On the coincidental reflection groups

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1. A few of my favorite product formulas

2. Reflection group generalizations with invariant theory explanations

3. The coincidental groups and two theorems

1. Product formulas

Start with three that generalize the fact that the symmetric group

$$\mathfrak{S}_n = \left\{ \begin{array}{l} \text{permutations} \\ \qquad w = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ w_1 & w_2 & w_3 & \cdots & w_n \end{pmatrix} \end{array} \right\}$$

has cardinality

$$|\mathfrak{S}_n| = n! = 1 \cdot 2 \cdots n$$

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = [n]!_q$$

Mahonian distribution

$$:= [1]_q [2]_q \cdots [n]_q$$

where $[m]_q := 1 + q + q^2 + \ldots + q^{m-1}$

and $\mathrm{inv}(w) := \#\{1 \le i < j \le n : w_i > w_j\}$

= inversion number of w

$n=3$

w	$\mathrm{inv}(w)$
123	0
132	1
213	1
231	2
312	2
321	3

$$1 + 2q + 2q^2 + q^3$$

$$= 1 \cdot (1+q)(1+q+q^2)$$

$$= [1]_q [2]_q [3]_q$$

$$\sum_{w \in \mathfrak{S}_n} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$$= \sum_k c(n,k)\, q^k$$

signless Stirling numbers of the 1st kind

$n=3$

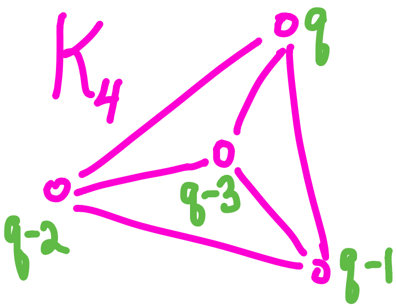
w	$\#\text{cycles}(w)$
$123 = (1)(2)(3)$	3
$132 = (1)(23)$	2
$213 = (12)(3)$	2
$231 = (123)$	1
$312 = (132)$	1
$321 = (13)(2)$	2

$$q^3 + 3q^2 + 2q^1 = q(q+1)(q+2)$$

$$\sum_{w\in\mathfrak{S}_n} \mathrm{sgn}(w)\, q^{\#\mathrm{cycles}(w)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$= \sum_k s(n,k)\, q^k$$

(signed) Stirling numbers of the 1st kind



= chromatic polynomial of complete graph K_n, counting proper q-colorings

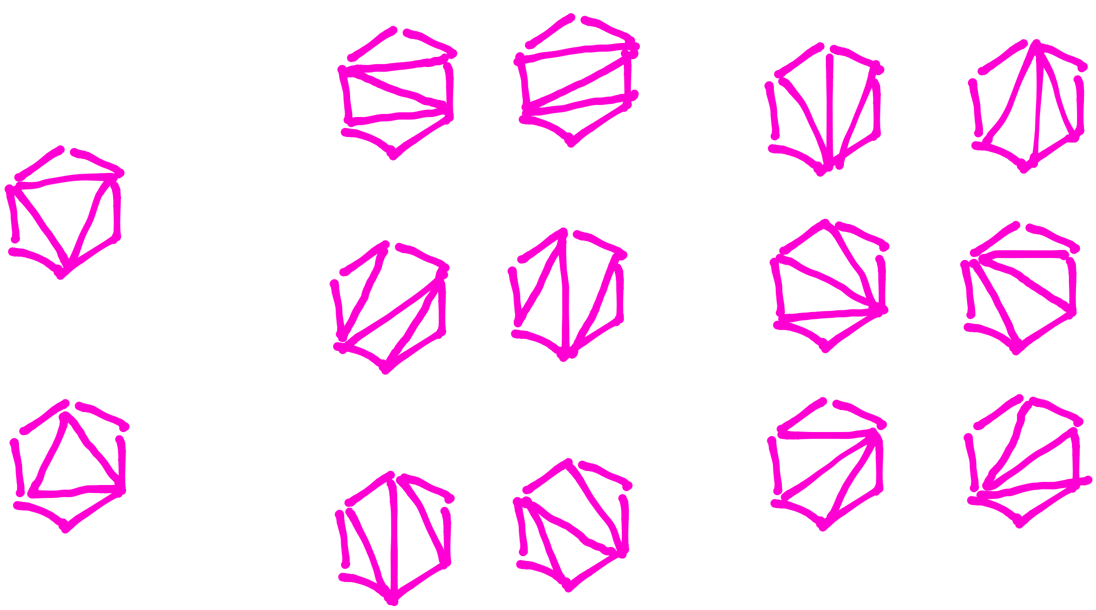
$q=-1$

$$(-1)^n\,|\mathfrak{S}_n| = (-1)(-2)\cdots(-n)$$

$$\#\text{triangulations of an } (n+2)\text{-gon} = \frac{1}{n+1}\binom{2n}{n}$$

$$=: \text{Catalan number} \qquad = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$

$n=4 \qquad \frac{1}{4+1}\binom{2\cdot 4}{4} = \frac{6\cdot 7\cdot 8}{2\cdot 3\cdot 4} = 14$

