# Ehrhart theory and a q-analogue

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(arXiv: 2407.06511)
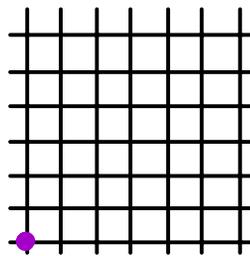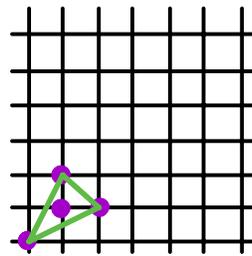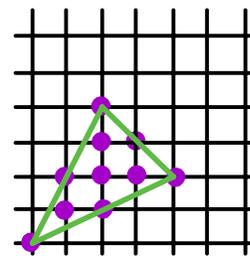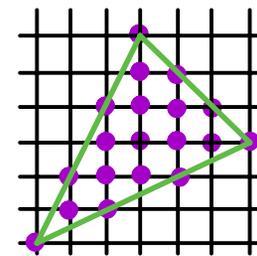
0·P   1·P   2·P   3·P

DePaul Math Colloquium   March 14, 2025

OUTLINE

1. Ehrhart theory review, including Greatest Hits

2. q-analogues and CONJECTURE

3. Harmonic algebra CONJECTURE

4. Ehrhart tidbits ⤳ q-tidbits

If time!

1. Ehrhart theory review

vertices in $\mathbb{Z}^n$

$P \subset \mathbb{R}^n$ a lattice polytope


P

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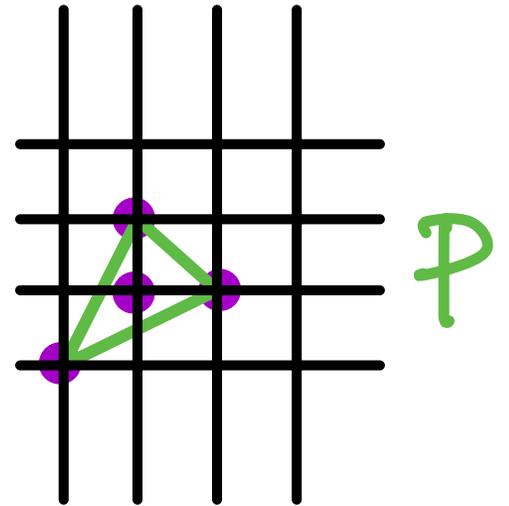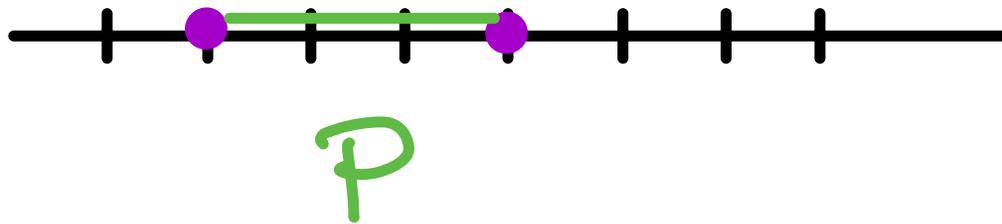
$\leadsto$ Ehrhart function

$$i_P(m) := \# \mathbb{Z}^n \cap mP$$

for $m = 0, 1, 2, \ldots$

# EXAMPLE

0. P $i_P(0) = 1$

1. P $i_P(1) = 4$

2. P $i_P(2) = 7$
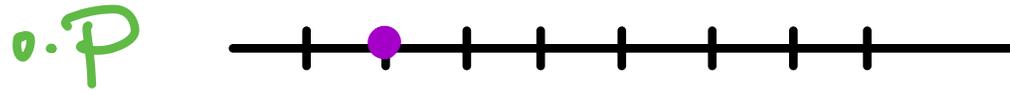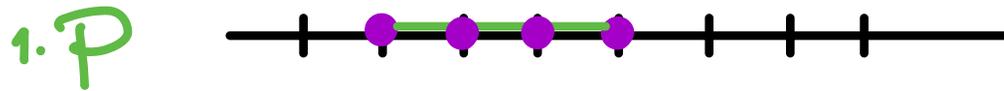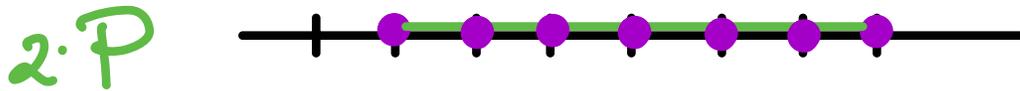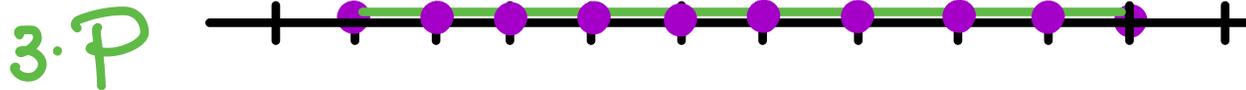
3. P $i_P(3) = 10$

$$i_P(m) = 1 + 3m$$

1-dimensional (Lebesgue) volume of P

EXAMPLE

0·P   1·P   2·P   3·P

1   4   10   19
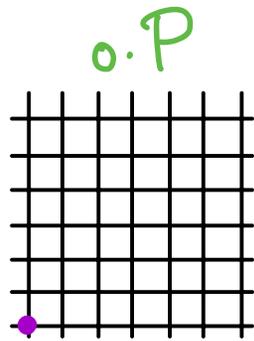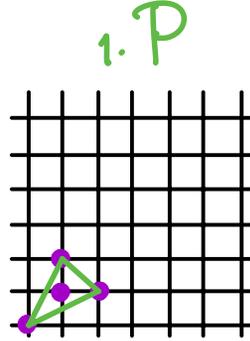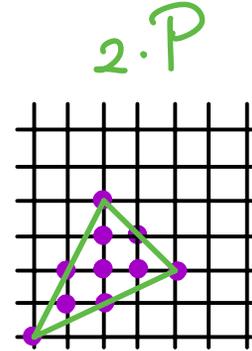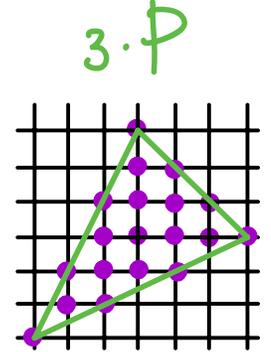
$i_P(0)$ $i_P(1)$ $i_P(2)$ $i_P(3)$

Guess $i_P(m) \approx$ area of $mP = m^2 \cdot ($area of $P)$

$\approx$ quadratic in m
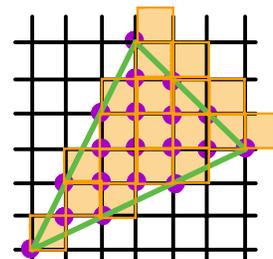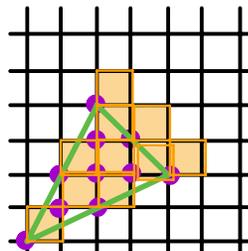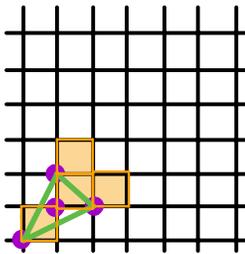
SURPRISE:

It's on the nose quadratic for polygons.

