overline{A}$.
CL2. $\overline{\overline{A}} = \overline{A}$.
CL3. $A_1 \subseteq A_2$ implies $\overline{A_1} \subseteq \overline{A_2}$.

(CL1,CL2,CL3 together say that $A \mapsto \overline{A}$ is a *closure operator* on 2^E.)

CL4. *(Exchange)* If $x, y \in E$ and $A \subset E$ have $y \in \overline{A \cup \{x\}} - \overline{A}$ then $x \in \overline{A \cup \{y\}}$.

Closed sets (those with $\overline{A} = A$) are called *flats* of M.

1.5.4. *The geometric lattice of flats.*

The poset $L(M)$ of all flats of M, ordered by inclusion, is a *geometric lattice*,

meaning that it is

- a *lattice*– it has *meets* $x \wedge y$, *joins* $x \vee y$,
- *(upper-)semimodular*– it is ranked, and satisfying

$$r(x \vee y) + r(x \wedge y) \leq r(x) + r(y),$$

- *atomic* – every $x \in L(M)$ is the join of the *atoms* below it.

Theorem 9. *(G. Birkhoff) Geometric lattices = posets of flats of* **simple** *matroids.*

A matroid is *simple* if it has no

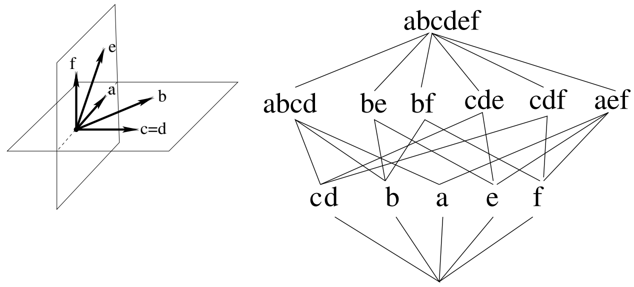


FIGURE 6

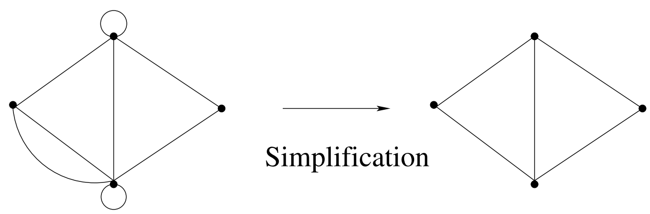


FIGURE 7

- *loops* (= elements $e \in E$ lying in **no** elements of $\mathcal{I}$, or equivalently, $e \in \overline{\varnothing}$), nor
- *parallel* elements (= elements $e, e' \in E$ with $e' \in \overline{\{e\}}$)

To every matroid M on ground set E, one can associate a simple matroid $\hat{M}$ by removing loops, and letting the ground set $\hat{E}$ of $\hat{M}$ be the set of parallelism classes of E, with obvious independent sets, bases, closure, etc. One calls $\hat{M}$ the *simplification* of M, and there is an obvious poset isomorphism $L(\hat{M}) \cong L(M)$.

One way of restating Birkhoff's Theorem 9 is as follows. There is a map $L \mapsto \hat{M}$ backward from geometric lattices to simple matroids, in which the ground set E of $\hat{M}$ is the set of atoms of L, and a subset of atoms I is independent if the rank in L of the join $\bigvee_{e \in I} e$ is the cardinality $|I|$. Then the composite map from matroids to matroids

$$M \mapsto L(M) \mapsto \hat{M}$$

is exactly the simplification map $M \mapsto \hat{M}$ from above.

Finite geometric lattices fit into a natural hierarchy for properties of finite lattices ...

LATTICE PROPERTY		... PLUS ATOMIC
distributive = ideals of a poset	$\Leftarrow$	atomic distributive = Boolean algebra
$\Downarrow$		$\Downarrow$
modular	$\Leftarrow$	atomic modular = products of *projective geometries* : $\begin{cases} \mathbb{P}(\mathbb{F}_q^r) & r \neq 3 \\ \text{proj. plane} & r = 3 \end{cases}$
$\Downarrow$		$\Downarrow$
semimodular	$\Leftarrow$	atomic semimodular i.e. *geometric* = simple matroid

From the viewpoint of geometric combinatorics, there are a few further interesting ways to characterize matroids. We mention some of these here.

1.6. **Simplicial complexes.** Recall that axioms $I1, I2$ for the independent sets $\mathcal{I} = \mathcal{I}(M)$ of a matroid M on E are equivalent to $\mathcal{I}$ being an *abstract simplicial complex* on E. One can replace the exchange axiom $I3$ with various others.

One relates to purity of its vertex-induced subcomplexes. Recall that a simplicial complex is *pure* (of dimension $r-1$) if every maximal face has the same cardinality r. For example, it can replaced (see Exercise 5) with this axiom:

> **I3′**. For every subset $A \subset E$, the restriction
> $$\mathcal{I}|_A := \{I \in \mathcal{I} : I \subset A\}$$
> is a *pure* simplicial complex.

Example 10.
The matroid on $E = \{a, c, d, e, f\}$ represented linearly by the columns of this matrix

$$\begin{matrix} a & c & d & e & f \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

has bases $\mathcal{B}(M) = \{ace, acf, ade, adf, cef, def\}$. Its simplicial complex of independent sets $\mathcal{I}$ is the boundary complex of the bipyramid shown in Figure 8.

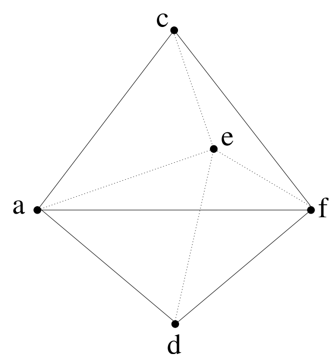


FIGURE 8.

Another characterization of $\mathcal{I}(M)$ relates to shellability. Recall that a simplicial complex is *shellable* if there exists an ordering $F_1, F_2, \ldots, F_t$ of its *facets* (= maximal faces) having the following property: for each $j \geq 2$, the facet F_j intersects the subcomplex generated by the previous facets $F_1, \ldots, F_{j-1}$ in a subcomplex of F_j which is *pure of codimension one* inside F_j. Then one can also replace $I3$ by this axiom:

> **I3$''$**. $\mathcal{I}$ is a pure simplicial complex, and every linear ordering on E makes the lexicographic ordering on its maximal elements (the bases $\mathcal{B}$) into a *shelling order* on $\mathcal{I}$.

For example, the ordering of $\mathcal{B}(M) = \{ace, acf, ade, adf, cef, def\}$ as in the previous example is lexicographic for $a < c < d < e < f$, and one can check that this shells the boundary of the bipyramid in Figure 8.

1.7. **The greedy algorithm.** Another characterization of $\mathcal{I}(M)$ relates to optimization. Consider the problem of finding a set $I \in \mathcal{I}$ of maximum weight

$$w(I) := \sum_{e \in I} w(e)$$

with respect to some arbitrary (nonnegative) weight function $w : E \to \mathbb{R}_+$.

> **(Kruskal's) greedy algorithm:**
> Initialize $I_0 = \varnothing$. Having constructed I_{j-1}, let $I_j := I_{j-1} \cup \{e_0\}$ where e_0 has the maximum weight $w(e_0)$ among all elements of the set
> $$\{e \in E - I_{j-1} : I_{j-1} \cup \{e\} \in \mathcal{I}\},$$
> assuming this set is non-empty. If this set is empty, stop and return $I = I_{j-1}$.

Then one can also replace $I3$ by this axiom:

> **I3′**. The greedy algorithm always finds a set $I \in \mathcal{I}$ achieving the maximum weight, regardless of the choice of nonnegative weight function w.

Example 11.
Here is a *non-example*, that is a simplicial complex which looks perfectly nice, but is not a matroid complex because it fails to satisfy any of the axioms $I3', I3'', I3'''$.

Let Δ be the pure 2-dimensional simplicial complex with facets $\{124, 245, 235\}$. Then one can check that

- the restriction $\Delta|_{\{1,2,3,4\}}$ to vertex set $\{1,2,3,4\}$ has facets $\{124, 23\}$, so is *not* pure.
- the lex order on the facets would order them $(F_1, F_2, F_3) = (124, 235, 245)$, which is *not* a shelling order: the intersection of $F_2 = 235$ with the subcomplex generated by $F_1 = 124$ is the vertex 2, and this does not have codimension 1 within F_2.
- if one weights the vertices $1, 2, 3, 4, 5$ by $99, 1, 100, 98, 2$ then the greedy algorith will try to build up a maximum weight subset I of Δ as follows: $I_0 = \varnothing, I_1 = 3, I_2 = 35, I_3 = 235$. This finds the subset $I_3 = 235$ of weight $1 + 100 + 2 = 103$, but *misses* the (unique) maximum weight subset of Δ, namely $I_{\max} = 124$ of weight $99 + 1 + 98 = 198$.

The last axiomatization relates to the greedy algorithm and bases. Let ω be an arbitrary linear ordering on E, and then define the associated *Gale ordering* on its r-subsets $\binom{E}{r}$ to be the following: let

$$B = \{b_1 <_\omega \cdots <_\omega b_r\}$$
$$B' = \{b'_1 <_\omega \cdots <_\omega b'_r\}$$

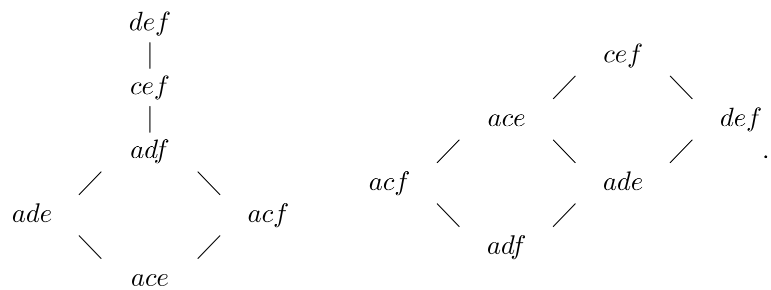
and say $B \leq_\omega B'$ if $b_i \leq b'_i$ for $i = 1, 2, \ldots, r$.

Then the exchange axiom $B2$ for bases of a matroid can be replaced by this axiom:

> **B2′**. The collection $\mathcal{B}$ consists of elements of a fixed cardinality r, and for every choice of linear ordering ω on E, the collection $\mathcal{B}$ has a maximum (and minimum) element under the associated Gale ordering on $\binom{E}{r}$.

Example 12.
For the bases $\mathcal{B}(M) = \{ace, acf, ade, adf, cef, def\}$ of the matroid M in Example 10, the Gale ordering on $\mathcal{B}(M)$ associated with two different linear orderings ω_1, ω_2 is shown. In both cases, there is a unique maximum and minimum base.

$\omega_1 : a < c < d < e < f$ $\qquad\qquad$ $\omega_2 : a < d < f < c < e$



These last two axiomatizations have been the source of multiple threads in matroid theory, including

- matroids and semimodular functions in *optimization*,
- *greedoids* – set systems that are not quite simplicial complexes but still satisfy $I3'''$; see [15].
- *Coxeter matroids* – one replaces $\binom{E}{r}$ with the cosets W/W_J of a parabolic subgroup W_J in a Coxeter system (W, S) and replaces the Gale ordering associated to ω with a W-translate of the Bruhat order on W/W_J. See the recent monograph by Borovik, Gelfand and White [4].

1.8. **Oriented matroids.** In two of our motivating families of examples of matroids, the circuits $\mathcal{C}$ could have recorded more data about *signs/orientations*, as we now explain.

In a graphic matroid coming from a graph $G = (V, E)$, one must first pick an arbitrary orientations for the edges to make it a directed graph. One then obtains the *signed circuits* $\mathcal{C}$ from the directed cycles in G; the plus/minus signs tell whether edges are traversed in the directed cycle agreeing/disagreeing with the chosen inital orientation. E.g., if G is the digraph shown in Figure 10, then

$$\mathcal{C} = \left\{ \overset{+-+}{a\,b\,c}, \overset{+-+}{a\,b\,d}, \overset{+-+}{a\,e\,f}, \overset{+-}{c\,d}, \overset{+--+}{b\,c\,e\,f}, \overset{+--+}{b\,d\,e\,f}, \right.$$
$$\left. \overset{-+-}{a\,b\,c}, \overset{-+-}{a\,b\,d}, \overset{-+-}{a\,e\,f}, \overset{-+}{c\,d}, \overset{-++-}{b\,c\,e\,f}, \overset{-++-}{b\,d\,e\,f} \right\}$$

In a matroid represented over $\mathbb{R}$, one obtains signed circuits $\mathcal{C}$ from recording the signs of coefficients in the minimal linear dependences. For example, from the vector configuration in Figure 9, one has these minimal dependences and corresponding signed circuits:

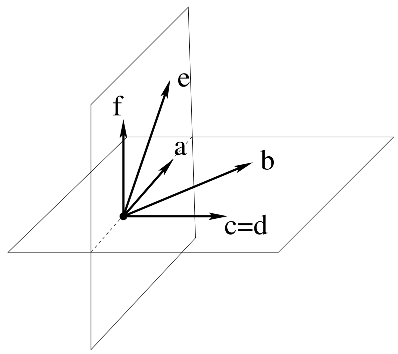


FIGURE 9.

$$0 = a - b + c \quad \rightsquigarrow \quad \begin{matrix} + & - & + \\ a & b & c \end{matrix}$$

$$0 = -a + b - c \quad \rightsquigarrow \quad \begin{matrix} - & + & - \\ a & b & c \end{matrix}$$

$$0 = a - e + f \quad \rightsquigarrow \quad \begin{matrix} + & - & + \\ a & e & f \end{matrix}$$

$$0 = c - d \quad \rightsquigarrow \quad \begin{matrix} + & - \\ c & d \end{matrix}$$

$$0 = b - c - e + f \quad \rightsquigarrow \quad \begin{matrix} + & - & - & + \\ b & c & e & f \end{matrix}$$

$$0 = b - d - e + f \quad \rightsquigarrow \quad \begin{matrix} + & - & - & + \\ b & d & e & f \end{matrix}$$

How do the circuit axioms for matroids morph into signed circuit axioms for oriented matroids? First let's establish some alternate terminology for signed circuits: a signed circuit C can be thought of as a *signed subset* $C = (C^+, C^-)$ of E, as illustrated in the example $C = \begin{matrix} + & - & + \\ a & b & c \end{matrix} = (\{a, c\}, \{b\})$. Its underlying subset is $\underline{C} = C^+ \sqcup C^-$, e.g. $\underline{C} = \{a, b, c\}$ in this example.

Definition 13.
A collection $\mathcal{C}$ of signed subsets of a finite set E forms the *circuits* of an *oriented matroid* $\mathcal{M}$ on E (and say $\mathcal{C} = \mathcal{C}(\mathcal{M})$) if it satisfies these three axioms:

C0. $\varnothing \; (:= (\varnothing, \varnothing)) \notin \mathcal{C}$.
C1. $\mathcal{C} = -\mathcal{C}$.
C2. $C_1, C_2 \in \mathcal{C}$ and $\underline{C_1} \subset \underline{C_2}$ implies $C_1 = \pm C_2$.

C3. (*Signed circuit elimination*) $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq -C_2$ and $e \in C_1^+ \cap C_2^-$ implies there exists $C_3 \in \mathcal{C}$ with

$$C_3^+ \subset (C_1^+ \cup C_2^+) - \{e\},$$
$$C_3^- \subset (C_1^- \cup C_2^-) - \{e\}.$$

Example 14.
In our example of circuit elimination from before:

$$C_1 : 0 = a - e + f$$
$$C_2 : 0 = -b + c + e - f$$
$$\text{implies}$$
$$C_3 : 0 = a - b + c$$

one sees how the signs get carried along: $C_3 = (C_3^+, C_3^-) = (\{a,c\},\{b\})$ has

$$C_3^+ = ac \subset (C_1^+ \cup C_2^+) - \{e\} = acf,$$
$$C_3^- = b \subset (C_1^- \cup C_2^-) - \{e\} = bf.$$

Some other axiomatizations of oriented matroids ...

1.9. **Covectors.** Given the vector configuration $\mathcal{V} = \{v_e\}_{e \in E}$ in a real vector space V, consider the hyperplane arrangement $\mathcal{A}$ in the dual space $V*$ whose hyperplanes $H_e = v_e^\perp$ are defined by $(f, v_e) = 0$ (where here $(-,-)$ denotes the canonical pairing $V^* \times V \to \mathbb{R}$.) This hyperplane arrangement decomposes regions of V^* into various cones/cells according the sign pattern in $\{\pm 1, 0\}^E$ of the various functionals f as they evaluate on the vectors v_e for $e \in E$. The sign patterns which occur are called the *covectors* of the oriented matroid $\mathcal{M}$ associated to $\mathcal{V}$.

Example 15.
Figure 11 below depicts the arrangement associated with the vectors $\mathcal{V}$ given by the columns of this matrix:

$$\begin{matrix} a & b & c & d & e & f \end{matrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Here the hyperplanes H_e (and the regions into which they decompose V^*) are shown as intersected with a sphere about the origin in V^*, with a small vector indicating the side of the hyperplane H_e on which the functionals f have $f(v_e) > 0$. A few of the covectors have also been labelled by their sign pattern.

The collection of all covectors of $\mathcal{M}$ satisfy a set of axioms (the *covector axioms*, which we won't write down here, but can be found in [3]) which can also be used as an equivalent characterization/definition of oriented matroids.

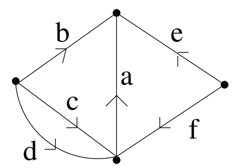


FIGURE 10.