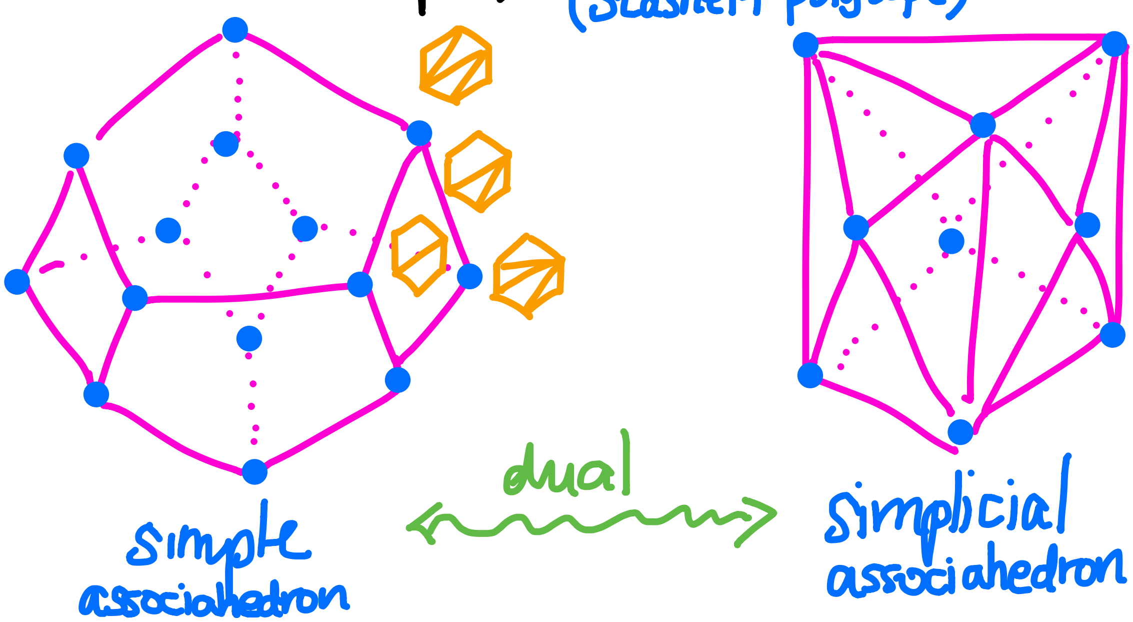


More generally,

Kirkman-Cayley or 'little' Schröder numbers

#dissections of $(n+2)$-gon using $n-1-k$ diagonals $= \dfrac{1}{n}\dbinom{n}{k}\dbinom{2n-k}{n-k-1}$

gives the f-vector (= face numbers) for the (simple) associahedron (Stasheff polytope)



$(f_0, f_1, f_2, f_3) = (14, 21, 9, 1)$

vertices, edges, polygons, 3-face

The corresponding h-vector entries are celebrated products too!

f-vector

$(f_0, f_1, f_2, f_3) = (14, 21, 9, 1)$ $\quad f(t) = \sum_{i=0}^{n-1} f_i t^i$

$= 14 + 21t + 9t^2 + t^3$

$\updownarrow$ $\qquad t \mapsto t-1 \qquad t \mapsto t+1$

h-vector

$(h_0, h_1, h_2, h_3) = (1, 6, 6, 1)$ $\quad h(t) = \sum_{i=0}^{n-1} h_i t^i$

$= 1 + 6t + 6t^2 + t^3$

$$h_k = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

Narayana numbers

Catalan, Kirkman-Cayley, Narayana numbers also have q-analogues with (essentially) same product formulas, that hide counts for objects with cyclic symmetry in their $q = e^{2\pi i/d}$ evaluations. [cyclic sieving phenomenon; R.-Stanton-White 2004]

E.g. q-Catalan $\dfrac{[n+2]_q [n+3]_q \cdots [2n]_q}{[2]_q [3]_q \cdots [n]_q}$

evaluated at $q = \left(e^{\frac{2\pi i}{n+2}}\right)^d$

counts triangulations of $(n+2)$-gon fixed by $\frac{2\pi}{n} \cdot d$ rotation

$n=4$

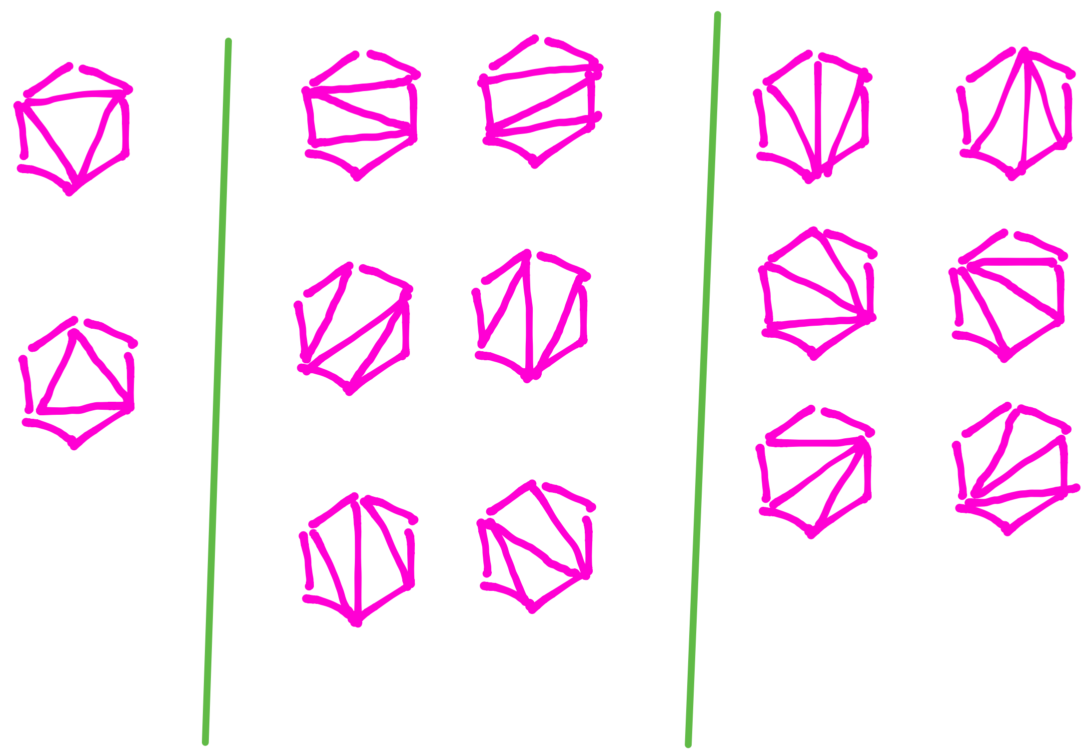
$$\frac{[6]_q[7]_q[8]_q}{[2]_q[3]_q[4]_q} = 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