$0 \cdot P$ $1 \cdot P$ $2 \cdot P$ $3 \cdot P$

1   4   10   19
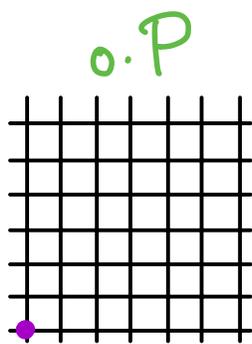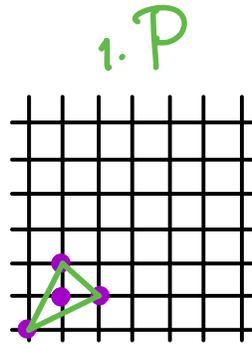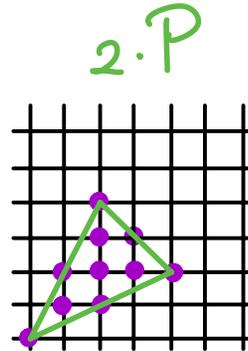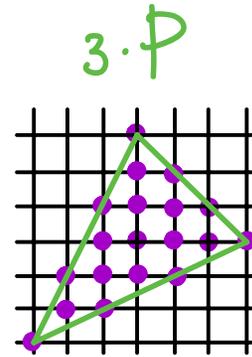
$i_P(0)$ $i_P(1)$ $i_P(2)$ $i_P(3)$

$$i_P(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$$

In general, $\quad i_P(m) = 1 + am + Am^2$

area of P

It is also convenient to encode $i_P(m)$ via the Ehrhart series

$$E_P(t) := \sum_{m=0}^{\infty} t^m \cdot i_P(m) = \sum_{m=0}^{\infty} t^m \cdot \#(\mathbb{Z}^n \cap mP)$$

EXAMPLES

P
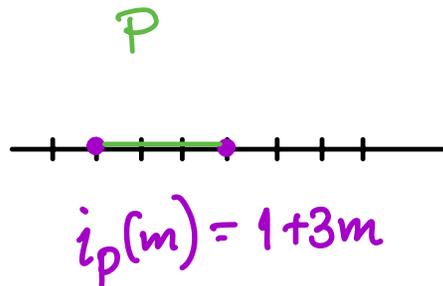
$i_P(m) = 1 + 3m$

$$E_P(t) = \sum_{m=0}^{\infty} t^m (1 + 3m) = \frac{1 + 3t}{(1-t)^2}$$

$0 \cdot P$ $1 \cdot P$ $2 \cdot P$ $3 \cdot P$
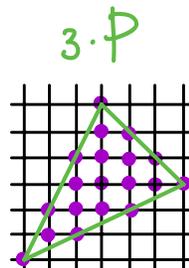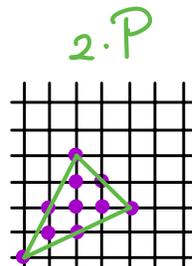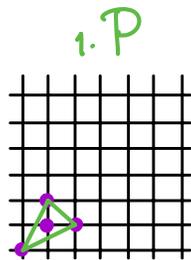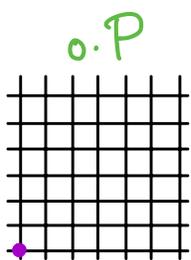
$$E_P(t) = 1 + 4t^1 + 10t^2 + 19t^3 + \ldots = \sum_{m=0}^{\infty} t^m \cdot \left(1 + \frac{3}{2}m + \frac{3}{2}m^2\right)$$

$$= \frac{1 + t + t^2}{(1-t)^3}$$

Classical Ehrhart Theory's Greatest Hits:

THEOREM
(Ehrhart 1962)

$i_P(m) := \# \mathbb{Z}^n \cap mP$ is a **polynomial** in m, of degree $d := \dim(P)$

or equivalently,

$$E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m) = \frac{h_0^* + h_1^* t + \ldots + h_d^* t^d}{(1-t)^{d+1}}$$

EE
1959

EXAMPLE

P
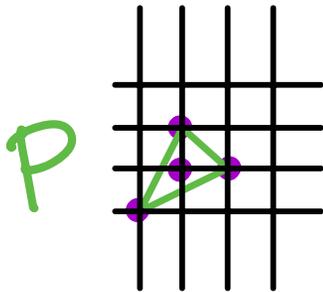
$d = 2$

$i_P(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$

$$E_P(t) = \frac{1 + t + t^2}{(1-t)^3}$$

In the numerator of

$$E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m) = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}},$$

the h^*-vector entries $(h_0^*, h_1^*, \dots, h_d^*)$

are nonnegative.

EXAMPLE

P
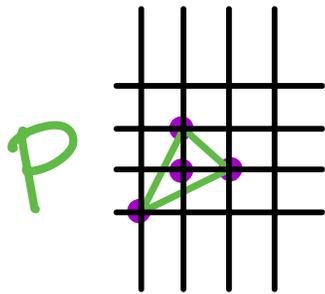
$$E_P(t) = \frac{1 + t + t^2}{(1-t)^3}$$

$$(h_0^*, h_1^*, h_2^*) = (1, 1, 1)$$

One can also count **interior** lattice points...

$$\overline{i_P}(m) := \#\mathbb{Z}^n \cap \text{interior}(mP)$$

$$\overline{E_P}(t) := \sum_{m=1}^{\infty} t^m \cdot \overline{i_P}(m)$$

interior$(1 \cdot P)$ interior$(2 \cdot P)$ interior$(3 \cdot P)$
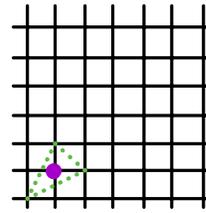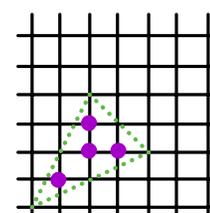
1   4   10
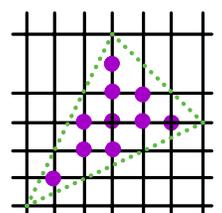
THEOREM (Ehrhart-Macdonald reciprocity)