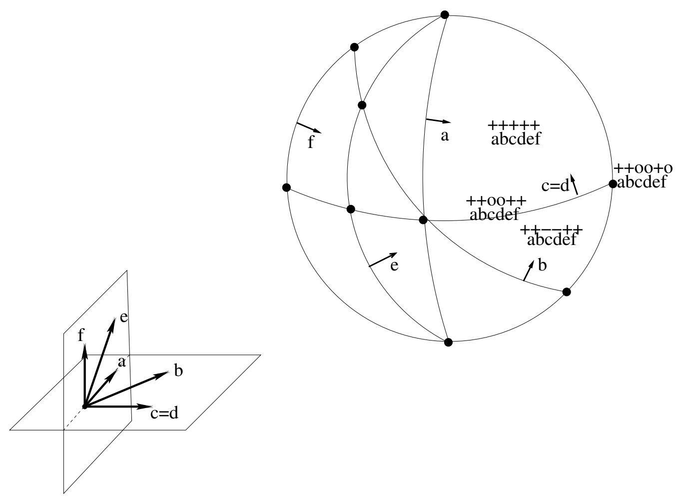


FIGURE 11.

What is perhaps more important to point out here is the connection between oriented matroids and the theory of *hyperplane arrangements*. Given an arbitrary hyperplane arrangement $\mathcal{A}$ in a real vector space $\mathbb{R}^r$, there are various combinatorial, geometric, topological and algebraic invariants one can associate to it. In particular, if one considers the same (*complexified*) arrangement $\mathcal{A}_{\mathbb{C}}$ within the complex space $\mathbb{C}^r$, the complement $\mathbb{C}^r - \mathcal{A}_{\mathbb{C}}$ can have very interesting topology, and has received much scrutiny since the second half of the 20th century. A central theme in this subject has been the question(s) of which of these various invariants can be computed purely in terms of the matroid data, or in terms of the oriented matroid data, associated to the arrangement. As examples of some famous answers (to be discussed a little further in a later lecture) for the complex complement $\mathbb{C}^r - \mathcal{A}_{\mathbb{C}}$,

- the integral cohomology ring structure depends only upon the (unoriented) matroid data, and has a very simple presentation given by the Orlik-Solomon algebra, while

- its entire homotopy type can be computed from the oriented matroid data using a simplicial complex defined by Salvetti, and even its homeomorphism type can be recovered from a construction of Björner and Ziegler, however,
- already its fundamental group *cannot* be computed purely from the (unoriented) matroid, as shown originally by an example of Rybnikov.

It is worth mentioning also that the *Folkman-Lawrence* representation theorem, to be discussed later, shows that oriented matroids are *almost* the same things as real hyperplane arrangements: the covectors of an oriented matroid of rank r always come from an arrangement of *pseudospheres* (= "wiggly, non-linear" codimension 1 spheres) inside a sphere of dimension $r-1$.

1.10. **Chirotopes.** Another way to record finer sign data than just linear dependence is to look at signs of determinants. Without loss of generality, if our oriented matroid $\mathcal{M}$ has rank r and comes from a vector configuration $\mathcal{V} = \{v_e\}_{e \in E}$ in a real vector space V, we may take $V = \mathbb{R}^r$, and assume that the v_e are the columns of some $r \times n$ matrix over $\mathbb{R}$, where $n := |E|$. By abuse of notation, call this matrix $\mathcal{M}$. Then the *chirotope* data associated to $\mathcal{M}$ is the following function

$$\chi : \{ \text{ ordered sequences } (e_1, \ldots, e_r) : e_i \in E\} \to \{\pm 1, 0\}$$

defined by

$$\begin{aligned} \chi(e_1, \ldots, e_r) &= \operatorname{sign} \det \left(\mathcal{M}|_{\text{columns } e_1, \ldots, e_r}\right) \\ &= \begin{cases} \pm 1 & \text{if } \{e_1, \ldots, e_r\} \in \mathcal{B}(M) \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

This data satisfies certain axioms, coming from the fact that determinants are *alternating*, and that they satisfy certain *Plücker syzygies*. We won't state these here (see [3]), but for example, the syzygy

$$\det(a, b, e) \det(c, e, f) = \det(c, b, e) \det(a, e, f) + \det(f, b, e) \det(c, e, a)$$

implies that if abe, cef are both bases of the matroid, then either cbe, aef are bases, or both bef, ace are bases. And taking into account signs, if the chirotope for the oriented matroid satisfies

$$\chi(a, b, e)\chi(c, e, f) = -1,$$

then either

$$\chi(c, b, e)\chi(a, e, f) = -1$$

or

$$\chi(f, b, e)\chi(c, e, a) = -1$$

(or both) must also hold.

Remark 16.
The reference to Plücker syzygies suggests a useful alternate viewpoint on vector configurations $\mathcal{V}$ of n vectors in $\mathbb{F}^r$. The row space of the $r \times n$ matrix $\mathcal{M}$ is an r-plane in $\mathbb{F}^n$, which can be viewed as a point in the *Grassmannian* $\mathbb{G}r(r, \mathbb{F}^n)$. The *Plücker embedding* embeds $\mathbb{G}r(r, \mathbb{F}^n) \hookrightarrow \mathbb{P}_{\mathbb{F}}^{\binom{n}{r}-1}$ by writing down the $\binom{n}{r}$ homogeneous Plücker coordinates

$$p_{e_1,\ldots,e_r}(\mathcal{M}) := \det\left(\mathcal{M}|_{\text{columns } e_1,\ldots,e_r}\right).$$

The basis form of the *matroid* data associated to $\mathcal{M}$ simply records which Plücker coordinates are non-zero. If $\mathbb{F} = \mathbb{R}$ or any ordered field, then the chirotope form of the *oriented matroid* data simply records the signs of the Plücker coordinates. From this viewpoint, matroids and oriented matroids give a natural way of decomposing Grassmannians into "strata". We'll return to this later.

2. Lecture 2: Constructions, representations, and realizations

In this lecture we'll look at the most basic constructions for building new matroids OM's from old ones. This highlights the notion of *minors*, which play a role in deciding what kinds of representations are possible for a matroid or OM.

2.1. The most basic constructions.

2.1.1. *Direct sum (boring).*

Given matroids M_1, M_2 on ground sets E_1, E_2, their *direct sum* $M_1 \oplus M_2$ is the matroid on ground set $E_1 \sqcup E_2$ having independent sets

$$\mathcal{I}(M_1 \oplus M_2) = \mathcal{I}(M_1) \times \mathcal{I}(M_2)$$

or bases

$$\mathcal{B}(M_1 \oplus M_2) = \mathcal{B}(M_1) \times \mathcal{B}(M_2)$$

It models vector configurations $\mathcal{V}_1, \mathcal{V}_2$ in vector spaces V_1, V_2 being put together as $(\mathcal{V}_1 \oplus 0) \sqcup (0 \oplus \mathcal{V}_2)$ inside $V_1 \oplus V_2$. For two graphs G_1, G_2, it models the *disjoint union* $G_1 \sqcup G_2$ or the *wedge* $G_1 \vee_v G_2$ (obtained by gluing G_1 and G_2 at some common vertex v):

$$M(G_1 \sqcup G_2) = M(G_1 \vee_v G_2) = M(G_1) \oplus M(G_2).$$

For the associated lattices of flats, one has

$$L(M_1 \oplus M_2) = L(M_1) \times L(M_2).$$

Equally simple/boring things happen in the oriented matroid setting.

2.1.2. *Deletion (boring).*

Given a matroid M on E, and $e \in E$, one says that e is an *isthmus* (or *coloop*) if e lies in every base $B \in \mathcal{B}(M)$, or if it can be added to every independent set I, with $I \cup \{e\}$ remaining independent. If e is **not a coloop**, define the *deletion* $M \backslash e$ to be the matroid on ground set $E - \{e\}$ having independent sets

$$\mathcal{I}(M \backslash e) = \{I \in \mathcal{I}(M) : e \notin I\}$$

or bases

$$\mathcal{B}(M \backslash e) = \{B \in \mathcal{B}(M) : e \notin B\}.$$

More generally, one can delete a subset from a matroid or restrict a subset from a matroid: Given $A = \{e_1, \ldots, e_k\} \subset E$, one has the deletion

$$M \backslash A := ((M \backslash a_1) \backslash a_2) \cdots \backslash a_k$$

or the *restriction*

$$M|_A := M \backslash (E - A).$$

Deletion clearly models removing vectors from a vector configuration, or removing edges from a graph. If it happens that A is a flat of the matroid, then on the level of lattices of flats one has that $L(M|_A)$ is the *lower interval* $[\hat{0}, A]$

within the lattice $L(M)$. Again, the oriented matroid counterparts are equally simple/boring.

2.1.3. *Contraction (seems, a piori, less boring).*

Given a matroid M on E, and $e \in E$, one says that e is a *loop* if e lies in none of the independent sets $I \in \mathcal{I}(M)$, or equivalently, in none of the bases $B \in \mathcal{B}(M)$. If e is **not a loop**, define the *contraction* M/e to be the matroid on ground set $E - \{e\}$ having independent sets

$$\mathcal{I}(M/e) = \{I - \{e\} : e \in I \in \mathcal{I}(M)\}$$

or bases

$$\mathcal{B}(M/e) = \{B - \{e\} : e \in B \in \mathcal{B}(M)\}.$$

More generally, one can contract on a subset: Given $A = \{e_1, \dots, e_k\} \subset E$, one has the contraction

$$M/A := ((M/a_1)/a_2) \cdots /a_k.$$

The terminology comes from graph theory, where one can contract a (non-loop) edge e from a graph $G = (V, E)$ to form a contracted graph G/e; see Figure 12(a). The forests/trees in G/e biject with the forests/trees in G that contain e, that is, $M(G/e) = M(G)/e$.

For the matroid M associated to a vector configuration $\mathcal{V} = \{v_e\}_{e \in E}$ in a vector space V, contraction models quotients or projections: given $A \subset E$, let V_A be the linear span of $\{v_e\}_{e \in A}$, and $\pi : V \to V/V_A$ the canonical quotient mapping (or if one prefers, $\pi : V \to V_A^\perp$ is orthogonal projection with respect to some nondegenerate bilinear form on V). Then the contracted matroid M/A is the matroid associated to the vector configuration $\{\pi(v_e)\}_{e \in E - A}$ in the quotient space V/V_A (or in $V_A^\perp$). See Figure 12(b).

Thinking in terms of the hyperplane arrangement $\mathcal{A}$ in V^* associated to $\mathcal{V}$, the contraction M/e corresponds to the *restriction* hyperplane arrangement $\mathcal{A}|_{H_e}$ within the codimension 1 linear subspace H_e, whose hyperplanes are $\{H_{e'} \cap H_e\}_{e' \in E - \{e\}}$. See Figure 12(c). This also suggests how one achieves contraction on e at the oriented matroid level: the covectors of $\mathcal{M}/e$ should be obtained from the covectors f of $\mathcal{M}$ having $f(e) = 0$ by restricting them to their values on $E - \{e\}$.

On the level of lattices of flats, one has that $L(M/A)$ is the *upper interval* $[\overline{A}, \hat{1}]$ within the lattice $L(M)$, where we recall that $\overline{A}$ is the closure of (or flat spanned by) A.

2.1.4. *Duality/Orthogonality (***Fascinating!***)*

Given a matroid M on ground set E, its *dual* (or *orthogonal*) matroid $M^\perp$ is defined by

$$\mathcal{B}(M^\perp) := \{E - B : B \in \mathcal{B}(M)\}$$

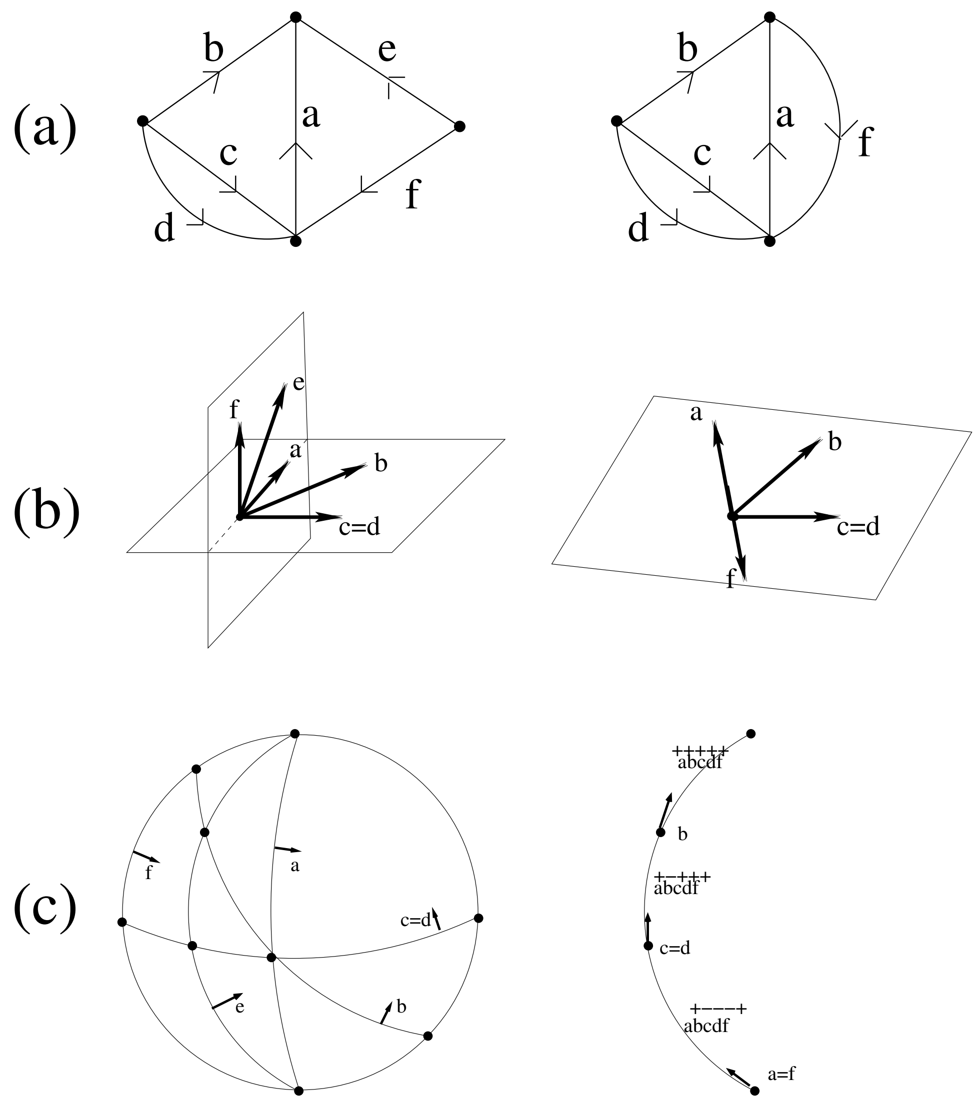


FIGURE 12. Three views on the contraction $\mathcal{M}/e$: (a) Contracting the edge e from the (directed) graph G. (b) Quotienting by the span of v_e, or projecting on $v_e^{\perp}$. (c) Restricting the arrangement to the hyperplane H_e.

It is *not* trivial to check that the basis axioms still hold for $\mathcal{B}(M^{\perp})$. One must show that the basis axioms (or independent set axioms or circuit axioms) for M imply a different version of the basis exchange axiom for M: given bases B_1, B_2

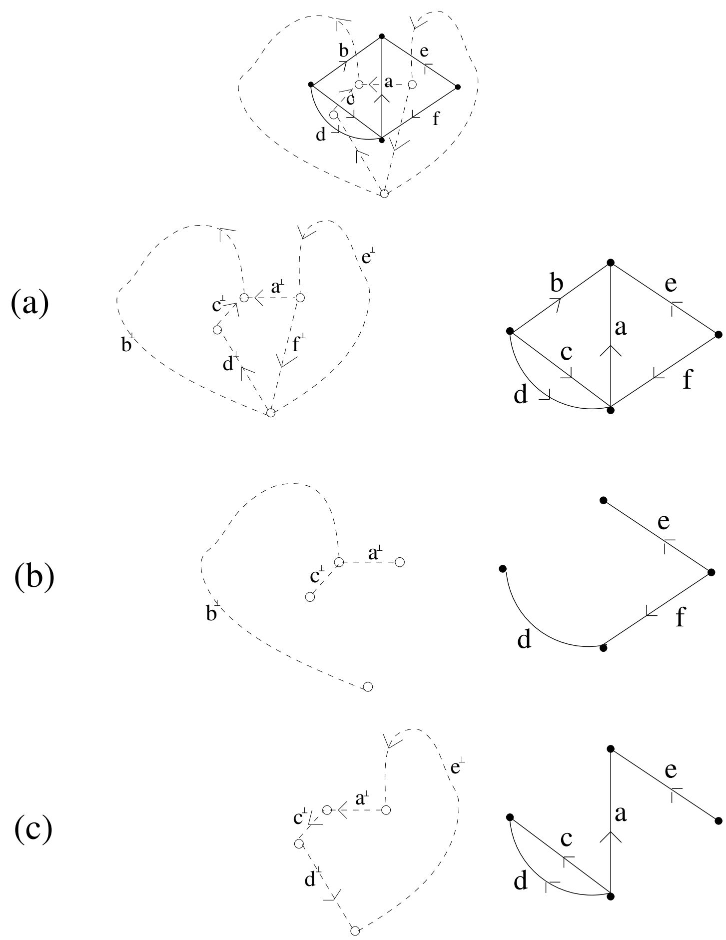


FIGURE 13. Duality of planar (oriented) graphs. A (directed) graph G, and its planar dual $G^{\perp}$. (a) The associated dual orientation, which is *totally cyclic*, because the original orientation is *acyclic*. (b) A pair of dual spanning trees. (c) A directed *cycle* and its dual directed *bond*.

in $\mathcal{B}(M)$ and $b_2 \in B_2$, there exists $b_1 \in B_1$ for which $(B_1 - \{b_1\}) \cup \{b_2\}$ is again a base in $\mathcal{B}(M)$.