$q=e^{\frac{2\pi i}{6}}$ $\rightsquigarrow$ 0

$q=e^{\frac{2\pi i}{3}}$ $\rightsquigarrow$ 2

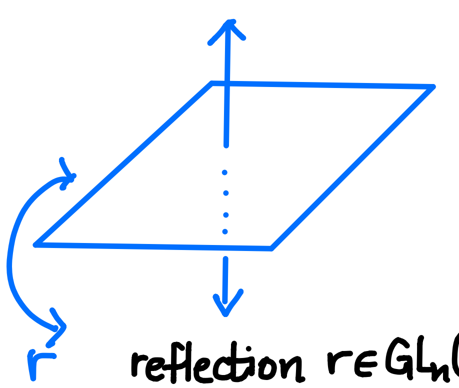
$q=-1$ $\rightsquigarrow$ 6

$q=1$ $\rightsquigarrow$ 14



2. Reflection group generalizations

A reflection group is a finite subgroup $W \subset GL_n(\mathbb{C}) = GL(V)$ generated by reflections $r \in W$, which are elements whose fixed space $V^r := \{v \in V : r(v) = v\}$ is a hyperplane.



reflection $r \in GL_n(\mathbb{R})$

r diagonalizes to

$$\begin{bmatrix} \zeta & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

in $GL_n(\mathbb{C})$

Shephard & Todd (1955) classified the irreducible reflection groups into

- one infinite family $G(de, e, n)$
- 34 exceptional groups

on their way to proving this:

THEOREM: When a finite subgroup $W \subset GL_n(\mathbb{C})$ acts on the polynomials $S = \mathbb{C}[x_1, \ldots, x_n]$, the W-invariant subalgebra S^W is again a polynomial algebra $S^W = \mathbb{C}[f_1, \ldots, f_n]$

$\iff$ W is a reflection group

In this case, the degrees

$$\begin{array}{ccccc} d_1 \le & d_2 \le & \ldots & \le d_n \\ \| & \| & & \| \\ \deg(f_1) & \deg(f_2) & & \deg(f_n) \end{array}$$

in $S^W = \mathbb{C}[f_1, \ldots, f_n]$ are unique, and give

$$\mathrm{Hilb}(S^W, q) := \sum_{d=0}^{\infty} q^d \cdot \dim_{\mathbb{C}} (S^W)_d$$

$$= \prod_{i=1}^{n} \frac{1}{1-q^{d_i}}$$

EXAMPLE $\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$

degrees $\begin{array}{cccc} d_1 \le & d_2 \le & \ldots & \le d_n \\ \| & \| & & \| \\ 1 & 2 & \ldots & n \end{array}$

and $\mathrm{Hilb}(S^{\mathfrak{S}_n}, q) = \dfrac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$

Shephard and Todd deduced this:

COROLLARY: For a finite ref'n group W,

$$d_1 d_2 \cdots d_n = |W|$$

and more generally,

$$[d_1]_q [d_2]_q \cdots [d_n]_q = \mathrm{Hilb}\left(S/(f_1, \ldots, f_n), q\right)$$

the coinvariant algebra, a graded version of the regular rep'n $\mathbb{C}W$

$$= \sum_{w \in W} q^{\ell(w)}$$

when W is a real ref'n group

$\ell(w)$ = Coxeter group length of w

$W = B_3$

Shephard & Todd also noted this...

THEOREM: W a reflection group has

$$\sum_{w \in W} q^{\dim(V^w)} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

where the exponents $(e_1, e_2, \ldots, e_n)$ can be defined by $e_i = d_i - 1$

EXAMPLE $W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$ has exponents $(0, 1, 2, \ldots, n-1)$, and so

$$\sum_{w \in \mathfrak{S}_n} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+n-1)$$

since $\dim(V^w) = \#\text{cycles}(w)$

This formula wasn't well-understood until...

THEOREM (Solomon 1963)

W a reflection group has $(S \otimes \wedge^{\cdot} V^*)^W$ an S^W-exterior algebra on exterior generators $df_1, \ldots, df_n$

where $df := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \otimes x_j$

$$\Downarrow$$

df_i contributes $q^{e_i} t$

$$\mathrm{Hilb}\Big((\overset{q}{S} \otimes \overset{t}{\wedge^{\cdot}} V^*)^W, q, t \Big) = \prod_{i=1}^{n} \frac{1+q^{e_i} t}{1-q^{d_i}}$$

f_i contributes q^{d_i}

The missing pieces are Molien formulas:
for finite subgroups $W \subset GL_n(\mathbb{C})$,

$$\frac{1}{|W|} \sum_{w \in W} \mathrm{Trace}_V(w) = \dim(V^W)$$

$$\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1-qw)} = \mathrm{Hilb}(S^W, q)$$

$$\frac{1}{|W|} \sum_{w \in W} \frac{\det(1+tw)}{\det(1-qw)} = \mathrm{Hilb}\left((S \otimes \Lambda V^*)^W, q, t\right)$$

From here, the product formulas come from coefficient extraction.