CONJECTURE   PROOF
1959   1971

I.G. Macdonald
1928 - 2023

$$\overline{i_P}(m) = (-1)^d \, i_P(-m) \qquad \text{for } m = 1, 2, 3, \ldots$$

or equivalently,

$$\overline{E_P}(t) = (-1)^{d+1} E_P\left(\tfrac{1}{t}\right)$$

EXAMPLE

$0 \cdot P$

$1 \cdot P$

$2 \cdot P$

$3 \cdot P$
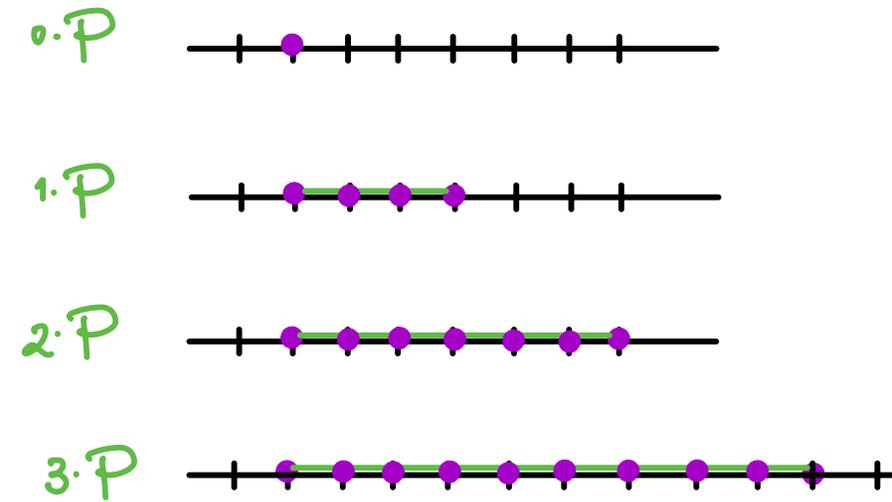
interior$(1 \cdot P)$

interior$(2 \cdot P)$
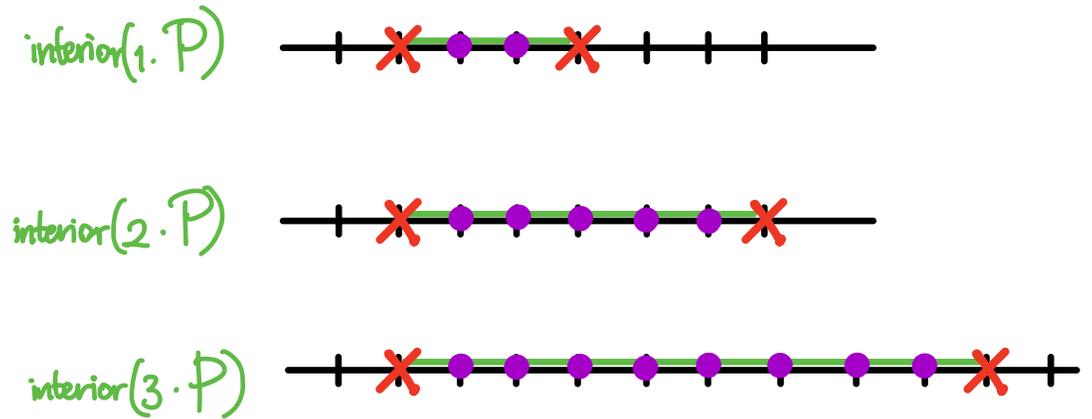
interior$(3 \cdot P)$

$$i_p(m) = 1 + 3m$$

$$\bar{i}_p(m) = -1 + 3m$$
$$= (-1)^1 (1 - 3m) = (-1)^1 i_p(-m)$$

$$E_\phi(t) = \frac{1 + 3t}{(1-t)^2}$$

$$\bar{E}_\phi(t) = \frac{3t + t^2}{(1-t)^2} = (-1)^2 E_p(1/t)$$

EXAMPLE

$i_p(0) = 1$
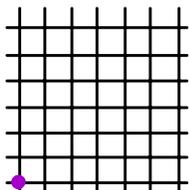
0·P

$i_p(1) = 4$
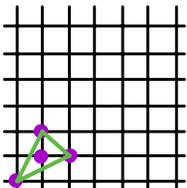
1·P

$i_p(2) = 10$

2·P

$i_p(3) = 19$

3·P

interior(1·P)

$\bar{i}_p(1) = 1$

interior(2·P)
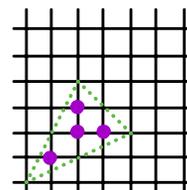
$\bar{i}_p(2) = 4$

interior(3·P)
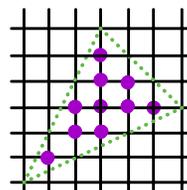
$\bar{i}_p(3) = 10$

$i_p(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$

$\bar{i}_p(m) = 1 - \frac{3}{2}m + \frac{3}{2}m^2 = (-1)^2 i_p(-m)$

$E_p(t) = \frac{1 + t + t^2}{(1-t)^3}$

$\bar{E}_p(t) = \frac{t + t^2 + t^3}{(1-t)^3} = (-1)^3 E_p(1/t)$
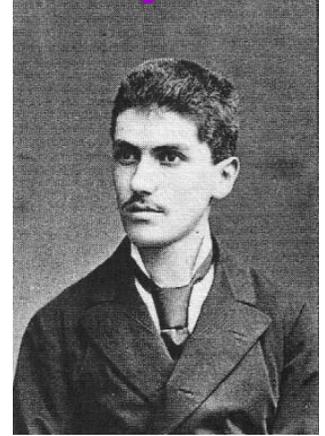
(A traditional...) COROLLARY

(Pick's Theorem 1899)

Georg Pick

(2-dimensional)

Any lattice polygon P has

$$\underset{A}{\text{area}} = \underset{\text{points}}{\#\text{interior}} + \#\text{boundary points}/2 - 1$$

EXAMPLE
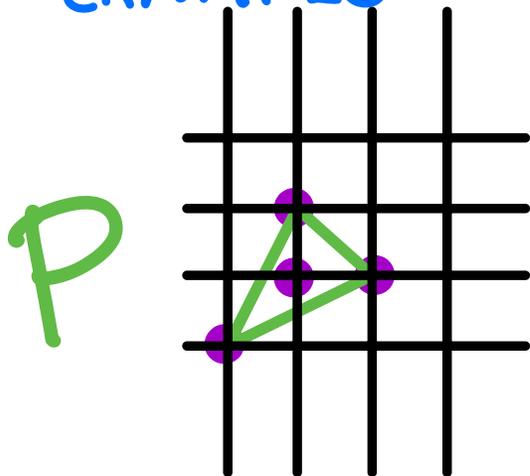
$$\underset{A}{\frac{3}{2}} = 1 + 3/2 - 1$$
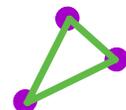
PROOF:

Want $\text{area } A \overset{?}{=} \underbrace{\#\text{interior points}}_{i_P(1)} + \underbrace{\#\text{boundary points}}_{i_P(1) - \overline{i}_P(1)} / 2 - 1$

Pick

Ehrhart-Macdonald reciprocity

$$i_P(m) = 1 + am + Am^2$$

$$\overline{i}_P(m) = (-1)^2 \, i_P(-m) = 1 - am + Am^2$$

set $m=1$

$$\left. \begin{array}{l} i_P(1) = 1 + a + A \\[2mm] \overline{i}_P(1) = 1 - a + A \end{array} \right\}$$
solve for a, A
$\rightsquigarrow$
$$\left\{ \begin{array}{l} 2A = i_P(1) + \overline{i}_P(1) - 2 \\[3mm] 2a = i_P(1) - \overline{i}_P(1) \end{array} \right.$$

Pick's formula, re-written

Two proof methods (for the Greatest Hits)