At the oriented matroid level, one can define the associated *dual chirotope* $\chi_{\mathcal{M}^{\perp}}$. One first prescribes an arbitrary linear ordering ω on E, and then defines for $\{e_1, \ldots, e_{n-r}\} \subset E$ having complementary set $\{e'_1, \ldots, e'_r\}$

$$\chi_{\mathcal{M}^{\perp}}(e_1, \ldots, e_{n-r}) := \operatorname{sign}(e_1, \ldots, e_{n-r}, e'_1, \ldots, e'_r)\chi_{\mathcal{M}}(e'_1, \ldots, e'_r) \tag{1}$$

where the sign above is the sign of the permutation that sorts $(e_1, \ldots, e_{n-r}, e'_1, \ldots, e'_r)$ into the ω-order.

This models duality of planar graphs $G, G^\perp$, even taking into account edge orientations. One must use the convention that every edge e of G and its crossing edge $e^\perp$ in $G^\perp$ are oriented compatibly, so that locally at their crossing point they look like a positively oriented basis for the plane $\mathbb{R}^2$. See Figure 13.

Many of the wonderful features of duality of planar graphs extend to matroids and OM's:

- *Spanning trees/bases* are complementary to *dual spanning trees/dual bases*: a subset $T \subset E(G)$ forms a spanning tree for G if and only if the complementary set
$$(E - T)^\perp := \{e^\perp : e \in E - T\}$$
forms a spanning tree for $G^\perp$.
- *Deletion* is dual to *contraction*: $(M/e)^\perp = M^\perp \backslash e^\perp$.
- *Loops* are dual to *isthmes* (*coloops*): e lies in every base of M if and only if it lies in no base of $M^\perp$.
- *Circuits* are dual to *cocircuits* (= covectors of minimal support). A circuit in the graphic (oriented) matroid $\mathcal{M}(G)$ corresponds to a directed cycle. In the dual $\mathcal{M}(G^\perp) = \mathcal{M}(G)^\perp$ this corresponds to a *directed bond*, that is, a collection of directed edges which go across a bipartition of the vertices of $G^\perp$ (directed from the vertices of $G^\perp$ inside the original cycle to those outside it).
- *Acyclic orientations* are dual to *totally cyclic orientations*. It is not hard to see (Exercise 2) that acyclic orientations of the edges of a graph G naturally biject with the top-dimensional cones/cells/chambers in the decomposition of space given by the associated graphic hyperplane arrangement; in the oriented matroid these correspond to covectors f of $\mathcal{M}(G)$ which are non-zero ($+$ or $-$) on every $e \in E$, also called *topes*.

  A *totally cyclic orientation* is one in which every edge lies in some directed cycle; the corresponding OM concept is that of a *vector* (= sign vector which is the "union" of signed circuits) which is non-zero on every $e \in E$. Acyclic orientations of a planar graph G are identified with totally cyclic orientations of the planar dual $G^\perp$; more generally, acyclic orientations of an oriented matroid $\mathcal{M}$ are identified with totally cyclic orientations of the oriented matroid dual $\mathcal{M}^\perp$.

Hopefully you are now convinced that, since every graph (planar or not) has an (oriented) matroid $\mathcal{M}$ and we have supplied a somewhat satisfactory "dual object", even for non-planar graphs!

For an OM $\mathcal{M}$ coming from a vector configuration $\mathcal{V} = \{v_e\}_{e \in E}$ in $\mathbb{R}^r$, duality (and particularly, the definition (1) of the dual chirotope) comes from an isomorphism between the Grassmannians $\mathbb{G}r(r, \mathbb{F}^n)$ and $\mathbb{G}r(n-r, \mathbb{F}^n)$ which sends an

r-subspace to its perpendicular $(n-r)$-subspace, on the level of Plücker coordinates. As before, let $\mathcal{M}$ also denote the $r \times n$ matrix having the v_e as columns, and let $\mathcal{M}^\perp$ be any $(n-r) \times n$ matrix whose row space is the perp to the row space of $\mathcal{M}$ within $\mathbb{R}^n$. Then the columns $\{v_e^\perp\}_{e \in E}$ of $\mathcal{M}^\perp$ turn out to realize the orthogonal oriented matroid $\mathcal{M}^\perp$. More precisely (see Exercise 11), the row spaces of $\mathcal{M}, \mathcal{M}^\perp$ have the following relation between their Plücker coordinates: there exists an overall scalar $c \in \mathbb{F}^\times$ such that for complementary sets $\{e_1 < \cdots < e_r\}, \{e'_1 < \cdots < e'_{n-r}\}$ one has

$$p_{e_1,\ldots,e_r}(\mathcal{M}) := c \cdot \mathrm{sign}(e_1, \cdots, e_r, e'_1, \cdots, e'_{n-r}) p_{e'_1,\ldots,e'_{n-r}}(\mathcal{M}^\perp). \tag{2}$$

Perhaps a word or two more is in order about the meaing of acyclic and totally cyclic orientation for vector configurations $\mathcal{V}$ in $\mathbb{R}^r$ and oriented matroids.

The vectors $\mathcal{V}$ are *acyclically oriented* if there is a hyperplane containing all the vectors in its positive (open) halfspace, that is, there is a *covector* in its oriented matroid $\mathcal{M}$ which is all $+$. Equivalently, by a version of Farkas' Lemma from the theory of linear inequalities, there is no linear dependence among the vectors that all has all $+$ coefficients, i.e., every signed circuit in $\mathcal{M}$ must contain both $+$ and $-$ signs.

Dually, the vectors are *totally cyclically oriented* if there is a linear dependence[2] among the vectors that has all $+$ coefficients. Equivalently, again by Farkas' Lemma, there is no hyperplane containing all the vectors in its nonnegative (closed) halfspace, that is, every *covector* of $\mathcal{M}$ must contain both $+$ and $-$ signs.

It turns out that these notions are consistent with our definitions for directed graphs to be acyclically/totally cyclically oriented, and that a vector configuration or OM is acyclically oriented if and only if its dual is totally cyclic oriented.

Example 17.
The vector configuration $\mathcal{V} = \{a, b, c, e, f\}$ given by the columns of this matrix $\mathcal{M}$

$$\begin{array}{ccccc} a & b & c & e & f \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

has a dual vector configuration $\mathcal{V}^\perp = \{a^\perp, b^\perp, c^\perp, e^\perp, f^\perp\}$ given by the columns of this matrix $\mathcal{M}^\perp$

$$\begin{array}{ccccc} a^\perp & b^\perp & c^\perp & e^\perp & f^\perp \end{array}$$
$$\begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

[2]We haven't defined here the technical OM term "vector" because the terminology is slightly confusing: a *vector* in an OM realized by a vector configuration over $\mathbb{R}$ means a sign pattern achieve by the coefficients of some linear dependence among the vectors! For general OM's, they are exactly the *covectors* of the dual OM.

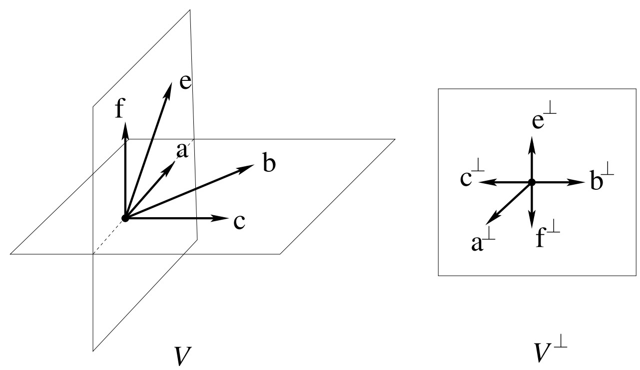


FIGURE 14. A configuration of 5 vectors $\mathcal{V}$ in $\mathbb{R}^3$, along with the dual configuration $\mathcal{V}^\perp$ in $\mathbb{R}^2$.

But one could also use any other matrix such that

$$\text{rowspace}(\mathcal{M}^\perp) = \text{rowspace}(\mathcal{M})^\perp.$$

Note that the subsets of columns of $\mathcal{M}^\perp$ which do not form bases for their span are the ones indexed by $\{b^\perp c^\perp, e^\perp f^\perp\}$, and these are exactly the ones complementary to the sets $\{aef, abc\}$ that index triples of vectors which do not form bases for the span of the columns of $\mathcal{M}$.

Note also that the vector configuration $\mathcal{V}$ is acyclically oriented, while its dual $\mathcal{V}^\perp$ is totally cyclically oriented.

2.2. **Duality in other guises.** Matroid/OM duality is a highly non-trivial operation, with many powerful applications. We mention some other instances where it arises, perhaps in disguised form.

2.2.1. *Dual linear codes.*

When working over finite fields, the relation among the row spaces $\mathcal{M}, \mathcal{M}^\perp$ is that of a *(linear) code* and its *dual* or *orthgonal code*. We will touch on this again in a later lecture when discussing how the Tutte polynomial specializes to give the weight enumerator of a linear code, and the MacWilliams identity.

2.2.2. *Linear programming duality.*

The theory of linear programming (including the simplex method, the duality theorem of linear programming, complementary slackness of optimal primal/dual solutions) all have a beautiful generalization to oriented matroids. In this theory, one considers a triple $(\mathcal{M}, f, g)$ of an oriented matroid $\mathcal{M}$ on ground set E with two distinguished elements f, g. The elements $E - \{f, g\}$ play the role of the inequalities that define the *feasible polyhedron*, a regular CW-ball whose faces are indexed by the covectors which are nonnegative on $E - \{f\}$. The element g plays

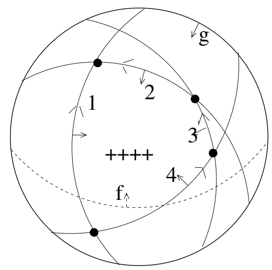


FIGURE 15. An oriented matroid program $(\mathcal{M}, f, g)$. The four pseudospheres $1, 2, 3, 4$ define a quadrangular feasible region with covector $+ + ++$, lying entirely on the $+$ side of the pseudosphere g "infinity". Since all four vertices of the feasible region lie *strictly* on the $+$ side of g, the feasible region is bounded. The "objective function" f indicates how to direct some 1-cells (edges) of the feasible region: as long as the unique pseudocircle containing the edge does not lie inside f, orient the semicircle in which this pseudocircle intersects the hemisphere H_g^+ from the $-$ side of f to the $+$ side, and then take the induced orientation on the edge.

the role of the *hyperplane at infinity*, converting between linear and affine hyperplanes/inequalities and defining the notion of unbounded/boundedness. The element f plays the role of the objective function, by (partially) orienting the edges of the *feasible polyhedron*; see Figure 15. The dual linear program then corresponds to the triple $(\mathcal{M}^\perp, g, f)$, in which the roles of g, f have been exchanged. See [3] for more on this.

2.2.3. *Gale transforms.*

Much of convex geometry deals with affine dependencies and convexity relations among a configuration of points $\mathcal{A} = \{a_1, \ldots, a_n\}$ in affine space, say of dimension $r - 1$. A frequently recurring example is where $\mathcal{A}$ is the vertex set of a convex polytope in $\mathbb{R}^{r-1}$. A venerable and useful trick is to encode the same information in a configuration of vectors in $\mathbb{R}^{n-r}$ as follows.

One considers the vector configuration $\mathcal{V} := \{(a_1, 1), \ldots, (a_n, 1)\}$ insider $\mathbb{R}^r$, whose oriented matroid $\mathcal{M}$ encodes all of the previous affine dependency/convexity data about $\mathcal{A}$. This same data is encoded in the dual oriented matroid $\mathcal{M}^\perp$, which corresponds to the dual configuration of vectors (called the *Gale transform*) $\mathcal{V}^\perp$, that is, the columns of any matrix whose row space is perpendicular to the row space of the matrix having the $\mathcal{V}$ as column.

Gale transforms have been useful, for example, in reducing the dimension of the problem under consideration (if the dimension r is large, but the *codimension* $n-r$

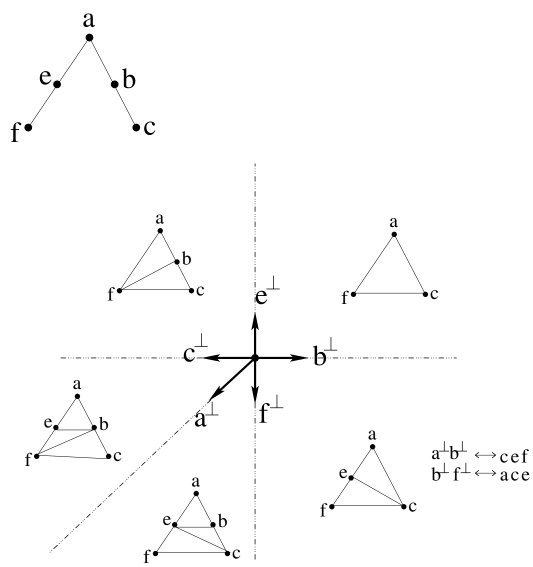


FIGURE 16. A point configuration $\mathcal{A} = \{a, b, c, e, f\}$ in $\mathbb{R}^2$, along with its Gale diagram, and the cones of its secondary fan labelled by its five triangulations, all of which are coherent triangulations.

is small), for encoding properties of particular matroids as properties of particular polytopes (including interesting pathological examples), and for understanding the set of triangulations of a point configurations $\mathcal{A}$. See Ziegler's book [31, Lecture 6].

Example 18.
We cannot resist illustrating the connection between the Gale diagram of a point configuration $\mathcal{A} = \{a_1, \ldots, a_n\}$ in $\mathbb{R}^{r-1}$ and its set of (coherent) triangulations of $\mathcal{A}$. A *triangulation* of $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a collection of geometric $(r-1)$-simplices, all spanned by subsets of $\mathcal{A}$, having pairwise disjoint interiors, which cover the convex hull of $\mathcal{A}$, and which meet pairwise along common faces. These triangulations need not use all of the a_i as vertices; see Figure 16 for an example.

A triangulation of $\mathcal{A}$ is called *coherent* if it can be achieved by the following geometric construction: lift the points $\mathcal{A}$ into $\mathbb{R}^r$ by appending an r^{th} coordinate to each a_i, then take the convex hull of the resulting points in $\mathbb{R}^r$, and project the *lower facets* (= those facets whose outward normal has negative r^{th} coordinate) back down into $\mathbb{R}^{r-1}$.

It turns out that the set of all coherent triangulations, along with certain natural moves connecting them (called *bistellar flips*), is very nicely structured:

it forms a graph which is the 1-skeleton (= vertices and edges) of an $(n-r)$-dimensional convex polytope, called the *secondary polytope* of $\mathcal{A}$. It also turns out that the normal fan of the secondary polytope (called the *secondary fan* of $\mathcal{A}$), and hence this same graph structure, can be constructed from the Gale diagram as follows.

Start with the vectors $\mathcal{V}^\perp$ in the Gale diagram for $\mathcal{A}$, and consider all of the simplicial cones one can obtain by taking the nonnegative span of some linearly independent subset of $\mathcal{V}^\perp$. Because the vectors $\mathcal{V}^\perp$ are totally cyclically oriented in $\mathbb{R}^{n-r}$ (due to the fact that their dual vectors $\mathcal{V}$ in $\mathbb{R}^r$ were acyclically oriented since they came from a point configuration $\mathcal{A}$ in $\mathbb{R}^{r-1}$), these simplicial cones will cover all of $\mathbb{R}^{n-r}$. Now take the common refinement of all of these simplicial cones, and this gives the secondary fan.

The correspondence between a top-dimensional cone σ in the secondary fan and a triangulation of $\mathcal{A}$ can be made explicit as follows. Write down the list of all matroid bases for $\mathcal{V}^\perp$ that span a simplicial cone containing σ as a subcone. Take the complements of these bases, which will give a list of matroid bases of $\mathcal{V}$. Then these bases span the simplices that make up a coherent triangulation of $\mathcal{A}$ (!) For example, in Figure 16, the lower right triangulation is shown along with a listing of the relevant bases for $\mathcal{V}^\perp$, and their complementary bases of $\mathcal{V}$ that span the simplices in the triangulation.

2.3. **Representability questions.**

Given a matroid M, it is natural to ask whether it falls into one of the classes that we've already considered: is it algebraic, linear representable, graphic, transversal, orientable? Answers to these questions sometimes can be phrased in terms of the *minors* of M (= matroids obtained by a sequence of deletions and/or contractions).