Later came this...

THEOREM (Orlik & Solomon 1980)

W a reflection group has

$$\sum_{w \in W} \det(w)\, q^{\dim(V^w)} = \sum_{\substack{\text{reflecting}\\ \text{hyperplane}\\ \text{intersections } X}} \mu(V,X)\, q^{\dim X}$$

characteristic polynomial for the reflection arrangement

$$= (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

where $(e_1^*, e_2^*, \ldots, e_n^*)$ are the co-exponents

giving the degrees of an

S^W-basis $\Theta_1, \Theta_2, \ldots, \Theta_n$ for $(S \otimes V)^W$

i.e. $\Theta_i = \sum_{j=1}^{n} \Theta_i^{(j)} \otimes y_j$

has polynomial degree e_i^*

For real refl'n groups W, one has $V \cong V^*$ and co-exponents e_i^* = exponents e_i

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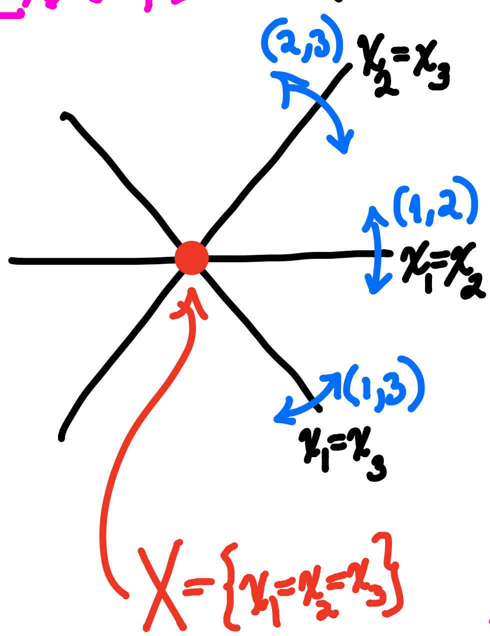
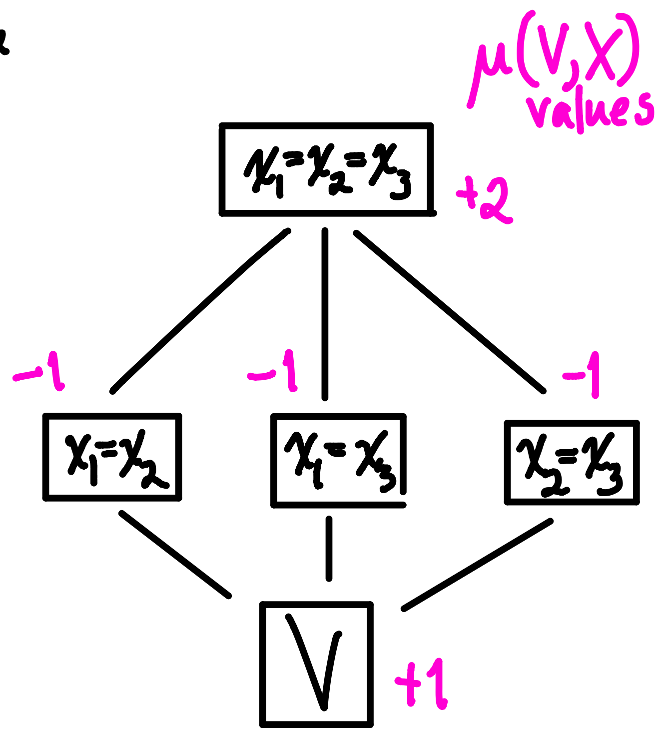
EXAMPLE $W = \mathfrak{S}_3$ acting on $V = \mathbb{R}^3$





$$\sum_X \mu(V,X)\, q^{\dim X} = q^3 - 3q^2 + 2q$$
$$= q(q-1)(q-2)$$

Orlik & Solomon deduced it from this result...

THEOREM W a reflection group has $(S \otimes \wedge^{\cdot} V)^W$ an S^W-exterior algebra on exterior generators $\theta_1, \theta_2, \ldots, \theta_n$

$\Downarrow$

θ_i contributes $q^{e_i^*} u$

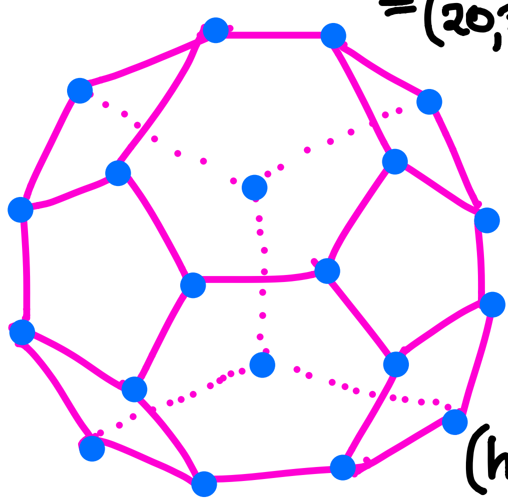
$$\mathrm{Hilb}\big((\overset{q}{S} \otimes \overset{u}{\wedge^{\cdot}} V)^W, q, u \big) = \prod_{i=1}^{n} \frac{1 + q^{e_i^*} u}{1 - q^{d_i}}$$

f_i contributes q^{d_i}

Catalan, Kirkman-Cayley, Narayana numbers generalize to real reflection groups W as f-vectors and h-vectors for finite type cluster complexes (Fomin & Zelevinsky 2001) and Cambrian fans (Reading 2004)