Method 1 ($\begin{smallmatrix}\text{Ehrhart}\\\text{Macdonald}\\\text{Stanley}\end{smallmatrix}$): Reduce to simplices via triangulations

P
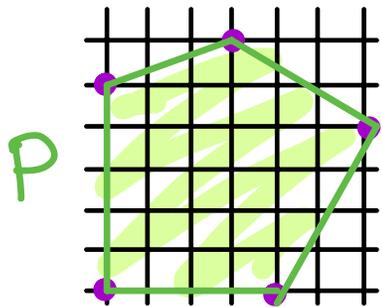
triangulate $\rightsquigarrow$
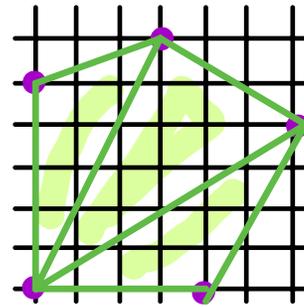
$E_P(t)$ is a valuative function of P:

$$E_P(t) = E_{P_1}(t) + E_{P_2}(t) + E_{P_3}(t) - E_{P_1 \cap P_2}(t) - E_{P_2 \cap P_3}(t)$$
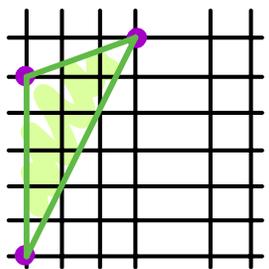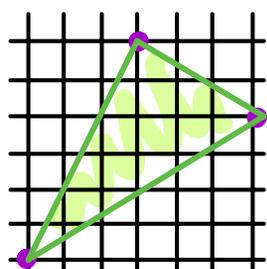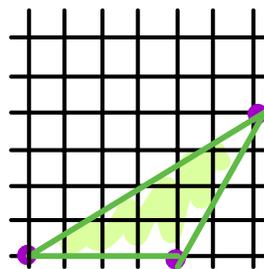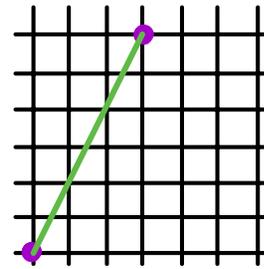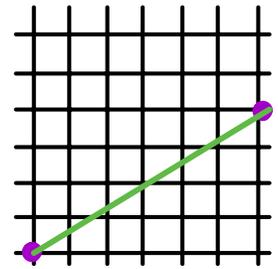
$P_1 \qquad P_2 \qquad P_3 \qquad P_1 \cap P_2 \qquad P_2 \cap P_3$

... and simplices have very explicit formulas:

PROPOSITION: For a lattice d-simplex $P \subset \mathbb{R}^n$ with vertices $v^{(1)}, v^{(2)}, \ldots, v^{(d+1)}$

$$E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

where $h_i^* = \#\left(\mathbb{Z}^n \times \{i\}\right) \cap \Pi$

semi-open parallelepiped
$$\Pi := \sum_j [0,1) \cdot \begin{bmatrix} v^{(j)} \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^{n+1}$$

EXAMPLE

$P \subset \mathbb{R}^2$

$\Pi \subset \mathbb{R}^3$

$\rightsquigarrow$

t^2

t

1

$\rightsquigarrow$
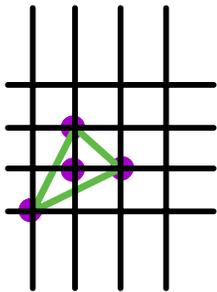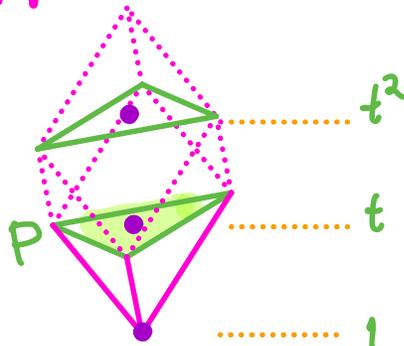
$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

Method 2 (Stanley): Commutative algebra

of the affine semigroup ring

$3P \times \{3\}$

y_0^3

$2P \times \{2\}$

y_0^2

$P \times \{1\}$

y_0^1
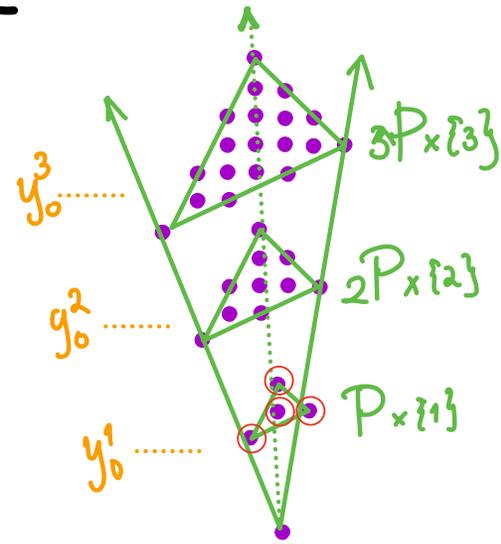
$$\Bbbk[\Lambda_P] := \text{span}_{\Bbbk}\left\{ y_0^m \, y_1^{a_1} \cdots y_n^{a_n} : \underline{a} \in \mathbb{Z}^n \cap mP, \; m = 0, 1, 2, \ldots \right\}$$

$$\subset \Bbbk[y_0, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

which has Hilbert series equal to the Ehrhart series:

$$\text{Hilb}(\Bbbk[\Lambda_P], t) := \sum_{m=0}^{\infty} t^m \cdot \dim_{\Bbbk} \Bbbk[\Lambda_P]_m = E_P(t)$$

EXAMPLES

$\mathbb{P}$
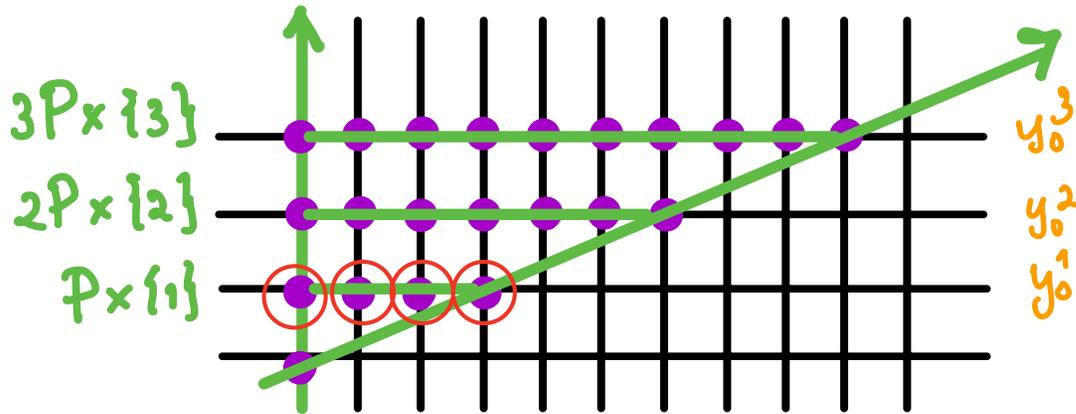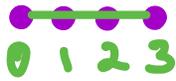
$$\mathbb{k}[\Lambda_P] = \mathbb{k}[\; y_0, \; y_0 y_1, \; y_0 y_1^2, \; y_0 y_1^3 \;]$$