2.3.1. *Algebraic representability.*

The question of which matroids are algebraic seems to be hard. Algebraic matroids are (obviously) closed under deletion, and also (but not obviously) closed under contraction, hence closed under minors. For a while it was not known whether there exist non-algebraic matroids at all, but an example of Vámos (see Figure 17(c)) was shown to be non-algebraic by Ingleton and Main (1975). Surprisingly the following question remains open:

Problem 19. *Are algebraic matroids closed under duality?*

2.3.2. *Linear representability.*

Linear representability is much better behaved in some ways: the discussion of basic constructions makes it clear that collection of matroids linearly representable as a vector configuration over a fixed field $\mathbb{F}$ is closed under minors and under duality. However, characterizing those matroids representable over at least

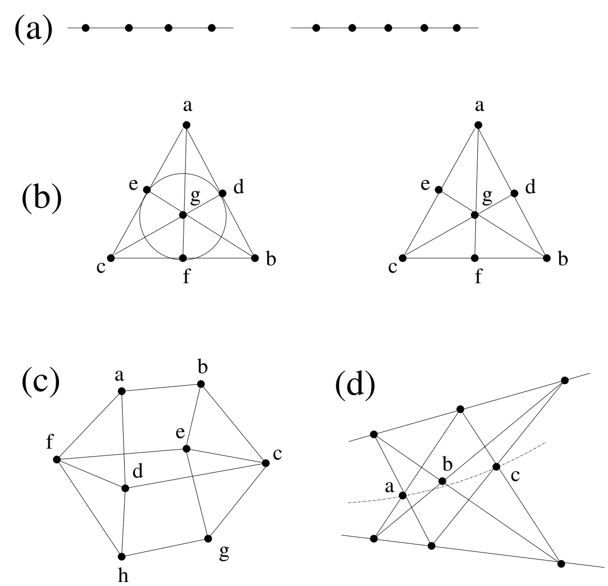


FIGURE 17. Some examples of matroids that are not representable over various fields. (a) The uniform matroids $U_{2,4}$ and $U_{2,5}$. (b) The Fano and non-Fano matroids, in which def are collinear (resp. non-collinear). (c) The Vámos matroid, in which $abcd, abef, cdgh, efgh$ are coplanar quadruples, but there are no other non-obvious coplanarities. (d) The non-Pappus matroid, in which abc are non-collinear.

one field or over a specific field can be quite difficult. We discuss some of the famous results/examples in this regard. In this discussion, it will be convenient to "draw" matroids of rank 2, 3, 4 as if they represent affine dependencies of point configurations on a line or in a plane, where non-obvious collinearities or coplanarities are indicated by drawing a line/plane through the points in question.

Example 20.
The *uniform matroid* $U_{r,n}$ of rank r on n elements has as bases all r-element subsets of its n-element ground set E; it can be represented over any field with sufficiently many elements, but for example, $U_{2,q+2}$ cannot be represented over the finite field $\mathbb{F}_q$ (because there exist only $q+1$ different slopes possible for vectors in $\mathbb{F}_q^2$). Note that uniform matroids are closed under duality: $U_(r,n)^\perp = U(n-r,n)$.

It turns out that every matroid on 7 or fewer elements is representable over at least one field, but the same example of Vámos mentioned above is a matroid of rank 4 on 8 elements (shown in Figure 21(c)) which cannot be represented over any field. Similarly the *non-Pappus matroid* (shown in Figure 21(d)) is not linearly representable. The *Fano* and *non-Fano* matroids shown in Figure 21(b) have the property that the former is representable only over a field of characteristic 2, while the latter is representable only over a field whose characteristic is not 2 (see Exercise 9). Consequently, their direct sum is also not linearly representable over *any* field.

Here is an omnibus sampling of same famous results characterizing representability of various kinds. In each case, one direction is easy, and the other direction is somewhat unpleasant (to varying degrees).

Theorem 21. (i) *A matroid is representable over* $\mathbb{F}_2$ *if and only if it has no minor isomorphic to* $U_{2,4}$ *(Tutte 1958).*

(ii) *A matroid is representable over* $\mathbb{F}_3$ *if and only if it has no minor isomorphic to* $U_{2,5}, U_{3,5}$*, the Fano matroid, or its dual (Bixby 1979, Seymour 1979).*

(iii) *A matroid is representable over every field*
if and only if it is regular, that is, it has a representation as the columns of a totally unimodular integer matrix (one with all minor subdeterminants $\pm 1, 0$*)*
if and only if it has no minor isomorphic to $U_{2,4}$*, the Fano matroid, or its dual (Tutte 1958).*

(iv) *A matroid is graphic if and only if it has no minor isomorphic to* $U_{2,4}$*, the Fano plane, its dual, the dual of* $M(K_5)$*, or the dual of* $M(K_{3,3})$ *(Tutte 1959).*

One might be tempted to conclude from the previous results that every minor-closed class of matroids has a characterization by a finite list of excluded minors– this would be analogous to the celebrated result of Robertson and Seymour in the mid 1990's showing that every minor-closed class of graphs has such a characterization. Unfortunately, for $\mathbb{F}$ any field of characteristic zero, the class of matroids representable over $\mathbb{F}$ has infinitely many minor-minimal counterexamples (Lazarson 1958).

Question 22. *Fix a finite field* $\mathbb{F}_q$*. Is there a finite list of excluded minors for linear representability over* $\mathbb{F}_q$*?*

For example, such a list is conjectured explicitly for representability over $\mathbb{F}_4$; see [18, §6.5].

2.3.3. *Digression: How much of a graph is captured by its matroid?*

Before turning to oriented matroids and representability questions over the reals, we briefly discuss one of the first deep results in matroid theory, Whitney's *2-isomorphism theorem*, which tells us exactly how much of the structure of a graph $G = (V, E)$ is captured by its graphic matroid $M(G)$; this is related to Tutte's characterization of graphic matroids in Theorem 21(iv) above.

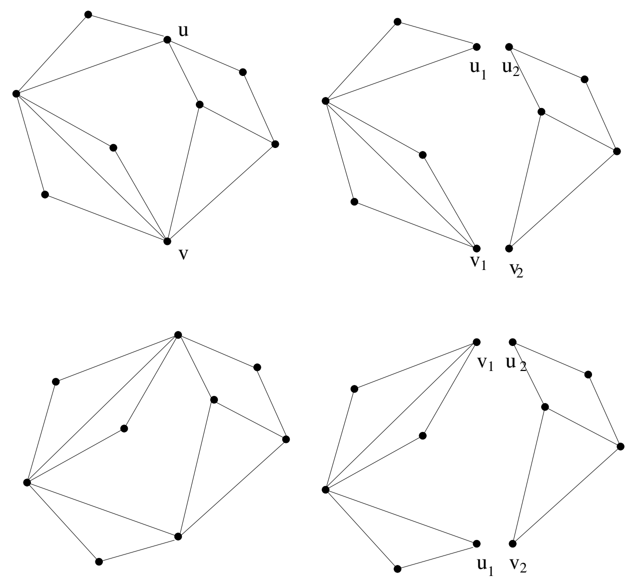


FIGURE 18. The twisting operation from Whitney's 2-isomorphism theorem, yielding isomorphic graphic matroids.

Since disjoint unions $G_1 \sqcup G_2$ of graphs as well as one-point wedges $G_1 \vee G_2$ both have matroids given by the direct sum $M(G_1) \oplus M(G_2)$, it is clear that two graphs obtained from each other by a sequence of replacements of $G_1 \sqcup G_2$ with $G_1 \vee G_2$, or vice-versa, in any order, will have the same matroids on the same edge set E.

Of course, this only produces matroid isomorphisms between graphs which are not 2-vertex connected, i.e. those which are either disconnected or can be disconnected by removing a single vertex. Whitney observed another operation that leaves the matroid $M(G)$ invariant, applicable to a graph which can be disconnected by removing two vertices, illustrated in Figure 18: if removing vertices $\{u, v\}$ disconnects G, then G can be viewed as a 2-vertex union of two other disjoint graphs G_1, G_2, in which one identifies vertices $u_1 \in G_1$ with $u_2 \in G_2$ as $u \in G$ and identifies vertices $v_1 \in G_1$ with $v_2 \in G_2$ as $v \in G$. Then the *twisting of* G *about* $\{u, v\}$ is obtained by instead identifying u_1 with v_2, and v_1 with u_2; one can check (see Exercise 10) that G and its twist about $\{u, v\}$ have the same matroid. Say that two graphs G, G' are *2-isomorphic* if they can be obtained from each other by a sequence of replacements of wedges with disjoint unions or vice-versa, and twists.

Theorem 23. *(Whitney's 2-isomorphism theorem, 1933) Two graphs* G *on edge set* E *have the same graphic matroid* $M(G)$ *on* E *if and only if they are 2-isomorphic.*

In particular, a graph $G = (V, E)$ which is 3-vertex-connected (that is, connected, not a wedge, and to which no twist is applicable) can be recovered uniquely from its matroid $M(G)$ on E; see Exercise 10.

An interesting application of Whitney's result was given recently by [24], who used it to show that the *critical group* of a graph is a matroid invariant.

2.4. **Orientability, topological representations and realizations.**

A related question to representability is that of *orientability*: when is a matroid M the underlying matroid of some oriented matroid $\mathcal{M}$? Clearly matroids representable over $\mathbb{R}$ have this property, and various constructions preserve this property, e.g. taking minors and duals.

Unfortunately, there is no good characterization known in general for orientable matroids, and the list of minor-minimal counterexamples is known to include infinitely many of rank 3 ! For example, the Fano plane is one such minor-minimal counterexample, and if one combines this with Tutte's characterizations of binary and of regular matroids (Theorem 21(i) and (iii)), one concludes a result of Bland and Las Vergnas asserting that binary matroids are orientable if and only if they are regular.

In the other direction, given an oriented matroid $\mathcal{M}$, one can ask whether it has a representation over $\mathbb{R}$, or some substitute for such a representation. A wonderfully useful substitute is provided by the *Folkman-Lawrence Topological Representation Theorem*, which says *every* OM comes from an arrangement of pseudospheres.

Definition 24.
A pseudosphere S inside a d-sphere S^d is a subspace such that the pair (S, S^d) is homeomorphic to a standard pair (S^{d-1}, S^d) of a $(d-1)$-sphere in a d-sphere. By the Jordan-Brouwer separation theorem, it divides S^d into two hemispheres S^+, S^-.

An *arrangement* $\mathcal{A} = \{S_e\}_{e \in E}$ *of pseudospheres* in S^d is a finite subset of pseudospheres such that

A1. Every non-empty intersection $S_A = \cap_{e \in A} S_e$ is homeomorphic to a sphere of some dimension.
A2. For every such intersection S_A and $e \in E$ with $S_A \not\subseteq S_e$, the intersection $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

Theorem 25. *Arrangements of pseudospheres $\mathcal{A}$ in S^{r-1} are in bijective correspondence with oriented matroids $\mathcal{M}$ of rank r, once one fixes a positive side for each pseudosphere. Under this correspondence, the covectors of $\mathcal{M}$ are exactly the sign patterns achieved by the different points of S^{r-1} with respect to pseudospheres.*

There are several different proofs of this result. The one given in the OM bible [3] applies the technique of *lexicographic shellability* to the poset $\mathcal{L}(\mathcal{M})$ of

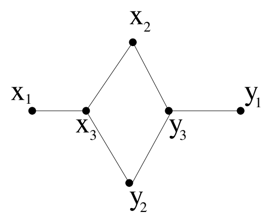


FIGURE 19. A representation, á la Swartz, of the matroid $U_{2,3}$ as an arrangement of three 0-spheres, $\{x_i, y_i\}$ inside a homotopy 1-sphere.

covectors of $\mathcal{M}$ ordered by $0 < +, -$ componentwise, which ends up being the face poset for the cell decomposition of S^{r-1} induced by the pseudospheres.

Inspired by this theorem (and running counter to the received wisdom about representability of matroids), E. Swartz recently proved [23] an analogous representation theorem for *all* matroids, in which spheres/pseudospheres are replaced by *homotopy spheres*; an example of a homotopy-sphere representation of $U_{2,3}$ is shown in Figure 19.

Not every oriented matroid $\mathcal{M}$ can be realized by a configuration of *linear pseudospheres*, that is, by a vector configuration over $\mathbb{R}$; the non-Pappus and Vámos matroids give rise to counterexamples beginning with rank 3 on 9 elements, or rank 4 with 8 elements. One can even perturb these example to obtain non-realizable oriented matroids $\mathcal{M}$ whose underlying matroid is a uniform matroid $U_{r,n}$.

2.4.1. *Realization spaces and OM strata in the Grassmannian.*

Given an oriented matroid $\mathcal{M}$ of rank r with n elements, one can define the *realization space* $\mathcal{R}(\mathcal{M})$ as the space of $r \times n$ matrices whose $r \times r$ realize the chirotope of $\mathcal{M}$, modulo the action of $GL_r(\mathbb{R})$ by left-multiplication; implicitly we topologize $\mathcal{R}(\mathcal{M})$ as a subspace of the Grassmannian $\mathbb{G}r(r, \mathbb{R}^n)$, where it is called the *oriented matroid* stratum of the Grassmannian corresponding to $\mathcal{M}$.

Note that $\mathcal{R}(\mathcal{M})$ is a semialgebraic subset (that is, defined by a conjunction of polynomial equalities and inequalities) inside $\mathbb{G}r(r, \mathbb{R}^n)$, and realizability for $\mathcal{M}$ is simply the question of whether $\mathcal{R}(\mathcal{M})$ is empty. The general question of whether a real semialgebraic set is empty is a decision problem usually called the *existential theory of the reals*, for which complexity bounds are known. A theorem of Mnëv (1988) shows that $\mathcal{R}(\mathcal{M})$ can have the homotopy type of an arbitrary real semialgebraic set; see [3, §8.6].

2.5. **Passing between polytopes and matroids and OM's.**

Tangentially related to the question of realizability is the fact that there are many ways to pass from a (realizable, oriented) matroid to a polytope, and

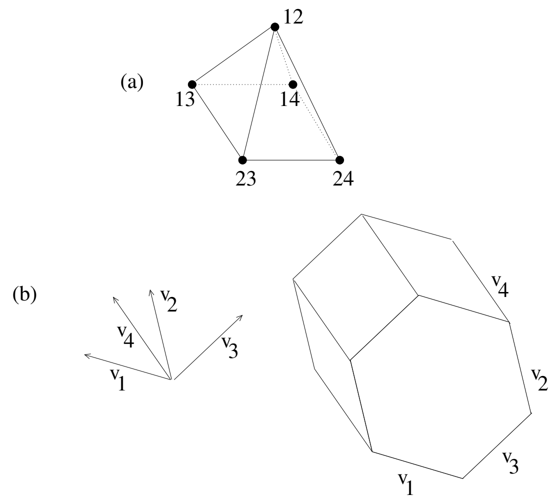


FIGURE 20. (a) The polytope P_M for the rank 2 matroid M having bases $\mathcal{B}(M) = \{12, 13, 14, 23, 24\}$; the vertices of P_M are the characteristic vectors of these bases. (b) A rank 3 vector configuration $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ in which v_1, v_2, v_3 are coplanar, and its associated zonotope $\mathbb{Z}(\mathcal{V})$. Some of the edges of $\mathbb{Z}(\mathcal{V})$ are labelled by their slope vector v_i.

vice-versa, making the two subjects closely related. We discuss a few of these constructions here.

2.5.1. *Independent set and basis polytopes.*

One of the historically earliest ways to get a polytope from a matroid was introduced by Edmonds (1970). Given a matroid M on ground set E, he considered the convex polytope $P(M)$ in $\mathbb{R}^E$ whose vertices are the characteristic vectors $\{0,1\}^E$ of the independent sets $\mathcal{I}(M)$. Edmonds described the facet inequalities of P_M, and proved that the intersection of two such polytopes $P_M \cap P_{M'}$ has vertices with integer coordinates (but this fails for triple intersections).

One of the facets of P_M, lying in the affine hyperplane of $\mathbb{R}^E$ where the sum of coordinates is $r = \text{rank}(M)$, is the convex hull of the characteristic vectors of the bases $\mathcal{B}(M)$; see Figure 20(a) for an example. The polytope P_M has nice properties, many of which generalize to the polytopes associated with *Coxeter matroids* [4].

2.5.2. *Zonotopes.* Given a vector configuration $\mathcal{V} = \{v_e\}_{e \in E}$ in $\mathbb{R}^r$, the *zonotope* generated by $\mathcal{V}$ is the convex polytope which is the *Minkowski sum* of line segments in the direction of the v_e:

$$\mathbb{Z}(\mathcal{V}) := \left\{ \sum_{e \in E} c_e v_e : c_e \in [-1, +1] \right\}.$$

In other words, $\mathbb{Z}(\mathcal{V})$ is the projection of the cube $[-1, +1]^E$ under the linear map $\mathbb{R}^E \to \mathbb{R}^r$ sending the standard basis vector of $\mathbb{R}^E$ indexed by $e \in E$ to v_e in $\mathbb{R}^r$. Aside from being projections of cubes, zonotopes have many other nice characterizations; see e.g. [3, Prop. 2.2.14].

The *normal fan of* $\mathbb{Z}(\mathcal{V})$ (= the decomposition of the linear functionals $(\mathbb{R}^r)^*$ according to the face of $\mathbb{Z}(\mathcal{V})$ on which they maximize) is the hyperplane arrangement $\mathcal{A} = \{v_e^\perp\}_{e \in E}$ that we mentioned before. Consequently the face poset of $\mathbb{Z}(\mathcal{V})$ is the poset of covectors of the oriented matroid $\mathcal{M}$ realized by $\mathcal{V}$, ordered by $+, - < 0$ componentwise, that is, the boundary complex of $\mathbb{Z}(\mathcal{V})$ is the regular CW-sphere polar dual to the Folkman-Lawrence sphere. More explicitly, the covector $f : E \to \{\pm 1, 0\}$ indexes the face F of $\mathbb{Z}(\mathcal{V})$ consisting of vectors of the form $\sum_{e \in E} c_e v_e$ with $c_e = f(e)$ if $f(e) \in \{\pm 1\}$ and with $c_e \in [-1, +1]$ if $f(e) = 0$. See Figure 20

For any oriented matroid $\mathcal{M}$, since the Folkman-Lawrence sphere is a PL-regular CW-sphere, it has a dual PL-regular CW-sphere, which plays the role of a (not always polytopal) "zonotope".