$W = B_3$

$(f_0, f_1, f_2, f_3) = (20, 30, 12, 1)$



$(h_0, h_1, h_2, h_3) = (1, 9, 9, 1)$

cyclohedron or Bott-Taubes polytope

$$f(t) = 20 + 30t + 12t^2 + t^3$$

$t \mapsto t-1$ $\downarrow$ $\uparrow$ $t \mapsto t+1$

$$h(t) = 1 + 9t + 9t^2 + t^3$$

These f_k have a nice q-analogue with an invariant theory connection:

THEOREM (Armstrong-R.-Rhoades 2015)
A real reflection group W with $h := d_n$ (Coxeter number) has $f_k = \lim_{q \to 1} f_k(q)$ where $f_k(q)$ is

$$\left. \frac{\mathrm{Hilb}\left(\left(S \otimes \wedge V^* \otimes \wedge^k V \right)^W, q, t \right)}{\mathrm{Hilb}(S^W, q)} \right|_{t = -q^{h+1}}$$

(not so well-understood, though!)

The $k=0$ case is the q-Catalan number for W
and has a product formula

$$f_0(q) = \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

derived from Solomon's Theorem

EXAMPLE For $W = \mathfrak{S}_n$,

$$f_0(q) = \frac{[n+2]_q [n+3]_q \cdots [2n]_q}{[2]_q [3]_q \cdots [n]_q} = q\text{-Catalan}$$

The $k=0$ case also was checked [Eu & Fu 2006]
to have a cyclic sieving phenomenon
for a natural cyclic action on clusters.

However the other $f_k(q)$ outside of the boundary cases

$k = 0, 1, n-1, n$

seem to have

- no product formula
- no cyclic sieving phenomenon

... except in the

coincidental

types W ...

3. The coincidental types

Shephard & Todd's classification dichotomy ...

- One infinite family

$G(de, e, n) = n \times n$ monomial matrices with nonzero entries in $\sqrt[de]{1}$ and product in $\sqrt[d]{1}$

$$\begin{bmatrix} 0 & 0 & \zeta^2 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & \zeta^4 & 0 & 0 \end{bmatrix}$$

- 34 exceptions

... is deceptive.

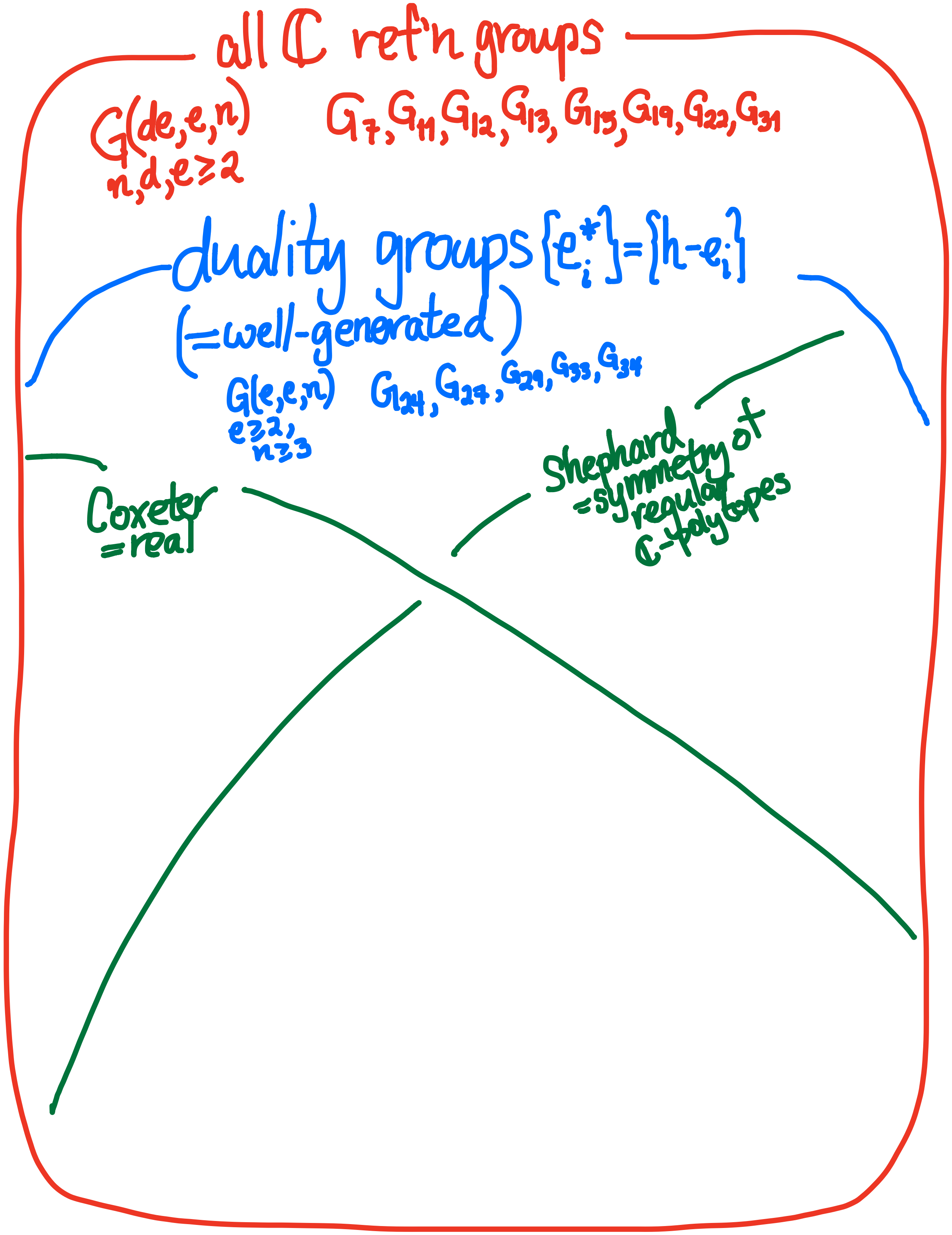
all $\mathbb{C}$ ref'n groups

all $\mathbb{C}$ ref'n groups

$G(de,e,n)$ $n,d,e \geq 2$

$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$

duality groups $\{e_i^*\} = \{h - e_i\}$ (= well-generated)