$$\subset \mathbb{k}[y_0, y_1^{\pm 1}]$$

$\mathbb{P}$
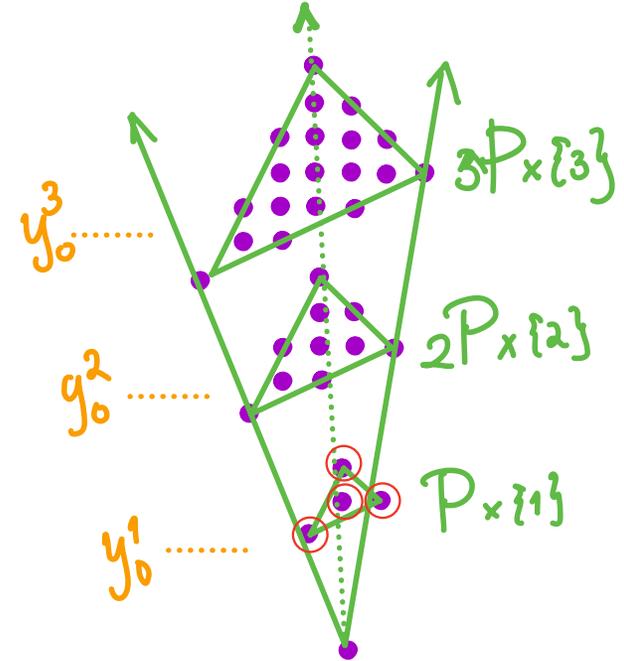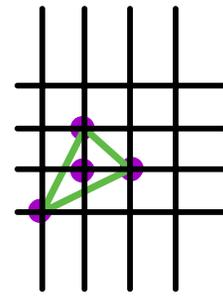
$$\mathbb{k}[\Lambda_P] = \mathbb{k}[\; y_0, \; y_0 y_1 y_2, \; y_0 y_1^2 y_2, \; y_0 y_1 y_2^2 \;]$$

$$\subset \mathbb{k}[y_0, y_1^{\pm 1}, y_2^{\pm 1}]$$

- $\Bbbk[\Lambda_P]$ is Noetherian

  (Gordan 1873)

- $\Bbbk[\Lambda_P]$ has a linear system of parameters

  (Noether 1926)

$$\Rightarrow \quad E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

- $\Bbbk[\Lambda_P]$ is Cohen-Macaulay

  (Hochster 1972)

$$\Rightarrow \quad h_i^* \geq 0 \quad \text{for } i = 1, 2, \ldots, d$$

- $\Omega \, \Bbbk[\Lambda_P] \cong \Bbbk[\Lambda_{\text{interior}(P)}]$

  canonical module

  (Danilov 1978)

$$\Rightarrow \quad \bar{E}_P(t) = (-1)^{d+1} E_P(1/t)$$

2. q-analogues

0·P   1·P   2·P   3·P
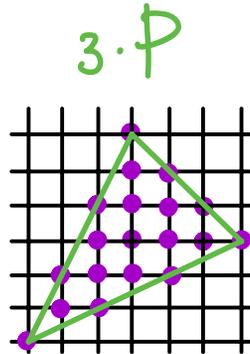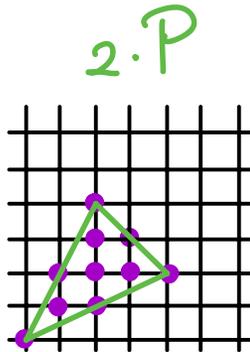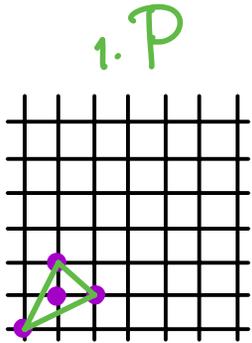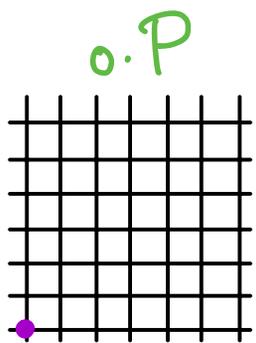
$$1 \quad + \quad 4t^1 \quad + \quad 10t^2 \quad + \quad 19t^3 + \ldots = E_P(t)$$

$$= \frac{1+t+t^2}{(1-t)^3}$$

q-analogue

$$1 \quad + \quad (1+2q+q^2)t^1 \quad + \quad \left(\begin{matrix}1+2q+3q^2\\ +3q^3+q^4\end{matrix}\right)t^2 \quad + \quad \left(\begin{matrix}1+2q+3q^2+4q^3\\ +5q^4+3q^5+q^6\end{matrix}\right)t^3 + \ldots = E_P(t,q)$$

$$= \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

Those q-analogues come from using the **point orbit method**
to **deform** the coordinate ring of $Z := \mathbb{Z}^n \cap mP \subseteq \mathbb{R}^n$
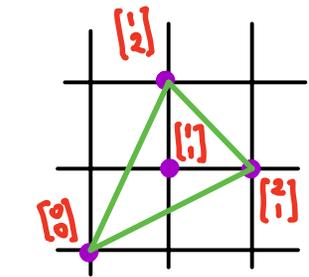
affine coordinate ring of Z as 0-dimensional variety

$Z := \mathbb{Z}^n \cap mP$

$\#Z = 4$

$\rightsquigarrow$

$$\mathbb{R}[Z] = \mathbb{R}[x,y] \big/ I(Z)$$

polynomials $f(x,y)$ vanishing on Z

$$= \mathbb{R}[x,y] \big/ (x-0, y-0) \cap (x-1, y-1) \cap (x-1, y-2) \cap (x-2, y-1)$$

$\begin{bmatrix}0\\0\end{bmatrix}$ $\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}1\\2\end{bmatrix}$ $\begin{bmatrix}2\\1\end{bmatrix}$

Macaulay 2
computation

$$= \mathbb{R}[x,y] \big/ (2xy - y^2 - 2x + y,$$
$$x^2 - y^2 - 3x + y,$$
$$y^3 - 3y^2 + 2y)$$

$Z := \mathbb{Z}^n \cap mP$

$\#Z = 4$
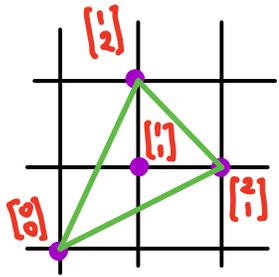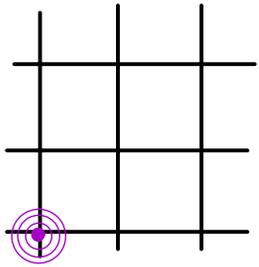
fat point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of multiplicity 4

affine coordinate ring

$$\mathbb{R}[Z] = \mathbb{R}[x,y] \Big/ I(Z)$$
$$= \mathbb{R}[x,y] \Big/ (2xy - y^2 - 2x + y, \; x^2 - y^2 - 3x + y, \; y^3 - 3y^2 + 2y)$$

DEFORM!