2.5.3. *Matroid polytopes.*

Consider a top-dimensional cell in the Folkman-Lawrence sphere of an oriented matroid $\mathcal{M}$ of rank r, and let's assume that $\mathcal{M}$ is acyclically oriented so that this cell is indexed by the all $+$ covector. The boundary the cell in question is a (shellable, regular CW) $(r-2)$-sphere, with the face poset of the cell (called the *Edmonds-Mandel face lattice* $\mathcal{F}_{em}(\mathcal{M})$) given by the componentwise $0 < +, -$ order on the nonnegative covectors of $\mathcal{M}$.

These cells model (polar duals of) convex $(r-1)$-polytopes in the following way. Given a convex $(r-1)$ polytope P in $\mathbb{R}^{r-1}$ with vertex set $\{a_1, \ldots, a_n\}$, its *polar dual* polytope in $(\mathbb{R}^{r-1})^*$ may be defined by

$$P^\Delta := \{f \in (\mathbb{R}^{r-1})^* : f(a_i) \geq -1, \ i = 1, \ldots, n\}.$$

If we consider the vector configuration $\mathcal{V} := \{(a_1, 1), \ldots, (a_n, 1)\}$ in $\mathbb{R}^r$. Then the chamber with all $+$ covector in the associated hyperplane arrangement $\mathcal{A} = \{(a_i, i)^\perp\}$ in $(\mathbb{R}^r)^*$ is linearly isomorpic to the cone over P^Δ. Thus the face lattice of P^Δ is $\mathcal{F}_{em}(\mathcal{M})$ for the associated oriented matroid $\mathcal{M}$. Consequently, its opposite poset, called the *Las Vergnas face lattice* $\mathcal{F}_{lv}(\mathcal{M})$.

Because every acyclic oriented matroid has $\mathcal{F}_{em}(\mathcal{M})$ equal to the face lattice of a shellable (and hence PL) sphere, the opposite poset $\mathcal{F}_{lv}(\mathcal{M})$ is the face lattice of a sphere which is at least PL.

Question 26. *Is the sphere having face lattice* $\mathcal{F}_{lv}(\mathcal{M})$ *always shellable?*

These spheres, known as *matroid polytopes*, therefore generalize the boundaries of convex polytopes. The generalization can be shown to be strict using a construction of J. Lawrence to be discussed next.

2.5.4. *Lawrence polytopes.*
Lawrence (1980) showed how to encode the structure of vector configuration (resp. oriented matroid) into the face lattice of a convex polytope (resp. matroid polytope). The construction for a configuration $\mathcal{V} = \{v_e\}_{e \in E}$ of n vectors of rank r in $\mathbb{R}^r$ proceeds by

- forming the Gale transform $\mathcal{V}^\perp = \{v_e^*\}_{e \in E}$ in $\mathbb{R}^{n-r}$,
- doubling the Gale transform by adding in (disjoint) copies of the negatives of each Gale transform vector: $\mathcal{V}^\perp \sqcup -\mathcal{V}^\perp$, still in $\mathbb{R}^{n-r}$ but now with $2n$ vectors, and
- taking the Gale transform back again:

$$\Lambda(\mathcal{V}) := \left(\mathcal{V}^\perp \sqcup -\mathcal{V}^\perp\right)^\perp \subset \mathbb{R}^{n+r}$$

In what way is $\Lambda(\mathcal{V})$ a polytope? The central symmetry of the configuration $\mathcal{V}^\perp \sqcup -\mathcal{V}^\perp$ implies that it is *totally cyclic* (there is a linear dependence among having + sign on every vector), and every open halfspace contains at least two of its vectors, or in other words, its cocircuits all contain at least two + entries. This means that its Gale transform $\Lambda(\mathcal{V})$ is a configuration of vectors $\mathbb{R}^{n+r}$ which is *acyclic*, and which has all circuits containing at least two + entries. In other words, $\Lambda(\mathcal{V})$ is a configuration of vectors lying in a halfspace and in which every vector spans an extreme ray of their convex hull, so that they can be rescaled to lie in on the vertices of some convex $(n + r - 1)$-polytope lying in an affine hyperplane of $\mathbb{R}^{n+r}$.

This whole construction can be mimicked without the vectors $\mathcal{V}$, just using the oriented matroid $\mathcal{M}$, and then the Lawrence lifting of $\mathcal{M}$ is defined to be the OM

$$\Lambda(\mathcal{M}) = (\mathcal{M}^\perp \sqcup -\mathcal{M}^\perp)^\perp.$$

The oriented matroid $\Lambda(\mathcal{M})$ has the property that it can be recovered entirely from the Las Vergnas face lattice $\mathcal{F}_{lv}(\Lambda(\mathcal{M}))$, that is, from the matroid polytope. This leads to various counterexample constructions, such as matroid polytopes that cannot be realized as convex polytopes in $\mathbb{R}^r$, or convex polytopes that cannot be realized with all vertex coordinates in $\mathbb{Q}$.

The Lawrence construction also plays a prominent role in Richter-Gebert's version [20] of Mnëv's universality theorem for realization spaces of polytopes: the realization spaces of 4-dimensional polytopes can have the homotopy type of an arbitrary semialgebraic set. This is in contrast to the realization spaces of 3-dimensional polytopes, which are always contractible by a version of Steinitz's Theorem [31, Lecture 4]

3. Lecture 3: Invariants (enumerative, topological, algebraic)

3.1. Enumerative invariants.

The mother-of-all-matroid-invariants is surely the Tutte polynomial, introduced in equivalent forms by Whitney (1932) and Tutte (1947) for graphs, and then generalized to matroids by Crapo (1969).

Theorem 27. *There exists an isomorphism invariant of matroid M on ground set E in the form of a polynomial $T_M(x,y)$ in two variables x,y, called the Tutte polynomial of M, having these properties:*

T1. *If $e \in E$ is neither a loop nor coloop of M, then*

$$T_M(x,y) = T_{M \backslash e}(x,y) + T_{M/e}(x,y)$$

T2. *If e is a coloop of M, then $T_M(x,y) = xT_{M/e}(x,y)$. If If e is a loop of M, then $T_M(x,y) = yT_{M\backslash e}(x,y)$.*

T2′ $T_{M_1 \oplus M_2}(x,y) = T_{M_1}(x,y)T_{M_2}(x,y)$.

T3. *When $E = \{e\}$ has cardinality 1,*

$$T_M(x,y) = \begin{cases} x & \text{if } e \text{ is a coloop/isthmus.} \\ y & \text{if } e \text{ is a loop.} \end{cases}$$

T4. $T_{M^\perp}(x,y) = T_M(y,x)$.

Furthermore, $T_M(x,y)$ is characterized by properties T1, T2, T3 above, or alternatively by the properties T1, T2′, T3.

Proof. (sketch) Check that the above properties are satisfied by $T_M(x,y) := S_M(x-1,y-1)$, where $S_M(x,y)$ is the *Whitney corank-nullity polynomial*

$$S_M(x,y) = \sum_{A \subset E} x^{\text{rank}(M)-r(A)} y^{|A|-r(A)}. \tag{3}$$

Once one knows there exists at least one polynomial with these properties, then it is uniquely computable from the above properties using induction on $|E|$. □

The fact that many graph, matroid and oriented matroid invariants satisfy a deletion-contraction recurrence similar to $T1$ and respect direct sums as in $T2'$ makes them specializiations of $T_M(x,y)$.

Proposition 28. *Let $\Psi(M)$ be any ismorphism invariant of matroids taking values in a commutative ring R, satisfying these properties:*

- $\Psi(M_1 \oplus M_2) = \Psi(M_1)\Psi(M_2)$.
- *When e is neither a loop nor an isthmus of M,*

$$\Psi(M) = a\Psi(M\backslash e) + b\Psi(M/e).$$

- *When $E = \{e\}$ has cardinality 1,*

$$\Psi(M) = \begin{cases} c & \text{if } e \text{ is a coloop/isthmus.} \\ d & \text{if } e \text{ is an loop.} \end{cases}$$

Then

$$\Psi(M) = a^{r(M^{\perp})} b^{r(M)} T_M\left(\frac{c}{b}, \frac{d}{a}\right).$$

Here are a series of examples of graph/matroid/OM invariants that are Tutte polynomial specializations that are generally easy to prove, either directly from (3) or using Proposition 28. Many of them are in fact specializations (for M or $M^{\perp}$) of the single variable *characteristic polynomial*

$$\begin{aligned}\chi_M(t) &:= \sum_{X \in L(M)} \mu(\hat{0}, X) t^{\text{rank}(M)-\text{rank}(X)} \\ &= (-1)^{\text{rank}(M)} T_M(1-t, 0)\end{aligned}$$

where $\mu(-,-)$ denotes the *Möbius function* in the lattice of flats $L(M)$.

3.1.1. *Independent sets, spanning sets.*

Recall that $\mathcal{I}(M)$ denotes the independent sets of M. Let $\mathcal{S}(M)$ denote the *spanning subsets* of M, that is, those $S \subseteq E$ for which $\overline{S} = E$. Then

$$\begin{aligned}T(1+t, 1) &= \sum_{I \in \mathcal{I}(M)} t^{\text{rank}(M)-|I|} \\ T(1, 1+u) &= \sum_{S \in \mathcal{S}(M)} u^{|S|-\text{rank}(M)}.\end{aligned}$$

3.1.2. *Basis activities.*

Let M be a matroid on ground set E, and fix an arbitrary linear ordering ω of E. Given a base B of M and $e \in B$ (resp. $e \notin B$), say that e is *internally (resp.externally) active* with respect to B if

$$\begin{aligned}e &= \min_{\omega}\{e' \in E : B - \{e\} \cup \{e'\} \in \mathcal{B}(M)\} \\ (\text{resp. } e &= \min_{\omega}\{e' \in E : B \cup \{e\} - \{e'\} \in \mathcal{B}(M)\}).\end{aligned}$$

The *internal (resp. external) activity* of B, denoted $ia_{\omega}(B)$ (resp. $ea_{\omega}(B)$), is the number of elements which are internal (resp. externally) active with respect to B. Then

$$T_M(x, y) = \sum_{B \in \mathcal{B}(M)} x^{ia_{\omega}(B)} y^{ea_{\omega}(B)}.$$

Note that it is not a priori clear from these definitions that the polynomial on the right is independent of the choice of ω. But once one proves that it coincides with the Tutte polynomial $T_M(x, y)$, this independence follows.

3.1.3. *Chromatic and flow polynomials.*

For a graph $G = (V, E)$ having $c(G)$ connected components, and t, u nonnegative integers,
(4)

$$\begin{aligned} \chi_{M(G)}(t) &= t^{-c(G)} |\{ \text{ proper vertex colorings of } G \text{ with } t \text{ colors}\} \\ \chi_{M(G)^\perp}(u) &= |\{ \text{ nowhere-zero } \mathbb{Z}/u\mathbb{Z}\text{-valued flows on the edges of } G\}| \\ T_{M(G)}(1-t, 1-u) &= (-t)^{\mathrm{rank}(M)} (-1)^{|V|} \sum_{(x,y)} (-1)^{|\mathrm{supp}(y)|} \end{aligned}$$

in which the last sum is over pairs (x, y) with x a vertex t-coloring x of G, y a $\mathbb{Z}/u\mathbb{Z}$-valued flow[3] on the edges of G, and where x, y have complementary supports: $\mathrm{supp}(x) \sqcup \mathrm{supp}(y) = E$.

3.1.4. *Finite field intepretations.*

More generally for any matroid M represented over $\mathbb{Q}$ by an integer matrix M, and for all pairs of finite field $\mathbb{F}_p, \mathbb{F}_q$ with sufficiently large characteristics (large enough so that all non-zero minor subdeterminants of M are invertible),

$$T_M(1-p, 1-q) = (-1)^{\mathrm{rank}(M)} \sum_{(x,y)} (-1)^{|\mathrm{supp}(y)|}$$

where the sum is over pairs $(x, y) \in \mathbb{F}_p^E \times \mathbb{F}_q^E$ of complementary support, with x in the $\mathbb{F}_p$-row-space of M, and y in the$\mathbb{F}_q$-kernel of M.

In particular, if $\mathcal{A}$ is a hyperplane arrangement over $\mathbb{F}_p$ with normal vectors given by the columns of $M \in \mathbb{F}_p^{d \times n}$, then

$$\begin{aligned} \chi_M(p) &= |\{ \text{ nowhere zero vectors in the } \mathbb{F}_p\text{-row space of } M\}| \\ &= p^{\mathrm{rank}(M)-d} |\mathbb{F}_p^d - \mathcal{A}|. \end{aligned}$$

This interpretation (and the next one) have been used extensively under the name of the *finite field method* to write down explicit characteristic polynomials and Tutte polynomials explicitly for various infinite families of hyperplane arrangements of interest.

3.1.5. *Two-variable coloring.*
There is a slightly different generalization of the chromatic polynomial interpretation. For a graph $G = (V, E)$ having $c(G)$ connected components, and t a

[3]A *flow* is specified by first fixing an orientation of the edges of G, and then assigning a value in $\mathbb{Z}/u\mathbb{Z}$ to each directed edge in such a way that at every vertex, the sum of the values on incoming edges equals the sum of the values on outgoing edges. Alternatively, a flow is a 1-cycle on the (directed) edges of G with coefficients in $\mathbb{Z}/u\mathbb{Z}$.

nonnegative integer,

$$T_{M(G)}\left(\frac{u+t-1}{u-1}, u\right) = \frac{1}{t^{c(G)}(u-1)^{|V|-c(G)}} \sum_f u^{\mathrm{mono}(f)}$$

where the sum ranges over all vertex t-colorings f of G, and $\mathrm{mono}(f)$ is the number of *monochromatic/improper* edges in E, that is, those whose endpoints receive the same color under f.

3.1.6. *Acyclic and totally cyclic orientations.*

For a graph $G = (V, E)$ with $c(G)$ connected components,
(5)

$$T_{M(G)}(2,0) = (-1)^{|V|-c(G)} \chi_{M(G)}(-1) = |\{\text{ acyclic orientations of } G\}|$$
$$T_{M(G)}(0,2) = (-1)^{|E|-|V|+c(G)} \chi_{M(G)^\perp}(-1) = |\{\text{ totally cyclic orientations of } G\}|$$

and more generally for any oriented matroid $\mathcal{M}$ with underlying matroid M,

$$T_M(2,0) = (-1)^{\mathrm{rank}(M)} \chi_M(-1) = |\{\text{ acyclic orientations of } \mathcal{M}\}|$$
$$T_M(0,2) = (-1)^{\mathrm{rank}(M)^\perp} \chi_{M^\perp}(-1) = |\{\text{ totally cyclic orientations of } \mathcal{M}\}|$$

For an oriented matroid coming from an arrangement of vectors over $\mathbb{R}$, the acyclic orientations/topes correspond to the top-dimensional cells (=chambers/regions) in the associated hyperplane arrangement, and as mentioned earlier (see Exercise 2), in the special case where this arrangement is graphic, these correspond naturally to acyclic orientations of the graph.

3.1.7. *Weight enumerators of linear codes.*

Given an r-dimensional subspace of $\mathbb{F}_q^n$, thought of as an $\mathbb{F}_q$-linear code $\mathcal{C}$ consisting of codewords of codelength n, let M be its $r \times n$ *generator matrix* having rowspace equal to the code $\mathcal{C}$. Then the *weight enumerator*

$$W_{\mathcal{C}}(t) := \sum_{x \in \mathcal{C}} t^{\mathrm{supp}(x)}$$

is an evaluation of the Tutte polynomial for the matroid M represented by the columns of M:

$$W_{\mathcal{C}}(t) = t^{n-r}(1-t)^r T_M\left(\frac{1+(q-1)t}{1-t}, \frac{1}{t}\right).$$

Property $T4$ of the Tutte polynomial then gives an immediate proof the *MacWilliams identity* from coding theory, that determines the weight enumerator of a code from that of its dual:

$$W_{\mathcal{C}^\perp}(t) = \frac{(1+(q-1)t)^n}{q^r} W_{\mathcal{C}}\left(\frac{1-t}{1+(q-1)t}\right).$$

A curious coding-theory Tutte evaluation relates to *binary codes* and *bicycles* (= codewords in the intersection $\mathcal{C} \cap \mathcal{C}^{\perp}$):

$$T_M(-1,-1) = |\mathcal{C} \cap \mathcal{C}^{\perp}|.$$

3.1.8. *Reliability polynomials.*

For a graph $G = (V, E)$ with $c(G)$ connected components, if we choose a random edge-subgraph G' of G by including each edge $e \in E$ with the same probability p in $(0,1)$, then the probability that this subgraph has $c(G') = c(G)$ is

$$(1-p)^{|E|-|V|+c(g)} p^{|V|-c(g)} T_{M(G)}\left(1, \frac{1}{1-p}\right).$$

More generally, for a matroid M on ground set E, the probability that a random subset $A \subseteq E$ has $\text{rank}(A) = \text{rank}(M)$ if each $e \in E$ is included with probability p is

$$(1-p)^{\text{rank}(M^{\perp})} p^{\text{rank}(M))} T_M\left(1, \frac{1}{1-p}\right).$$

Here are two more enumerative invariants thrown in for the fun of it ...

3.1.9. *Crapo's beta-invariant.*

We will have more to say later about interpretations of the characteristic polynomial $\chi_M(t)$, but we mention here one of its particularly interesting specializations: Crapo's *beta-invariant*

$$\begin{aligned}\beta(M) &:= (-1)^{\text{rank}(M)-1}\left[\frac{d}{dt}\chi_M(t)\right]_{t=1} \\ &= (-1)^{\text{rank}(M)} \sum_{X \in L(M)} \mu(\hat{0}, X)\text{rank}(X).\end{aligned}$$

This has some nice properties, such as

- $\beta(M) \geq 0$ for any matroid M,
- $\beta(M) = 0$ if and only if M is disconnected (i.e. a non-trivial direct sum), or just a single loop,
- $\beta(M) \leq 1$ if and only if M is the graphic matroid for a series-parallel graph (Brylawski 1971), and
- $\beta(M^{\perp}) = \beta(M)$ (except when $|M| = 1$).