all $\mathbb{C}$ ref'n groups
$G(de,e,n)$
$n,d,e \geq 2$
$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$
duality groups $\{e_i^*\} = \{h - e_i\}$
(=well-generated)
$G(e,e,n)$
$e \geq 2,$
$n \geq 3$
$G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$
Coxeter
=real
Shephard
=symmetry of
regular
$\mathbb{C}$-polytopes

all $\mathbb{C}$ ref'n groups

$G(de,e,n)$
$n,d,e \geq 2$

$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$

duality groups $\{e_i^*\} = \{h - e_i\}$
(=well-generated)

$G(e,e,n)$
$e \geq 2,$
$n \geq 3$

$G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$

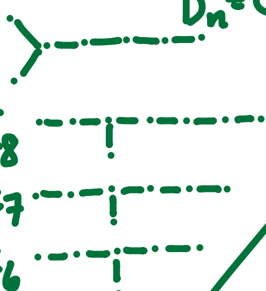
Coxeter
=real

$D_n = G(2,2,n)$

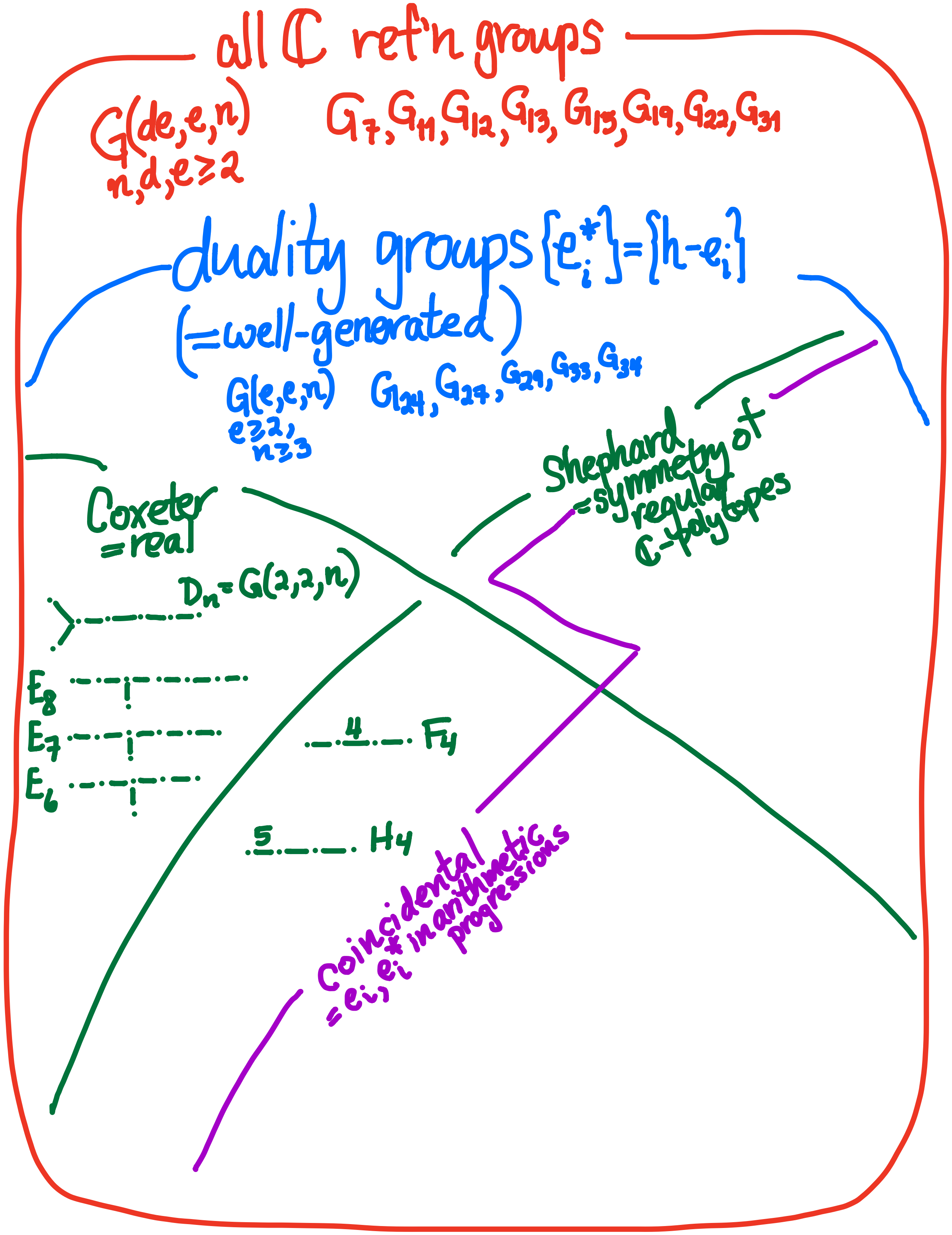
E_8

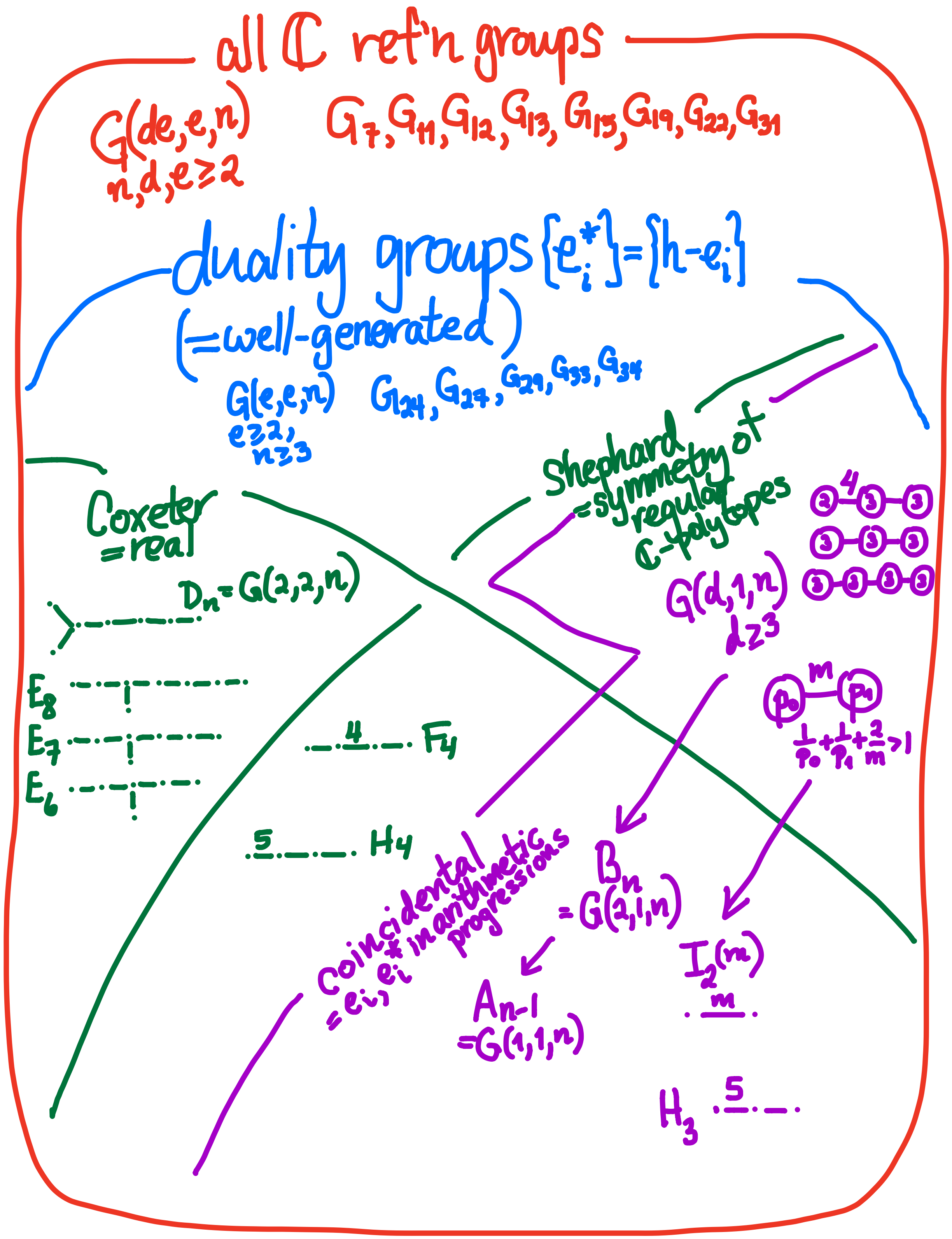
E_7

E_6



Shephard
=symmetry of
regular
$\mathbb{C}$-polytopes


all ℂ ref'n groups
G(de,e,n)