(= by taking top degree components of all $f(x,y)$ in $I(Z)$)

associated graded ring

$$\text{gr } \mathbb{R}[Z] = \mathbb{R}[x,y] \Big/ \text{gr } I(Z)$$
$$= (2xy - y^2, \; x^2 - y^2, \; y^3)$$

$$\text{Hilb}\left(\text{gr } \mathbb{R}[Z], q\right) = 1 + 2q + q^2 \qquad \xrightarrow{q=1} \qquad 4 = \#Z$$

= the q-analogue of $\#Z$

REMARK: Instead of Hilbert series for the fat point coordinate ring $\mathbb{R}[x_1,\ldots,x_n]/gr\,I(Z)$, would have used its **Macaulay inverse system:**

$$V_Z := \left\{ g(\underline{y}) \in \mathbb{R}[y_1,\ldots,y_n] : f\left(\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_n}\right)g(\underline{y}) = 0 \quad \forall f(\underline{x}) \in gr\,I(Z) \right\}$$

since $\quad \text{Hilb}(V_Z, q) = \text{Hilb}\left(\mathbb{R}[x_1,\ldots,x_n]/gr\,I(Z),\ q\right)$

EXAMPLE
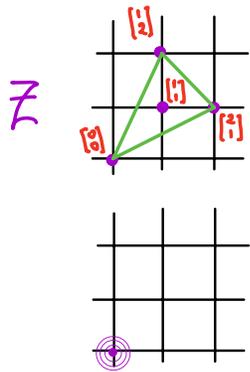
$$\mathbb{R}[Z] = \mathbb{R}[x_1,x_2] \Big/ (2x_1x_2 - x_2^2 - 2x_1 + y_1,\ x_1^2 - x_2^2 - 3x_1 + x_2,\ x_2^3 - 3x_2^2 + 2x_2)$$

$$gr\,\mathbb{R}[Z] = \mathbb{R}[x_1,x_2] \Big/ (2x_1x_2 - x_2^2,\ x_1^2 - x_2^2,\ x_2^3)$$

$$V_Z = \text{span}_{\mathbb{R}}\left\{ 1,\ y_1,\ y_2,\ y_1^2 + y_1y_2 + y_2^2 \right\} \subset \mathbb{R}[y_1,y_2]$$

$$\text{Hilb}(V_Z, q) = 1 + 2q + q^2 \quad \xrightarrow{q=1} \quad 4 = \#Z$$

MAIN DEFINITION:

For a lattice polytope $P \subset \mathbb{R}^n$, define its q-Ehrhart function

$$i_P(m;q) = \text{Hilb}\left(q_n \mathbb{R}\left[\mathbb{Z}^n \cap mP\right], q\right)$$
$$= \text{Hilb}\left(V_{\mathbb{Z}^n \cap mP}, q\right)$$

$$\xrightarrow{q=1} \quad i_P(m) = \#\mathbb{Z}^n \cap mP$$

and then its q-Ehrhart series in $\mathbb{R}[q][[t]]$

$$\boxed{E_P(t,q) := \sum_{m=0}^{\infty} t^m \cdot i_P(m;q)}$$

$$\xrightarrow{q=1} \quad E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m)$$

EXAMPLE

$0 \cdot P$ $1 \cdot P$ $2 \cdot P$ $3 \cdot P$
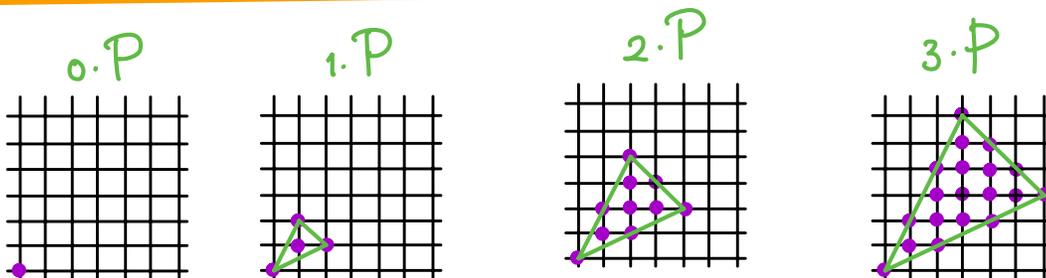
$$1 \quad + \quad 4t^1 \quad + \quad 10\,t^2 \quad + \quad 19\,t^3 + \ldots \quad = E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

$$1 \quad + \quad (q+2q+q^2)t^1 + \begin{pmatrix} 1+2q+3q^2 \\ +3q^3+q^4 \end{pmatrix} t^2 + \begin{pmatrix} q+2q+3q^2+4q^3 \\ +5q^4+3q^5+q^6 \end{pmatrix} t^3 + \ldots = E_P(t,q) = \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

normalized area of P	vertices of P	$h_P^*(t) =$ $1 + h_1^* t + h_2^* t^2$	$E_P(t,q)$
1	$(0,0),(1,0),(0,1)$	1	$\frac{1}{(1-t)(1-qt)^2}$
2	$(0,0),(1,0),(1,2)$	$1+t$	$\frac{1+qt}{(1-t)(1-qt)(1-q^2t)}$
2	$(0,0),(1,0),(0,1),(1,1)$	$1+t$	$\frac{1+qt}{(1-t)(1-qt)(1-q^2t)}$
3	$(0,0),(1,0),(1,3)$	$1+2t$	$\frac{1+qt+q^2t}{(1-t)(1-qt)(1-q^3t)}$
3	$(0,0),(1,0),(2,3)$	$1+t+t^2$	$\frac{(1+qt)(1+qt+q^2t^2)}{(1-t)(1-q^2t)(1-q^3t^2)}$
3	$(0,0),(1,0),(0,1),(-2,1)$	$1+2t$	$\frac{1+qt-q^2t^2-q^3t^2}{(1-t)(1-qt)(1-q^2t)^2}$
4	$(0,0),(1,0),(1,4)$	$1+3t$	$\frac{1+t(q+q^2+q^3)}{(1-t)(1-qt)(1-q^4t)}$
4	$(0,0),(1,0),(3,4)$	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2}$
4	$(0,0),(2,0),(0,2)$	$1+3t$	$\frac{1+2qt+q^2t}{(1-t)(1-q^2t)^2}$
4	$(0,0),(1,0),(0,1),(-3,1)$	$1+3t$	$\frac{1+qt+q^2t-q^2t^2-q^3t^2-q^4t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}$
4	$(0,0),(1,0),(0,2),(1,2)$	$1+3t$	$\frac{1+qt+q^2t-q^2t^2-q^3t^2-q^4t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}$
4	$(0,0),(2,0),(0,1),(1,-1)$	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2}$
4	$(0,0),(1,0),(1,2),(2,2)$	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2)}$