Brylawski has given intepretations for other small values of $\beta(M)$, such as when $\beta(M) = 2$.

3.1.10. *The rank partition.*

Our last enumerative invariant of matroids has been studied in recent years mainly by Dias da Silva, Fernandes, and Fonseca, and (I think) deserves even more study. The *rank partition*

$$\rho(M) = (\rho_1 \geq \rho_2 \geq \cdots \rho_\ell > 0)$$

is a partition of the number of non-loop elements in the ground set of E, defined uniquely by requiring for each j that the partial sum $\rho_1 + \rho_2 + \cdots + \rho_j$ is the maximum cardinality of a union $I_1 \cup I_2 \cup \cdots \cup I_j$ of independent sets $I_j \in \mathcal{I}(M)$. It is true, but not obvious, that this definition forces the parts of ρ to be weakly decreasing: $\rho_j \geq \rho_{j+1}$.

By definition, the first part ρ_1 of the rank partition $\rho(M)$ is just the rank(M). The length ℓ of $\rho(M)$ is sometimes called the *covering number* of M, or in the case where $M = M(G)$ is a graphic matroid, the *arboricity* of G; it is the minimum number of independent sets required to cover the ground set of M. Dias da Silva [9] has given a very interesting interpretation of $\rho(M)$ when M is represented in characteristic zero by a vector configuration $\mathcal{V} = \{v_1, \ldots, v_n\}$, in terms of the nonvanishing of *immanents* (= symmetrizations by characters of the symmetric group S_n) applied to the tensor $v_1 \otimes \cdots \otimes v_n$.

3.2. **Topological invariants.**
We begin with four simplicial complexes (three of them shellable) derived from a matroid M of rank r on ground set E, nicely discussed by Björner in [28, Chapter 7] (and where one can find much more detailed information about their topology, homology bases, face numbers, etc.)

3.2.1. *The independent sets $\mathcal{I}(M)$, and the nonspanning sets $\mathcal{NS}(M)$.*
As was noted early on, the independent sets of M form a pure $(r-1)$-dimensional simplicial complex which is shellable (Exercise 6). Since a pure shellable complex is homotopy equivalent to a wedge of spheres of the same dimension, its homotopy type is determined by the number of spheres in the wedge, which coincides with the absolute value of its (reduced) Euler characteristic $\tilde{\chi}(\mathcal{I}(M))$. This can be computed from knowledge of the number of independent sets of each cardinality, which was the first our Tutte polynomial evaluations. Straightforward calculation then shows that

$$\tilde{\chi}(\mathcal{I}(M)) = T_M(0,1) = T_{M^\perp}(1,0) = \mu_{L(M^\perp)}(\hat{0}, \hat{1}).$$

Example 29.

Let's re-examine Example 10. There

$$\mathcal{B}(M) = \{acf, adf, ace, ade, cef, def\}$$

and hence

$$\mathcal{B}(M^\perp) = \{de, cd, df, cf, ad, ac\}.$$

This means that $M^\perp$ is a direct sum of two of its rank 1 flats: the flat containing the parallel elements c, d, and the flat containing the parallel elements a, e, f. The lattice $L(M^\perp)$ is then a Boolean algebra of rank 2, having $\mu_{L(M^\perp)}(\hat{0}, \hat{1}) = +1$. This is consistent with the fact that $\mathcal{I}(M)$ is the boundary 2-sphere of a bipyramid, as in Figure 8.

The collection of *nonspanning sets*

$$\mathcal{NS}(M) := \{A \subset E : \overline{A} \neq E\}$$

forms another natural simplicial complex on the set E associated with the matroid M, which is closely related to $\mathcal{I}(M)$ by *Alexander duality*. Given an simplicial complex Δ on vertex set E, its *canonical Alexander dual* is the simplicial complex

$$\Delta^\vee := \{A \subset E : E - A \not\in \Delta\}.$$

$\Delta^\vee$ is an Alexander dual to Δ in the following sense: assuming that neither of $\Delta, \Delta^\vee$ is the full simplex 2^E, they both can be naturally embedded inside the (barycentric subdivision) of the boundary $(|E|-2)$-sphere of this $(|E|-1)$-simplex, in such a way that one is a deformation retraction of the complement of the other within this sphere. As a consequence, the Alexander duality theorem asserts that their (co-)homology groups determine each other as follows:

$$\tilde{H}^i(\Delta^\vee, \mathbb{Z}) \cong \tilde{H}_{|E|-3-i}(\Delta, \mathbb{Z}).$$

It is not hard to check (Exercise 13) that $\mathcal{NS}(M) = \mathcal{I}(M^\perp)^\vee$, and hence the results about the homotopy type of $\mathcal{I}(M^\perp)$ tell us about the homology calculation for $\mathcal{NS}(M)$. However, one can say more about the homotopy type of $\mathcal{NS}(M)$; see the discussion of $L(M)$ below.

3.2.2. *The (non-reduced and reduced) broken circuit complexes* $NBC(M), \overline{NBC}(M)$. For simplicity in this discussion, assume that the matroid M is *simple*, that is, it has no loops nor parallel elements. Fix a total ordering ω of the ground set E, and define a *broken circuit* of M to be a subset of E of the form $C - \{c\}$ where C is a circuit and c is its ω-minimum element. A subset of E containing no broken circuits will be called an *nbc-set*, and the collection of all *nbc* sets forms a simplicial complex on E called $NBC(M)$. A moments thought shows that the simplicial complex $NBC(M)$ has the ω-minimum element e_0 of E as a cone vertex, and hence is contractible. It is therefore more interesting topologically to look at the *reduced broken circuit complex* $\overline{NBC}(M)$ which is the base of the cone $NBC(M)$.

It turns out (see Exercise 15) that $NBC(M)$ is a pure $(r-1)$-dimensional shellable complex, so that $\overline{NBC}(M)$ is a pure shellable $(r-2)$-complex. The generating function for *nbc*-sets counted by cardinality turns out to be a rescaling of the characteristic polynomial $\chi_M(t)$ (see Exercise 15):

$$\chi_M(t) = \sum_{\text{nbc-sets } I} (-1)^{|I|} t^{\text{rank}(M)-|I|}. \tag{6}$$

In the case of a graphic matroid $M(G)$, this is Whitney's result intepreting the coefficients of the chromatic polynomial of G. Setting $t = 1$ in (6), one deduces that $\overline{NBC}(M)$ is homotopy equivalent to wedge of $\beta(M)$ spheres of dimension $(r-2)$, where $\beta(M)$ is Crapo's beta-invariant defined above.

Example 30.

Let M be the matroid of rank 3 on ground set $\{a, b, c, d, e\}$ having the circuits $\mathcal{C} = \{abc, cde\}$, shown in Figure 21. Using the alphabetic linear order $a < b < c < d < e$, the broken circuits are $\{bc, de\}$, and the nbc-sets are

$$\{\varnothing, a, b, c, d, e, ab, ac, ad, ae, bd, be, cd, ce, abd, abe, acd, ace\}.$$

This forms a simplicial complex $NBC(M)$ which is a cone having a as apex, with base $\overline{NBC}(M)$ equal to the 4-cycle of edges bd, cd, ce, be.

The characteristic polynomial is easily computed to be

$$\chi_M(t) = t^3 - 5t^2 + 8t - 4$$

whose coefficients count the nbc-sets by cardinality.

3.2.3. *The order complex of the geometric lattice $L(M)$.* Geometric lattices are in particular, (upper-)semimodular lattices, which are some of the original examples of *EL-shellable* graded posets. An *EL-labelling* of a graded poset L having top and bottom elements is a labelling of the edges in the Hasse diagram using labels from some totally ordered set Λ in such a way that

- every interval $[x, y]$ in L has a unique saturated chain on which the edge labels are weakly increasing as one goes up the chain,
- the sequence of labels read on this unique increasing chain come lexicographically earlier than any label sequence for other saturated chains in $[x, y]$.

In the case of an uppersemimodular lattice L, one can take the totally ordered set Λ to be the set of join-irreducibles elements of L, totally ordered by any linear extension of their induced partial order from L; the edge-labelling assigns to an edge $x < y$ in the Hasse diagram the Λ-smallest join-irreducible j having the property that $x \vee j = y$.

In the special case where $L = L(M)$ is the geometric lattice of flats of a (simple) matroid M, the join-irreducible elements of L correspond to the ground set elements E, and the linear order Λ is just a total ordering ω on E.

For any EL-shellable poset L of rank r, the order complex $\Delta := \Delta(L - \{\hat{0}, \hat{1}\})$ of its proper part is shellable, where a shelling order on the facets is given by listing the saturated chains in L according the Λ-lexicographic order on their label sequences. Thus Δ is homotopy equivalent to a wedge of $(r-2)$-spheres, and one can show that the number of spheres is the number of saturated chains in L whose label sequence is *strictly decreasing*. It turns out that (see [28, Chapter 7]) when $L = L(M)$ for a matroid M, these strictly decreasing label sequences

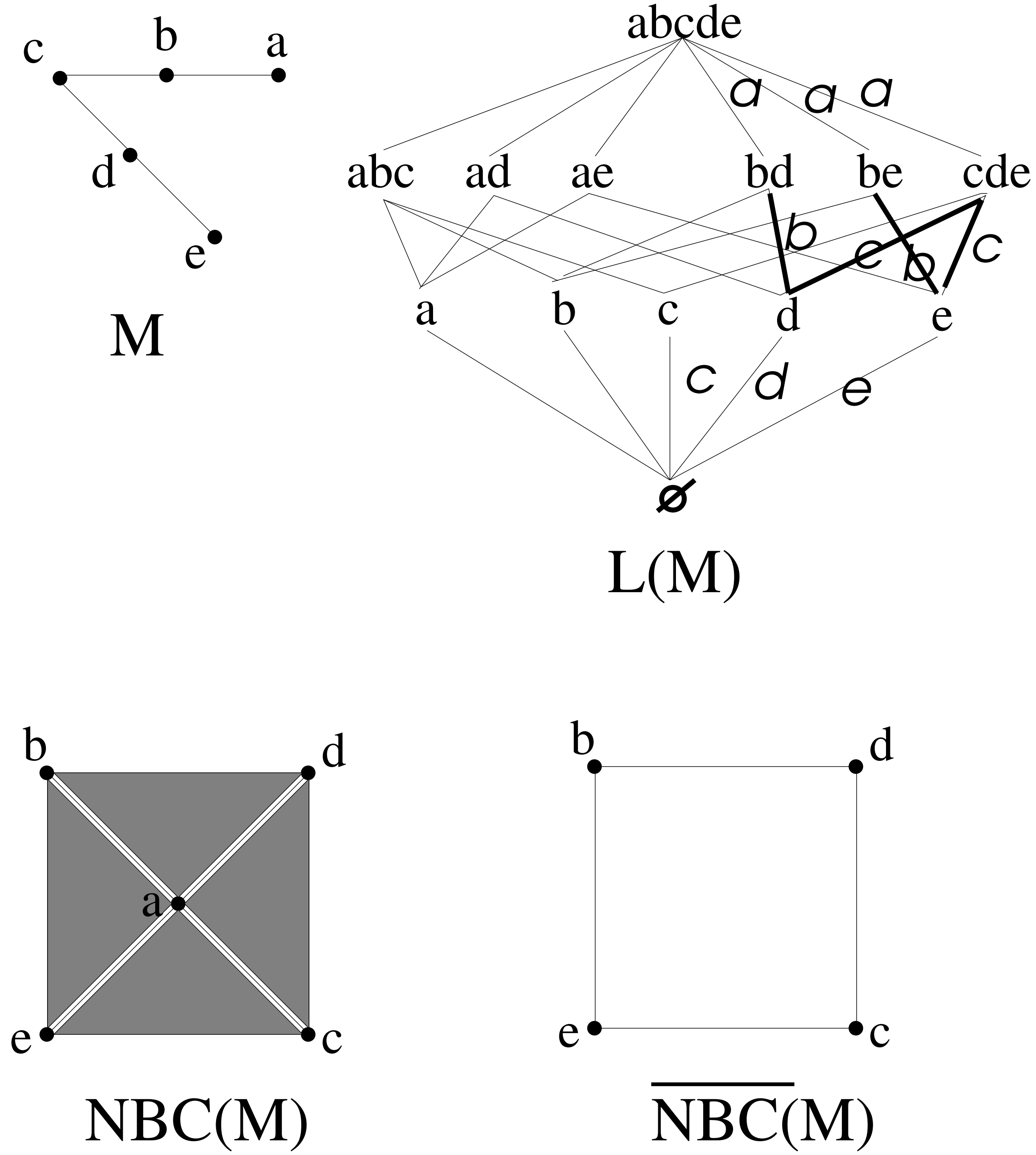


FIGURE 21. A matroid M, its lattice of flats $L(M)$, and reduced NBC complex $\overline{NBC}(M)$.

are exactly the *nbc*-bases of M, and hence the number of spheres in the wedge is

$$|\tilde{\chi}(\Delta)| = |\mu_{L(M)}(\hat{0}, \hat{1})| = |\{\text{nbc-bases of } M\}|.$$

Example 31.
In Figure 21, the matroid M from Example 30 has a few of its edges of the geometric lattice $L(M)$ labelled according the above scheme. The strictly decreasing label sequences occur on the maximal chains that are also darkened, and have label sequences $\{(d,b,a),(e,b,a),(d,c,a),(e,c,a)\}$, corresponding exactly to the four *nbc*-bases $\{abd, abe, acd, ace\}$ of M computed before.

The order complex $\Delta(L(M) - \{\hat{0}, \hat{1}\})$ also turns out to have the same homotopy type as the complex $\mathcal{NS}(M)$ of non-spanning sets of M; the latter is the *cross-cut complex* for the former, if one uses the atoms of $L(M)$ as the cross-cut. See [2] for the technique of cross-cuts.

Recently, the order complex of the proper part $L(M)$ has come up in the theory of *tropical geometry* as it is homeomorphic to the *Bergman fan/complex* of a linear variety/subspace whose associated matroid is M; see [1].

3.2.4. *Some oriented matroid complexes.* In our discussion of the Folkman-Lawrence sphere, matroid polytopes and the Lawrence construction, we've already discussed some cell complexes which turn out to be homeomorphic to spheres. We next discuss a trio of well-behaved (and homotopy equivalent) complexes/posets defined in terms of an oriented matroid $\mathcal{M}$: the *convex, free, and acyclic* sets.

When $\mathcal{M}$ is the oriented matroid realized by a vector configuration $\mathcal{V} = \{v_e\}_{e \in E}$ over $\mathbb{R}$, a subset $A \subset E$ is *acyclic* if it lies on the strictly positive side of some hyperplane, or equivalently (in OM terms) if the restricted oriented oriented $\mathcal{M}|_A$ is acyclic. Define the *convex hull* of $A \subset E$ to be the set of all vectors v_e with $e \in E$ which lie in the cone positively spanned by $\{v_a\}_{a \in A}$; in OM terms, e is in the convex hull of a if every covector f of $\mathcal{M}$ which has $f(a) = +$ for all a in A also has $f(e) = +$. Say A is convex if it equals its own convex hull. Lastly, say that A is free if every element a in A has the property that it does not lie in the convex hull of $A - \{a\}$. It turns out that the convex subsets form a semilattice $L_{\text{convex}}(\mathcal{M})$ under inclusion, while the acyclic sets and the free sets form simplicial complexes, $\Delta_{\text{acyclic}}(\mathcal{M}), \Delta_{\text{free}}(\mathcal{M})$. Figure 22 shows the example of a vector configuration $\mathcal{V} = \{a, b, c, d\}$ which are the columns of

$$\begin{array}{cccc} a & b & c & d \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

along with the three objects $L_{convex}(\mathcal{M}), \Delta_{\text{acyclic}}(\mathcal{M}), \Delta_{\text{free}}(\mathcal{M})$.

It turns out [10] that the three have the same (predictable) homotopy type: the face poset of $\Delta_{\text{free}}(\mathcal{M})$ turns out to be a deformation retraction of (the proper part of) $L_{\text{convex}}(\mathcal{M})$, which in turn is a deformation retraction of the face poset of $\Delta_{acyclic}(\mathcal{M})$. But then $\Delta_{acyclic}(\mathcal{M})$ turns out to be the *nerve* of the good

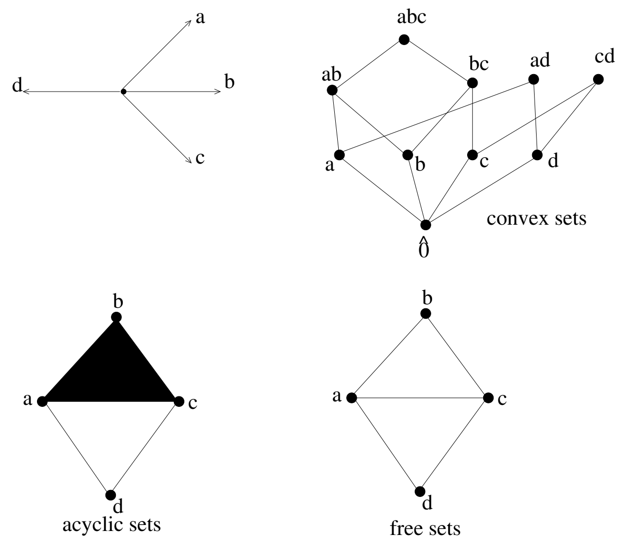


FIGURE 22. A vector configuration $\mathcal{V} = \{a, b, c, d\}$ in $\mathbb{R}^2$, along with the semilattice $L_{\text{convex}}(\mathcal{M})$ of convex sets, and the simplicial complexes of acyclic sets $\Delta_{\text{acyclic}}(\mathcal{M})$ and $\Delta_{\text{free}}(\mathcal{M})$ sets.

covering of the union of all the strictly positive hemispheres inside the Folkman-Lawrence sphere, giving all of three objects homotopy equivalent to the the full Folkman-Lawrence $(\text{rank}(\mathcal{M}) - 1)$-sphere if $\mathcal{M}$ is totally cyclic, and contractible otherwise.