n,d,e ≥ 2
G7, G11, G12, G13, G15, G19, G22, G31
duality groups {e*i} = {h − ei}
(= well-generated)
G(e,e,n)
e ≥ 2,
n ≥ 3
G24, G27, G29, G33, G34
Coxeter
= real
Dn = G(2,2,n)
E8
E7
E6
4
F4
5
H4
Shephard
= symmetry of
regular
ℂ-polytopes
coincidental
= ei, e*i in arithmetic
progressions



all ℂ ref'n groups
$G(de,e,n)$
$n,d,e \geq 2$
$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$
duality groups $\{e_i^*\} = \{h - e_i\}$
(=well-generated)
$G(e,e,n)$
$e \geq 2,$
$n \geq 3$
$G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$
Coxeter
=real
$D_n = G(2,2,n)$
E_8
E_7
E_6
4
F_4
5
H_4
Shephard
=symmetry of regular ℂ-polytopes
2 4 3 3
3 3 3
3 3 3 3
$G(d,1,n)$
$d \geq 3$
p_0 m p_1
$\frac{1}{p_0} + \frac{1}{p_1} + \frac{2}{m} > 1$
coincidental
$= e_i, e_i^*$ in arithmetic progressions
B_n
$= G(2,1,n)$
A_{n-1}
$= G(1,1,n)$
$I_2(m)$
m
H_3
5

DEFINITION:

A reflection group $W \subset GL_n(\mathbb{C})$ is

coincidental

if its exponents and coexponents form arithmetic progressions

$$(e_1, \ldots, e_n) = (e_1, e_1+a, e_1+2a, \ldots, e_1+(n-1)a)$$

$$(e_1^*, \ldots, e_n^*) = (1, 1+a, 1+2a, \ldots, 1+(n-1)a)$$

with the same

gap $a := e_i - e_{i-1}$
$= e_i^* - e_{i-1}^*$

EXAMPLES

$A_{n-1} = G_n$ has $(e_1, e_2, \ldots, e_{n-1})$

$= (e_1^*, e_2^*, \ldots, e_{n-1}^*)$

$= (1, 2, \ldots, n-1)$ so $a = 1$

$B_n = G(2,1,n)$ has $(e_1, e_2, \ldots, e_n)$

$= (e_1^*, e_2^*, \ldots, e_n^*)$

$= (1, 3, 5, \ldots, 2n-1)$ so $a = 2$

$G(d,1,n)$ has $(e_1, \ldots, e_n) = (d-1, 2d-1, \ldots, nd-1)$

$(e_1^*, \ldots, e_n^*) = (1, d+1, 2d+1, \ldots, (n-1)d+1)$

so $a = d$

THEOREM (R.-Shepler-Sommers 2019)

For coincidental W,

$$\mathrm{Hilb}\left(\left(S\otimes\wedge V^*\otimes\wedge^r V\right)^W, q, t\right) = \sigma_r\left(q^{e_1^*},\dots,q^{e_n^*}\right)\frac{\prod_{i=1}^{r}\left(1+q^{-e_i^*}t\right)\prod_{i=1}^{n-r}\left(1+q^{e_i}t\right)}{\prod_{i=1}^{n}\left(1-q^{d_i}\right)}$$

elementary symmetric function

set $t=0$

$r=0$

Solomon's Thm.

Orlik-Solomon Thm.

It is really a product formula that depends only on e_1 and a:

$$\mathrm{Hilb}\left(\left(S \otimes \Lambda^{\cdot} V^* \otimes \Lambda^r V\right)^W, q, t\right) =$$

$$q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{-1}; q^{-a})_r \, (-tq^{e_1}; q^a)_{n-r}}{(q^{e_1+1}; q^a)_n}$$

where $(z;q)_n := (1-z)(1-zq)\cdots(1-zq^{n-1})$

and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{[n]!_q}{[k]!_q \, [n-k]!_q}$

Thus if we call

$$f_r(W,q,t) := q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{-1};q^{-a})_r \, (-tq^{e_1};q^{a})_{n-r}}{(q^{e_1+1};q^{a})_n}$$

then it gives a product formula for the f-vector entries of the cluster complex or Cambrian fan in coincidental types (by setting $t = -q^{h+1}$ and then $q=1$).

Is there a product formula for the h-vector entries in coincidental types?

THEOREM (R.-Shepler-Sommers 2019)

For coincidental W with gap a, setting

$$h_r(W,q,t) := (-tq^{-ar-1})^{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq\,;q^{-a})_r}{(q^{e_1+1};q^a)_r}$$

one has

$$\sum_{r=0}^{n} f_r(W,q,t)\cdot s^r = \sum_{r=0}^{n} h_r(W,q,t)\cdot \underbrace{(-sq;q^a)_r}_{(1+sq)(1+sq^{a+1})\cdots(1+sq^{(r-1)a+1})}$$

(specializing to

$$\sum_{r=0}^{n} f_r \cdot s^r = \sum_{r=0}^{n} h_r (s+1)^r \Bigg)$$

Thus this $h_r(W,q,t)$ does give a product formula for the h-vector in the coincidental types (again by setting $t=-q^{h+1}$ and then $q=1$).

The q-analogue $h_r(W,q,t)\big|_{t=-q^{h+1}}$ is a q-Narayana number happily agreeing with precursors by...

- Fürlinger & Hofbauer
- R.-Sommers
- Krattenthaler & Wachs

... and exhibiting C.S.P.'s

The theorem on

$\mathrm{Hilb}\left((S \otimes \wedge^i V^* \otimes \wedge^k V)^W, q, t \right)$

is proven case-by-case

in coincidental types,

with $G(d,1,n)$ modeled on a result of

Kirillov & Pak 1990

The theorem generalizing

h-vector to f-vector uses a

q-transformation of F.H. Jackson

(thanks Dennis Stanton!)

REMARK: The formula

$$\mathrm{Hilb}\left(\left(S \otimes \Lambda^{i} V^{*} \otimes \Lambda^{r} V\right)^{W}, q, t\right) =$$

$$\sigma_r\left(q^{e_1^*}, \dots, q^{e_n^*}\right) \frac{\prod_{i=1}^{r}\left(1+q^{-e_i^*} t\right) \prod_{i=1}^{n-r}\left(1+q^{e_i} t\right)}{\prod_{i=1}^{n}\left(1-q^{d_i}\right)}$$

also suggests a structural

CONJECTURE

on $\left(S \otimes \Lambda^{i} V^{*} \otimes \Lambda^{r} V\right)^{W}$

(suppressed here).

MYSTERY: Why do coincidental types behave better not only here, but also

in formulas of Reading 2007 counting Coxeter element reflection factorizations $c = t_1 t_2 \cdots t_k w$ (and further work of Douvropoulos 2019)

in the Coxeter-biCatalan combinatorics of Barnard & Reading 2016

in doppelgängers of root posets, studied by Hamaker-Patrias-Pechenik-Williams 2016 and by S. Hopkins 2019

in work of Abramenko 1994
and Alex Miller 2018
on walls in Coxeter complexes
and Milnor fiber complexes

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in work of Alex Miller 2015
on Foulkes characters

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Thanks for
your
attention,
and thanks
Western!