vertices of P	$h_P^*(t) =$ $1 + h_1^* t + h_2^* t^2$	$E_P(t,q)$
$(0,0),(1,0),(1,5)$	$1+4t$	$\frac{1+t(q+q^2+q^3+q^4)}{(1-t)(1-qt)(1-q^5t)}$
$(0,0),(1,0),(2,5)$	$1+2t+2t^2$	$\frac{(1+qt)(1+t(q+q^2)+t^2(q^3+q^4))}{(1-t)(1-q^2t)(1-q^5t^2)}$
$(0,0),(1,0),(0,1),(-4,1)$	$1+4t$	$\frac{1+qt+q^2t+q^3t-q^2t^2-q^3t^2-q^4t^2-q^5t^2}{(1-t)(1-qt)(1-q^2t)(1-q^4t)}$
$(0,0),(2,0),(0,1),(-3,1)$	$1+4t$	$\frac{1+qt+2q^2t-q^2t^2-q^3t^2-q^4t^2-q^5t^2}{(1-t)(1-qt)(1-q^3t)^2}$
$(0,0),(1,0),(2,3),(2,1)$	$1+3t+t^2$	$\frac{1+2qt+2q^2t+2q^3t^2+2q^4t^2+q^5t^3}{(1-t)(1-q^2t)(1-q^5t^2)}$
$(0,0),(1,0),(1,2),(2,2),(0,-1)$	$1+3t+t^2$	$\frac{1+2qt+2q^2t+2q^3t^2+2q^4t^2+q^5t^3}{(1-t)(1-q^2t)(1-q^5t^2)}$

Since $E_P(t,q)$ is an Aff($\mathbb{Z}^n$)-invariant of P, can use database of lattice polytopes by Balletti 2021

EXAMPLE (V. Kurylenko)

Reeve tetrahedron P

of volume 3

seems to have
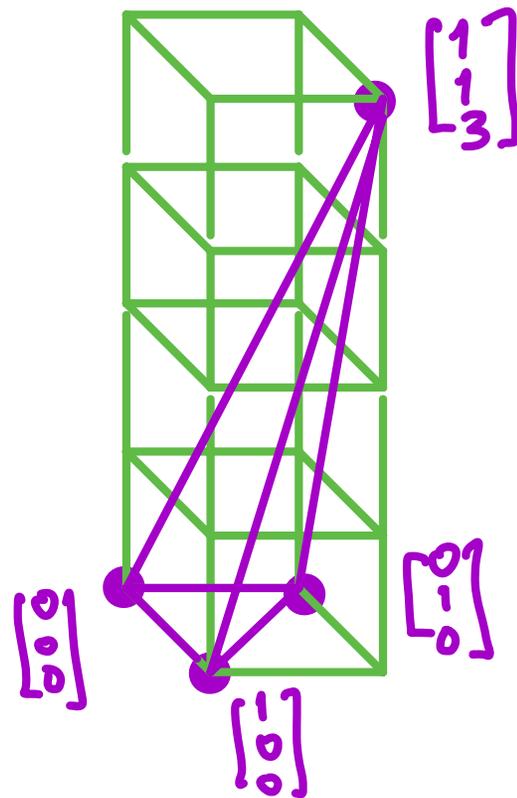
$$E_P(t,q) \stackrel{?}{=} \frac{(1-q^5t^4) \cdot (1+qt)(1+qt+(2q^2+q^3)t^2+2q^4t^3+(q^5+q^6)t^4)}{(1-t)(1-qt)(1-q^3t^2)(1-q^5t^3)(1-q^6t^4)}$$

q-Ehrhart theory CONJECTURE

First, recall the
CLASSICAL Ehrhart Theorems: For d-dimensional lattice polytopes P,

- $E_P(t) = \dfrac{\sum\limits_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$ (RATIONALITY)

- $E_P\left(\dfrac{1}{t}\right) = (-1)^{d+1} \, \overline{E}_P(t)$ (RECIPROCITY)

- For lattice simplices, (SIMPLEX NONNEGATIVITY)

$$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$$

- $h_i^* \geq 0$ for $i=1,2,\dots,d$ (GENERAL NONNEGATIVITY)

Classical Ehrhart THEOREMS

CONJECTURES (R.-Rhoades 2024)

- $E_P(t) = \dfrac{\sum\limits_{i=0}^{d} h_i^* \, t^i}{(1-t)^{d+1}}$

RATIONALITY

- $E_P(t,q) = \dfrac{N_P(t,q)}{\prod\limits_{i=1}^{\nu}(1-q^{a_i} t^{b_i})}$ with $N_P(t,q)$ in $\mathbb{Z}[t,q]$ and $\nu \geq d+1$

- $E_P\left(\dfrac{1}{t}\right) = (-1)^{d+1} \, \overline{E}_P(t)$

RECIPROCITY

- $E_P\left(\dfrac{1}{t}, \dfrac{1}{q}\right) = (-1)^{d+1} q^d \, \overline{E}_P(t, q)$

- For lattice simplices,

$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$

SIMPLEX NONNEGATIVITY

- For lattice d-simplices with $\nu = d+1$,

$N_P(t,q)$ lies in $\mathbb{N}[t,q]$

EXAMPLE P

$E_P(t,q) = \dfrac{(1 + tq + t^2 q^2)(1 + tq)}{(1-t)(1-tq^2)(1-t^2 q^3)}$

- $h_i^* \geq 0$ for $i = 1, 2, \dots, d$

GENERAL NONNEGATIVITY

- ??
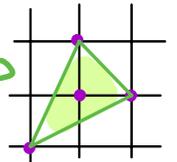
EXAMPLES: These conjectures are proven for

antiblocking polytopes $:=$ polytopes P inside $\left(\mathbb{R}_{\geq 0}\right)^n$

with $0 \leq z \leq z'$ and $z' \in P \Rightarrow z \in P$

$\underbrace{}$ componentwise comparison

P
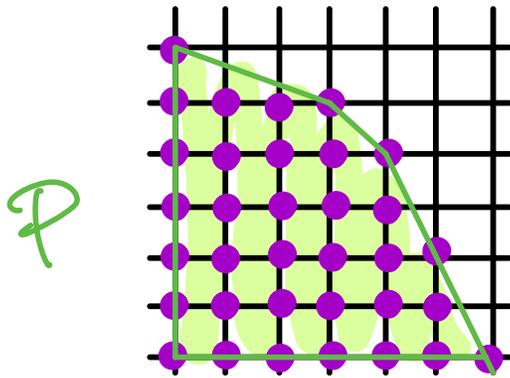
PROPOSITION: For P antiblocking, every ideal $\operatorname{gr} I(Z)$ for $Z := \mathbb{Z}^n \cap mP$

is a monomial ideal $\operatorname{gr} I(Z) = \operatorname{span}_{\mathbb{R}}\left\{ \underline{x}^{\underline{a}} : \underline{a} \notin Z \right\}$,

and $E_P(t,q) = \sum_{m=0}^{\infty} t^m \sum_{\underline{a} \in \mathbb{Z}^n \cap mP}^{\prime} q^{a_1 + \dots + a_n}$

$\qquad = \left[\operatorname{Hilb}\left(\Bbbk[\Lambda_P], y_0, y_1, \dots, y_n \right) \right]_{\substack{y_0 = t \\ y_1 = y_2 = \dots = y_n = q}}$