3.3. **Topology of hyperplane complements.** Arrangements of hyperplanes along with their elementary enumerative aspects and relations to matroids are discussed in the notes of Stanley [22]. A more topological viewpoint is dealt with thoroughly in Orlik and Terao's texts [16, 17]. We'll only touch on a few points here.

Let $\mathcal{A}$ be an arrangement hyperplanes in a vector space V over a field $\mathbb{F}$, with associated matroid M. The complement $V - \mathcal{A}$ has differing notions of "topology", depending upon the nature of the field $\mathbb{F}$...

3.3.1. *Finite fields: counting points.* When $\mathbb{F} = \mathbb{F}_q$ is a finite field of order q, we saw in our discussion of Tutte polytnomial evaluations that one can count the points in the complement by an evaluation of the characteristic polynomial:

$$|V - \mathcal{A}| = q^{\dim_{\mathbb{F}} V - \text{rank}(M)} \chi_M(q).$$

3.3.2. *The field $\mathbb{R}$: chambers, separating sets of hyperplanes, weak orders.* When $\mathbb{F} = \mathbb{R}$, the complement $V - \mathcal{A}$ decomposes into connected components (its chambers/regions), each convex and hence contractible, so without much topology.

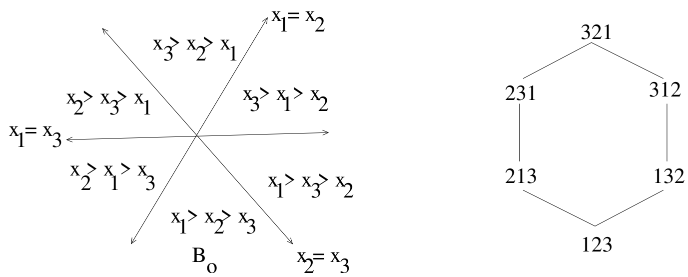


FIGURE 23. The arrangement of reflecting hyperplanes for the symmetric group $W = S_3$, acting in the subspace of $\mathbb{R}^3$ where $x_1 + x_2 + x_3 = 0$, along with the associated weak order on the chambers known as the *weak Bruhat order*. The base chamber B_0 is the one labelled by the identity permutation 123.

On the other hand, the number of these connected components is the number of acyclic orientations/topes of the oriented matroid, which we have seen is another evaluation of the characteristic polynomial

$$|\{ \text{ regions of } V - \mathcal{A}| = (-1)^{\text{rank}(M)} \chi_M(-1).$$

Some interesting features arise from keeping track of the set $\text{sep}(B, B')$ of hyperplanes H which separate two chambers B, B'. If one chooses a particular base region B_0, then the *weak order* $P(\mathcal{A}, B_0)$ on the chambers of $\mathcal{A}$ with respect to B_0 is the partial order on the chambers defined by $B \leq B'$ if $\text{sep}(B_0, B) \subseteq \text{sep}(B_0, B')$. The weak order has the pleasant "visual" feature that its Hasse diagram can be drawn as the 1-skeleton of the associated zonotope $\mathbb{Z}(\mathcal{V})$, where $\mathcal{V}$ is collection of vectors normal to the hyperplanes of $\mathcal{A}$.

When $\mathcal{A}$ is arrangement of reflecting hyperplanes for a finite reflection group W, all choices of base chamber B_0 are equivalent under the action of W, and this gives the usual *weak Bruhat order* on W. See Figure 23for the case where W is the symmetric group S_3.

The weak order construction works for arbitrary oriented matroids $\mathcal{M}$ along with the choice of a base tope B_0, giving the *tope poset* $T(\mathcal{M}, B_0)$, which plays an important role in one of the proofs of the Folkman-Lawrence representation theorem; the Folkman-Lawrence sphere can be shown to be a shellable regular cellular sphere, using as a shelling order on the topes any linear extension of the tope poset $T(\mathcal{M}, B_0)$.

The weak order/tope poset, although not shellable in general, does have known topology for its intervals (Edelman 1984). Say that two an interval $[B, B']$ of topes is *d-facial* if it consists of all of the topes containing some particular covector X that indexes a face of codimension r in the Folkman-Lawrence sphere. In the realizable case, this is equivalent to saying that $[B, B']$ correspond to the set of all vertices on an d-dimensional face of the zonotope $\mathbb{Z}(\mathcal{V})$. Then an open interval

(B, B') in a tope poset $T(\mathcal{M}, B_0)$ is non-contractible if and only if it is d-facial intervals for some d, in which case it is homotopy equivalent to a $(d-2)$-sphere[4]

As a last glimpse of the combinatorics of chambers and separating sets, we mention an amazing determinant evaluation of Varchenko (1993). For each hyperplane H of the arrangement, introduce an indeterminate a_H, and then define the *Varchenko matrix* $V_\mathcal{A}$ to have rows and columns indexed by the chambers of the arrangment $\mathcal{A}$, having (B, B') entry equal to $\prod_{H \in \text{sep}(B,B')} a_H$. Amazingly, its determinant factors as follows:

$$\det(V_\mathcal{A}) = \prod_{\hat{0} \neq X \in L(M)} \left(1 - \prod_{H \subset X} a_X^2\right)^{n(X)p(X)}$$

where $n(X)$ is the number of regions in the arrangement $\mathcal{A}$ *restricted* to the hyperplane X, or equivalently $n(X) = \sum_{Y \geq X} |\mu_{L(M)}(X, Y)|$, while $p(X)$ is Crapo's beta-invariant for the subarrangement of hyperplanes H containing X.

3.3.3. *The field $\mathbb{C}$: topology of the complexified complement.* When $\mathbb{F} = \mathbb{C}$, the complement $V - \mathcal{A}$ has more obvious topology, some of which is still recovered by the characteristic polynomial $\chi_M(t)$ of the matroid M.

It was conjectured by Arnold (1969) and proven by Brieskorn (1971) that the (co-)homology of $V - \mathcal{A}$ is torsion-free. Brieskorn also showed that that its *Poincaré series*

$$\text{Poin}(V - \mathcal{A}, t) := \sum_{i \geq 0} \text{rank}_\mathbb{Z} H^i(V - \mathcal{A}, \mathbb{Z})\, t^i$$

is simply a rescaling of the characteristic polynomial

$$\text{Poin}(V - \mathcal{A}, t) = t^{\text{rank}(M)} \chi_M(-\frac{1}{t}). \tag{7}$$

Arnold further conjectured that cohomology ring $H^*(V - \mathcal{A}, \mathbb{Z})$ should be generated by certain differential 1-forms in the deRham complex which are indexed by the hyperplanes H: if H is the zero set of the linear form $\ell_H \in V^*$ then

$$\omega_H := \frac{1}{2\pi i} \frac{d\ell_H}{\ell_H}.$$

Brieskorn proved this also, and it was eventually shown by Orlik and Solomon (1980) via a deletion-contraction induction, that the cohomology $H^*(V - \mathcal{A}, \mathbb{Z})$ has a very simple abstract presentation with respect to this set of generators. Consider an exterior algebra over $\mathbb{Z}$

$$\Lambda = \mathbb{Z}\langle e_H \rangle$$

[4]Here we are using a common topological combinatorics convention, interpreting a (-1)-sphere to mean a simplicial/cell complex $\{\emptyset\}$ containing only the empty face $\emptyset$, and a (-2)-spheres as an empty complex having no faces at all! Note that this is consistent with saying that the Möbius function value $\mu(x, y)$ is the reduced Euler characteristic of the order complex of the open interval (x, y), when one must deal with intervals of rank 1 (i.e. with only two elements) and of rank 0 (i.e. with only one element).

with degree one generators e_H indexed by the hyperplanes. Then the kernel of the map $\Lambda \to H^*(V - \mathcal{A}, \mathbb{Z})$ that sends $e_H \mapsto \omega_H$ turns out to be generated by some fairly obvious relations dictated by the underlying matroid M: for each circuit $C = \{H_1, \ldots, H_t\}$ in $\mathcal{C}(M)$, the kernel contains the relation

$$dC := \sum_{i=1}^{t} (-1)^{i-1} e_{H_1} \wedge \cdots \wedge \widehat{e_{H_i}} \wedge \cdots e_{H_t}$$

and the ideal I_M in Λ generated by $\{dC\}_{C \in \mathcal{C}(M)}$ generates the kernel. Hence the *Orlik-Solomon algebra* $A_M := \Lambda / I_M$ is isomorphic to $H^*(V - \mathcal{A}, \mathbb{Z})$, showing that the integer cohomology ring structure is a matroid invariant of $\mathcal{A}$.

Bearing this in mind, and comparing the interpretation of $\chi(M, t)$ in (7) with the interpretation in terms of broken circuits (6) suggests a purely algebraic/combinatorial result of Jambu and Terao [14]: the Orlik-Solomon algebra A_M has as a $\mathbb{Z}$-basis the monomials $\{e_A := \wedge_{H \in A} e_H\}_{A \text{ an nbc-set of } M}$. In fact, this can be proven (see [30]) using Gröbner bases for ideals in exterior algebras: one shows that $\{dC\}_{C \in \mathcal{C}(M)}$ form a Gröbner basis for I_A with respect to a certain term ordering, and that $\{e_A\}_{A \text{ an nbc-set of } M}$ are the associated standard monomials.

In the case where $\mathcal{A}$ is the complexification of a real arrangement, there is a simplicial complex due to Salvetti (see [3, §2.5]) having the same homotopy type as the complex complement $V - \mathcal{A}$, and which be written down very simply in terms of the covectors of the associated oriented matroid $\mathcal{M}$. Hence in this case, all homotopy invariants of the complement $V - \mathcal{A}$ can in principle be determined from the oriented matroid $\mathcal{M}$.

For a while in the 1980's it was wondered whether the homotopy type of a complex hyperplane arrangement complement $V - \mathcal{A}$ could be recovered from the weaker data of the underlying matroid M. An example of Rybnikov [21] finally showed that this is not true– even the fundamental group $\pi_1(V - \mathcal{A})$ is not determined by the matroid M alone.

3.4. **Algebraic invariants.**

3.4.1. *Resonance varieties and A_M as a chain complex.* There is much work recently that studies the Orlik-Solomon algebra A_M endowed as a chain complex; see Falk [11] for a nice survey. One first picks a vector $\lambda = (\lambda_H)_{H \in \mathcal{A}}$ in $\mathbb{C}^{\mathcal{A}}$ to define a linear form $e := \sum H a_H e_H \in A_M^1$, and considers the differential $A_M^i \xrightarrow{d_i} A_M^{i+1}$ that multiplies by e. Since $e \wedge e = 0$, the differential squared is zero, and so one can define cohomology groups $H^i(A_M, e) := \ker(d_i)/\mathrm{im}(d_{i+1})$. The p^{th} *resonance variety* $R^p(M)$ is defined as the following locus in $\mathbb{C}^{\mathcal{A}}$:

$$R^p(M) := \{\lambda \in \mathbb{C}^{\mathcal{A}} : H^i(A_M, e) \neq 0\}.$$

These loci have been useful in distinguishing matroids with non-isomorphic algebras A_M. $H^i(A_M, e)$ turns out to be the cohomology of a certain local system on the complement $V - \mathcal{A}$, and relates to the theory of hypergeometric integrals (see [17]).

Some well-behaved commutative rings/ideals are associated to matroids and OM's ...

3.4.2. *Basis monomial rings.* Given a matroid M on ground set $E = \{1, 2, \ldots, n\}$, fix a field $\mathbb{F}$, and consider the subalgebra R_M of the polynomial ring $\mathbb{F}[t_1, \ldots, t_n]$ generated by the monomials $\{t^B := \prod_{e \in B} t_e\}_{B \in \mathcal{B}(M)}$. These semigroup algebras R_M are sometimes called *basis monomial rings*, and were studied by N. White [29], who showed that they were *normal* (= integrally closed in their field of fractions, or equivalently, the semigroup generated by the monomials is saturated) using the theory of polymatroids. This immediately implies they are Cohen-Macaulay, by a result of Hochster [13]. White also conjectured that the ideal of syzygies among the t^B, that is, the (binomial) *toric ideal* I which is the kernel of the map

$$\begin{array}{ccc} S := \mathbb{F}[x_B]_{B \in \mathcal{B}(M)} & \longrightarrow & R_M := \mathbb{F}[t^B]_{B \in \mathcal{B}(M)} \\ x_B & \longmapsto & t^B \end{array}$$

is generated by quadratic binomials, a question which remains open. Several people (see Herzog and Hibi [12]) whether two successively stronger assertions hold:

- Is the ring $R_M = S/I$ a *Koszul algebra*?
- Does the toric ideal I have a quadratic Gröbner basis?

Example 32.

Let M be the rank 2 matroid with

$$\mathcal{B}(M) = \{ab, ac, ad, bc, bd\}$$

or in other words, a, b, c, d are represented by generic vectors in 2-dimensions except for c, d being parallel. Then

$$R_M = \mathbb{F}[ab, ac, ad, bc, bd] \subset \mathbb{F}[a, b, c, d]$$

and the toric ideal $I \subset \mathbb{F}[x_{ab}, x_{ac}, x_{ad}, x_{bc}, x_{bd}]$ turns out to be a principal ideal generated by the syzygy $x_{ad}x_{bc} - x_{ac}x_{bd}$, coming from the fact that $ad \cdot bc - ac \cdot bd = 0$. Note that this syzygy comes from an exchange of an element between the bases ad, bc; White's originally conjecture says (roughly) that the toric ideal is generated by such "basis-exchange" syzygies.

3.4.3. *Matroidal and oriented matroid ideals.* Given a matroid M on ground set $E = \{1, 2, \ldots, n\}$, Novik, Postnikov and Sturmfels [19] define the matroidal ideal I to be the following monomial ideal I in the polynomial ring $\mathbb{F}[x] := \mathbb{F}[x_1, \ldots, x_n]$:

$$I := \langle \prod_{i \notin F} x_i : F \text{ a flat of} M\} \rangle.$$

Given an affine oriented matroid $(\mathcal{M}, g)$, that is, an oriented matroid $\mathcal{M}$ on $E = \{1, 2, \ldots, n\}$ together with an extra element g representing the "hyperplane at infinity", they also define the oriented matroidal ideal J to be the following monomial ideal in the polynomial ring $\mathbb{F}[x, y] := \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$:

$$J := \left\langle \prod_{i \in E : f(i) = +} x_i \prod_{i \in E : f(i) = -} y_i : f \text{ a covector with } f(g) = + \right\rangle$$

They then produce combinatorially structured minimal free resolutions of I and J as modules over their ambient polynomial rings. In the special case where M is orientable (and furthermore augmented with an extra hyperplane g to make an affine oriented matroid $(\mathcal{M}, g)$), these resolutions of I, J have essentially the same simple and beautiful structure, and are *cellular resolutions* in the sense defined by Welker in his lectures for this summer school. In both cases, the cell complex which carries the resolution is the *bounded complex* within the Folkman-Lawrence sphere, whose cells are indexed by the covectors f for which $f(g) = +$. This bounded complex is known to contractible by a result of Björner and Ziegler; see [3, §4.5].

4. Lecture 4: New directions (maps and Hopf algebras)

This lecture might discuss a bit about weak maps and strong maps of matroids and oriented matroids. This could lead into a discussion of

- MacPhersonians and the recent result of D. Biss, and/or
- the Hopf algebra $\mathcal{M}at$ of matroids, including some of the recent work of Crapo and Schmitt, and even perhaps,
- work in progress with Billera on how the greedy algorithm leads to a natural Hopf morphism

  $$\mathcal{M}at \to \mathcal{QS}ym$$

  where $\mathcal{QS}ym$ is the Hopf algebra of the quasisymmetric functions.

(And as so often happens in mathematics, after starting the lectures, we discovered that the previous lectures weren't getting covered so quickly. Therefore, we never ended up having to write this fourth lecture, and simply covered as much as we could of the notes from the previous three!)

5. Problems to accompany the lectures

Most of the problems are quite straightforward; there are few real challenges. The starred problems are highly recommended to get a feeling for the subject.

5.1. **Lecture 1 problems.**

1*. (a) *(Graphic matroids are linearly representable)*
Given a graph $G = (V, E)$, show that one can linearly represent its graphic matroid over any field $\mathbb{F}$ as follows. In the vector space $\mathbb{F}^V$ having standard basis vectors ϵ_v indexed by the vertices v in V, represent the element $e = \{v, v'\}$ in E by the vector $\epsilon_v - \epsilon_{v'}$.

In other words, show that the linearly independent subsets of these vectors are indexed by the forests of edges in G.

(b) *(Transversal matroids are linearly representable)*
Given a bipartite graph G with vertex bipartition $E \cup F$, show that one can represent its transversal matroid in any field characteristic as follows. Let $\mathbb{F}(x_{e,f})$ be a field extension of the field $\mathbb{F}$ by transcendentals $\{x_{e,f}\}$ indexed by all edges $\{e, f\}$ of G. Then in the vector space $\mathbb{F}(x_{e,f})^F$ having standard basis vectors ϵ_f indexed by the vertices f in F, represent the element $e \in E$ by the vector

$$\sum_{f \in F : \{e,f\} \in G} x_{e,f} \; \epsilon_f .$$

In other words, show that the linearly independent subsets of these vectors are indexed by the subsets of vertices in E that can be matched into F along edges of G.

(c) *(Linearly representable matroids are algebraically representable)*
Given a matroid M of rank r linearly represented by a set of vectors $\{v_1, \ldots, v_n\}$ in the vector space $\mathbb{F}^r$, represent M algebraically by elements of the rational function field $\mathbb{F}(x_1, \ldots, x_r)$ as follows. If v_i has coordinates $(v_{i1}, \ldots, v_{ir})$ with respect to the standard basis for $\mathbb{F}^r$, then represent v_i by $f_i := \sum_{j=1}^{r} v_{ij} x_j$.

In other words, show that the algebraically independent subsets of these rational functions f_i are indexed the same as the linearly independent subsets of the v_i.

2*. *(Acyclic orientations and chambers)*
For a graph $G = (V, E)$, consider the graphic hyperplane arrangement $\mathcal{A}$ having hyperplanes of the form $\{x_i = x_j\}_{\{i,j\} \in E}$. Explain how each top-dimensional cell (or *chamber* or *region*) in the decomposition of $\mathbb{R}^V$ cut out by the hyperplanes is naturally labelled by an acyclic orientation of the edges of G, and why this gives a bijection between the chambers and the acyclic orientations.

3*. *(The greedy algorithm works for matroids)*
(a) Show that Kruskal's greedy algorithm (described in Lecture 1) always finds a maximum weight independent set in a matroid, regardless of the choice of weight function $w : E \to \mathbb{R}_+$.
(b) Show that that this property characterizes independent sets of matroids among all simplicial complexes. In other words, given a simplicial complex $\mathcal{I}$ for which the greedy algorithm always works, regardless of the weight function w, show that $\mathcal{I} = \mathcal{I}(M)$ for a matroid M. (Hint: One only needs to show that the exchange axiom I3 holds. To do this, given I_1, I_2 in $\mathcal{I}$ with $|I_2| = |I_1| + 1 = k + 1$, consider the weight function

$$w(e) := \begin{cases} \frac{k+1}{k+2} & \text{for } e \in I_1 \\ \frac{k}{k+1} & \text{for } e \in I_2 - I_1 \end{cases}$$

Explain why the greedy algorithm will build up I_1 first, and then at the next step, will exhibit an element of the form $I_1 \cup \{e\} \in \mathcal{I}$ with $e \in I_2 - I_1$. In particular, explain why the algorithm will not just stop after having found I_1 !)

4. *(Circuit axioms are equivalent to independent set axioms)*
Recall that the circuit axioms assert that $\mathcal{C} \subseteq 2^E$ forms the *circuits* of a matroid M on the finite set E if

- C1. $\varnothing \notin \mathcal{C}$.
- C2. If $C, C' \in \mathcal{C}$ and $C \subset C'$, then $C = C'$.
- C3. Given $C, C' \in \mathcal{C}$, with $C \neq C'$ and $e \in C \cap C'$, there exists some $C'' \in \mathcal{C}$ with $C'' \subseteq C \cup C' - \{e\}$.

Show that circuits give an equivalent axiomatization of matroids as do independents sets.

In other words, given a collection $\mathcal{I} \subseteq 2^E$ of sets satisfying the independent set axioms, show that the collection $\mathcal{C}$ of minimal subsets not in $\mathcal{I}$ satisfy $C1 - C3$, and conversely given a collection $\mathcal{C}$ satisfying $C1 - C3$, show that the collection $\mathcal{I}$ of subsets containing no subset from $\mathcal{C}$ satisfy the independent set axioms.

5. *(Matroids are hereditarily pure)* Show that restricting the independent sets of a matroid M on E to a subset E' of E always gives a pure simplicial complex on E'. Show that this property characterizes independent sets of matroids among all simplicial complexes.

5.2. **Lecture 2 problems.**

6. *(Independence complexes of matroids are characterized by lex shellings)*
For a matroid M on E, show that for any linear ordering on E, the the induced lexicographic ordering on bases of M gives a shelling order on the complex of independent sets $\mathcal{I}$. Show that this property characterizes independent sets of matroids among all simplicial complexes.

7*. *(Planar dual concepts from graph theory)*
Staring at Figure 13, explain why for a planar graph G and a planar dual $G^{\perp}$ (endowed with corresponding edge orientations as explained in § 2.1.4), one has that
(a) *deletion* of an edge e in G corresponds to *contraction* of the crossing edge $e^{\perp}$ in $G^{\perp}$,
(b) *loop* edges in G correspond to *isthmus* edges in $G^{\perp}$,
(c) a *directed cycle* in $G^{\perp}$ corresponds to a *directed bond* (= edges going from one side V' to other side $V - V'$ in a vertex partition $(V', V - V')$) of G,
(d) a set of edges $T \subset E$ forms a *spanning tree* in G if and only if the complementary set $(E - T)^{\perp} := \{e^{\perp} : e \notin T\}$ forms a *spanning tree* in $G^{\perp}$, and
(e) an *acyclic orientation* of the edges of G corresponds to a *totally cyclic orientation* (= one in which every directed edge lies in some directed cycle) of the edges of $G^{\perp}$.

8*. *(Independence complexes of matroids are vertex-decomposable)*
Let Δ be a simplicial complex on vertex set E. We do not assume that every $e \in E$ is actually used as a vertex of Δ. The concept of *vertex-decomposability* for a simplicial complex Δ on vertex set E is defined recursively: both the complex $\Delta = \emptyset$ having no faces at all (not even the empty face) and any complex Δ consisting of a single vertex are defined to be vertex-decomposable, and then Δ is said to be vertex-decomposable if it is pure, and there exists a vertex $e \in E$ for which both its *deletion* and *link*

$$\begin{aligned} \mathrm{del}_{\Delta}(e) &:= \{F \in \Delta : e \notin F\} \\ \mathrm{link}_{\Delta}(e) &:= \{F - \{e\} : e \in F \in \Delta\} \end{aligned}$$

are vertex-decomposable complexes.
(a) Show that vertex-decomposable complexes Δ are shellable. (Hint: Obviously one wants to use induction. Shell the facets in the deletion of e first, then those in the *star* of e, which is the cone over the link with apex e.)
(b) Show that for a matroid M with $\Delta = \mathcal{I}(M)$ and any non-loop, non-coloop element $e \in E$, one has

$$\begin{aligned} \mathrm{del}_{\Delta}(e) &:= \mathcal{I}(M \backslash e) \\ \mathrm{link}_{\Delta}(e) &:= \mathcal{I}(M/e) \end{aligned}$$

Deduce that independent set complexes $\mathcal{I}(M)$ of matroids are vertex-decomposable.

9*. *(Representability of Fano, non-Fano matroids)*
Here is an approach to showing that the Fano matroid is coordinatizable only in characteristic 2, and the non-Fano matroid is coordinatizable only in characteristic not 2.
(a) First try to show that in any coordinatization $\mathcal{V} = \{a, b, c, d, e, f, g\}$ of either the Fano or non-Fano matroids, with elements labelled as in Figure 21(b), one can use the action of $GL_3(\mathbb{F})$ along with scaling of individual vectors to assume that the representing matrix has columns looking like this:

$$\begin{matrix} a & b & c & d & e & f & g \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & \gamma \\ 0 & 1 & 0 & 1 & 0 & \alpha & \delta \\ 0 & 0 & 1 & 0 & 1 & \beta & \epsilon \end{bmatrix}$$

(b) Use some of the matroid dependencies to show that $\gamma = \delta = \epsilon$, and hence by scaling, $\gamma = \delta = \epsilon = 1$.
(c) Use some more of the matroid dependencies to show that $\alpha = \beta$.
(d) Use the last matroid dependence in the Fano (and its absence in the non-Fano) to decide whether or not the characteristic of $\mathbb{F}$ is 2.

10*. *(Whitney's twists leave a graphic matroid unchanged)*
(a) Explain why two graphs G, G' that differ by a twist along a pair of vertices will have the same matroids $M(G) = M(G')$.
(b) Can you see how to recover a 3-vertex-connected graph $G = (V, E)$ from the matroid $M(G)$? (Hint: for each vertex $v \in V$, can you show that the set of edges $H \subset E$ in the bond/cut that goes from $\{v\}$ to $V - \{v\}$ have the property that $M(G) \backslash H$ remains a connected matroid, and that these are the minimal edge cuts with respect to inclusion having this property).

11*.*(Duality of Plücker coordinates and matrix-tree-type theorems)*
Let M be an $r \times n$ matrix of (full) rank r over some field $\mathbb{F}$, and by abuse of notation, let M also denote the matroid on ground set $E := \{1, 2, \ldots, n\}$ represented by the columns of M. Let $M^\perp$ be $(n-r) \times n$ matrix whose rows span the nullspace/kernel of M, so that $M^\perp$ represents the orthogonal matroid $M^\perp$.

(a) Show that after re-ordering the column indices $E = \{1, 2, \ldots, n\}$, there exist $P \in GL_r(\mathbb{F}), Q \in GL_{n-r}(\mathbb{F})$ such that

$$\begin{aligned} PM &= [I_r | A] \\ QM^\perp &= [-A^T | I_{n-r}]. \end{aligned}$$

for some $r \times (n-r)$ matrix A.

(b) Prove the relationship (2) between complementary Plücker coordinates (Hint: one way is to use part (a) to reduce to the case where $M, M^\perp$ has the special form given on the right-hand sides of part (a)).

(c) Use the Binet-Cauchy Theorem to prove the following generalization of Kirchhoff's "matrix-tree theorem". If D is an $n \times n$ diagonal matrix of indeterminates $\{x_e\}_{e \in E}$, then

$$\det(M\,D\,M^T) = \sum_{\text{bases } B \in \mathcal{B}(M)} \det(M|_B)^2 \prod_{e \in B} x_e.$$

(d) Deduce that when M is represented over $\mathbb{F} = \mathbb{Q}$ by an *integer* matrix M, one has an upper bound

$$|\mathcal{B}(M)| \leq \det(M\,M^T)$$

with equality if and only if every maximal minor of M is ± 1 or 0.

Show that this equality occurs when M represents the graphic matroid $M(G)$ for a graph G as follows: M is a reduced version of its usual *node-edge signed incidence matrix* (that is, the cellular boundary map $C_1(G; \mathbb{Z}) \to C_0(G; \mathbb{Z})$, after orienting the edges arbitrarily) in which one deletes rows corresponding to a choice of one vertex chosen from each connected component of G.

(e) Deduce from (b), (c) a dual version of part (d):

$$c^2 \cdot \det(M^\perp\, D\, (M^\perp)^T) = \sum_{\text{bases } B \in \mathcal{B}(M)} \det(M|_B)^2 \prod_{e \notin B} x_e$$

where c is the overall scalar present in (2).

(f) Deduce from (b), (c) the following self-dual version. If $D(x)$ is an $n \times n$ diagonal matrix of indeterminates $\{x_e\}_{e \in E}$, as before, and $D(y)$ is the same matrix after substituting x_e for y_e, then

$$c \cdot \det \begin{bmatrix} M D(x) \\ M^\perp D(y) \end{bmatrix} = \sum_{\text{bases } B \in \mathcal{B}(M)} \det(M|_B)^2 \prod_{e \in B} x_e \prod_{e \notin B} y_e.$$

5.3. **Lecture 3 problems.**

12*. *(Various graphic Tutte specializations)*
This problem gives a few standard deletion-contraction recurrences for graph-theoretic quantities, that can be used to prove some of the Tutte polynomial specializations asserted in the lecture.
(a) Let $G = (V, E)$ be a graph, and for a positive integer t, let $p_G(t)$ denote the number of *proper colorings of V with t colors*, i.e. $p_G(t)$ is Birkhoff's *chromatic polynomial*. Show that for any non-loop edge $e \in E$, one has the deletion-contraction relation

$$p_{G-e}(t) = p_G(t) + p_{G/e}(t).$$

and then explain how to use Proposition 28 to deduce the first equation in (4). (Hint: for a proper coloring of $G - e$, did the endpoints $\{v, v'\}$ of edge e get colored with the same colors, or different colors?)
(b) Let $G = (V, E)$ be a graph, and for a positive integer t, let $p^{\perp}_G(t)$ denote the number of *nowhere-zero flows* on E with values in $\mathbb{Z}/t\mathbb{Z}$. Show that for any non-isthmus edge $e \in E$, one has the deletion-contraction relation

$$p^{\perp}_{G/e}(t) = p^{\perp}_G(t) + p^{\perp}_{G-e}(t)$$

and then explain how to use Proposition 28 to deduce the second equation in (4). (Hint: for a nowhere-zero flow on G/e, if one "uncontracts" the edge e to go from G/e to G, is the unique value that must be assigned to e to maintain the flow conditions equal to zero or not?)
(c) Let $G = (V, E)$ be a graph and a_G the number of *acyclic orientations* of E. Show that for any non-loop edge $e \in E$, one has the deletion-contraction relation

$$a_G = a_{G-e} + a_{G/e}.$$

and then explain how to use Proposition 28 to deduce the first equation in (5). One approach to this is as follows. Let α_i for $i = 0, 1, 2$ be the number of acyclic orientations of $G - e$ that can be extended to G in exactly i ways by orienting the edge e, and show that

$$\alpha_1 + 2\alpha_2 = a_G$$
$$\alpha_2 = a_{G/e}.$$

(d) Let $G = (V, E)$ be a graph and $a^{\perp}_G$ the number of *totally cyclic orientations* of E. Show that for any non-isthmus edge $e \in E$, one has the deletion-contraction relation

$$a^{\perp}_G = a_{G/e} + a_{G-e}.$$

and then explain how to use Proposition 28 to deduce the second equation in (5). One approach to this is as follows. Let α_i for $i = 0, 1, 2$ be the number of totally cyclic orientations of G/e that can be extended to G in exactly i ways by orienting the edge e, and show that

$$\alpha_1 + 2\alpha_2 = a^{\perp}_G$$
$$\alpha_2 = a^{\perp}_{G-e}.$$

You may find it useful here to note that an orientation of E is totally cyclic if and only if every connected component C in G is oriented in such a way that, as a digraph, it is *strongly connected*, i.e. there are directed paths in *both* directions between any pair of vertices in C.

13*. *(Nonspanning sets are Alexander dual to independent sets)*
Given a matroid M on ground set E, show that the simplicial complex of non-spanning sets $\mathcal{NS}(M)$ is the canonical Alexander dual to the simplicial complex of independent sets of $M^\perp$. In other words, show that $A \subset E$ is nonspanning ($\overline{A} \neq E$) if and only if $E - A$ is not independent in $M^\perp$.

14*. *(Boolean hyperplane arrangements)*
Boolean arrangements are the simplest examples of hyperplane arrangements. Let $\mathbb{F}$ be any field, and consider the *Boolean arrangement* $\mathcal{A}$ consisting of all coordinate hyperplanes $\{H_i\}_{i=1,\ldots,n}$ in $V = \mathbb{F}^n$ where H_i is defined by $x_i = 0$. Consider also the associated matroid M, the oriented matroid $\mathcal{M}$ when $\mathbb{F} = \mathbb{R}$, and the complement $V - \mathcal{A}$ (= the algebraic torus $(\mathbb{F}^\times)^n$).

(a) Compute the characteristic polynomial $\chi_M(t)$.

(b) Check that the complement $V - \mathcal{A}$ has

- for $\mathbb{F} = \mathbb{F}_q$, exactly $q^{\dim V - \mathrm{rank}(M)} \chi_M(q)$ points,
- for $\mathbb{F} = \mathbb{R}$, exactly $(-1)^{\mathrm{rank}(M)} \chi_M(-1)$ chambers,
- for $\mathbb{F} = \mathbb{C}$, Poincaré polynomial $(-t)^{\mathrm{rank}(M)} \chi_M\left(-\frac{1}{t}\right)$, and cohomology algebra structure agreeing with the Orlik-Solomon algebra A_M.

15. *(Broken circuit complexes)*
Let M be a matroid of rank r on ground set E, totally ordered by ω.

(a) Show that $NBC(M) = NBC(\hat{M})$ where $\hat{M}$ denotes the simplification of M.

(b) Show that $NBC(M)$ is pure of dimension $r - 1$ by showing that every *nbc*-set $I \subset E$ is independent, and that the ω-lexicographically first basis B of M containing I is actually an *nbc*-set (i.e. an *nbc*-basis).

(c) Show that the ω-lexicographic order on NBC-bases gives a shelling order for $NBC(M)$ (and hence also for $\overline{NBC}(M)$).

(c) Show that for any flat X in the geometric lattice of flats $L(M)$, one has

$$\mu_{L(M)}(\hat{0}, X) = (-1)^{r(X)} |\{ \text{ nbc-sets } I \subseteq E : \bar{I} = X\}|.$$

(Hint: Show the right-hand side satisfies the proper identity that defines $\mu_{L(M)}(\varnothing, F)$, via a sign-reversing involution).

(d) Deduce from (c) Whitney's interpretation of the coefficients of the chromatic/characteristic polynomial given in (6):

$$\chi_M(t) = \sum_{\text{nbc-sets } I} (-1)^{|I|} t^{\text{rank}(M)-|I|}.$$

(e) Conclude that Crapo's beta-invariant and the reduced Euler characteristic of $\overline{NBC}(M)$ are related by

$$\tilde{\chi}(\overline{NBC}(M)) = (-1)^r \beta(M).$$

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