$\qquad\qquad \nearrow$ affine semigroup ring for P

3. Harmonic algebra CONJECTURE

Method 1 of Ehrhart theory (= reduce to simplices via triangulation)
seems elusive, because

$$i_P(m; q) = \text{Hilb}\left(\text{gr } R[\mathbb{Z}^n_{\wedge m P}], q \right) \text{ is not valuative as a function of } P.$$

EXAMPLE

$$P = P_1 \cup P_2$$

$$i_P(1) = i_{P_1}(1) + i_{P_2}(1) - i_{P_1 \cap P_2}(1)$$

$$12 = 7 + 7 - 2$$

But
$$i_P(1; q) \neq i_{P_1}(1; q) + i_{P_2}(1; q) - i_{P_1 \cap P_2}(1; q)$$

$$= (1 + q + q^2) \cdot (1 + q + q^2 + q^3)$$

$$= 1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5$$

$$1 + 2q + 3q^2 + q^3 \qquad 1 + 2q + 3q^2 + q^3 \qquad 1 + q$$

Method 2 (= commutative algebra) looks promising ...
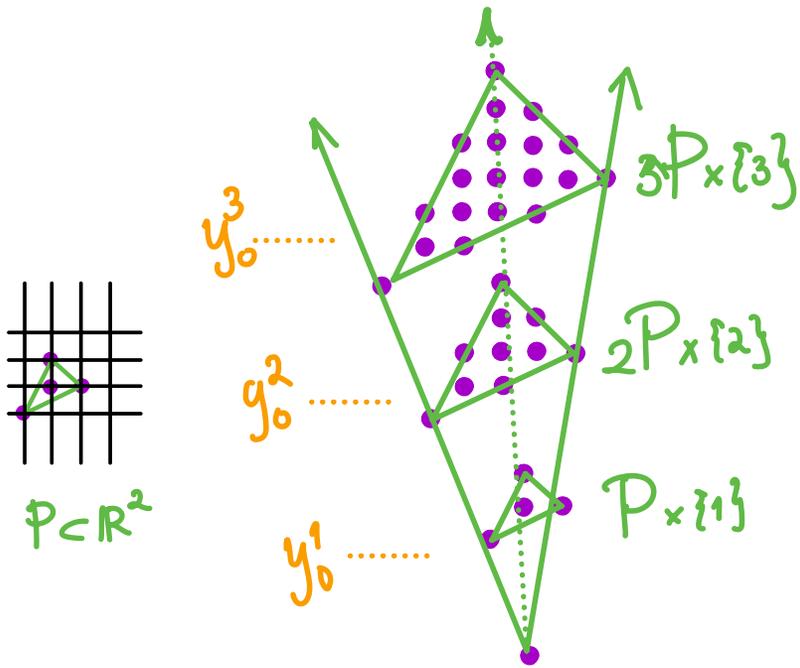
Recall

$$\Bbbk[\Lambda_P] := \bigoplus_{m=0}^{\infty} \mathrm{span}_{\Bbbk}\{y_0^m \, \underline{y}^{\underline{a}}\}_{\underline{a} \in \mathbb{Z}^n \cap mP}$$

affine
semigroup
ring

$$\subset \Bbbk[y_0, y_1, \dots, y_n]$$

$P \subset \mathbb{R}^2$

y_0^3 ......

$3P \times \{3\}$

y_0^2 ......

$2P \times \{2\}$

y_0^1 ......

$P \times \{1\}$

DEFINITION:

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP} \qquad \subset \mathbb{R}[y_0, y_1, \dots, y_n]$$

Harmonic
algebra

where V_Z = harmonic space/Macaulay inverse system

for $\mathbb{R}[y_1, \dots, y_n] / \mathrm{gr}_z I(Z)$

EXAMPLE
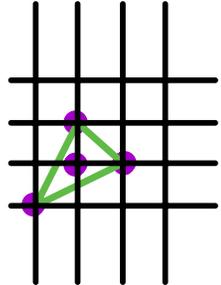
$P \subset \mathbb{R}^2$ has harmonic algebra

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP} \subset \mathbb{R}[y_0, y_1, y_2]$$

$$= \mathbb{R} \cdot 1 \oplus y_0 V_{\mathbb{Z}^n \cap 1P} \oplus y_0^2 V_{\mathbb{Z}^n \cap 2P} \oplus \cdots$$
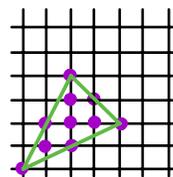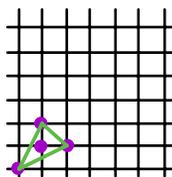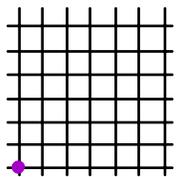
$$= \mathrm{span}_{\mathbb{R}} \left\{ 1, \quad \begin{array}{l} y_0, \\ y_0 y_1, \; y_0 y_2, \\ y_0(y_1^2 + y_1 y_2 + y_2^2) \end{array} \quad \begin{array}{l} y_0^2, \\ y_0^2 y_1, \; y_0^2 y_2, \\ y_0^2 y_1^2, \; y_0^2 y_1 y_2, \; y_0^2 y_2^2, \\ y_0^2 y_1^3, \; y_0^2(y_1^2 y_2 + y_1 y_2^2), \; y_0^2 y_2^3, \\ y_0^2(y_1^4 + 2y_1^3 y_2 + 3y_1^2 y_2^2 + 2y_1 y_2^3 + y_2^4) \end{array} \quad \cdots \right\}$$

Why is $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP}$ even an algebra ?!

For $\Bbbk[\Lambda_P] := \bigoplus_{m=0}^{\infty} \mathrm{span}_{\Bbbk} \{ y_0^m \, \underline{y}^{\underline{a}} \}_{\underline{a} \in \mathbb{Z}^n \cap mP}$ it came from

$$\mathbb{Z}^n \cap mP \;+\; \mathbb{Z}^n \cap m'P \;\subset\; \mathbb{Z}^n \cap (m+m')P$$

THEOREM For finite point sets Z and $Z' \subset \Bbbk^n$,

$$V_Z \cdot V_{Z'} \subseteq V_{Z+Z'}$$
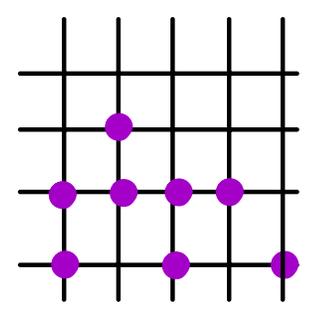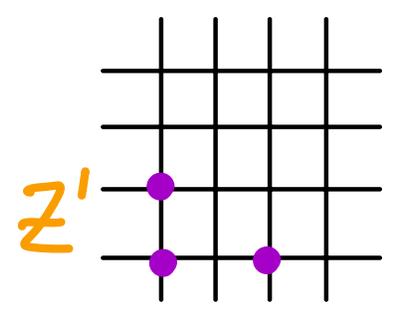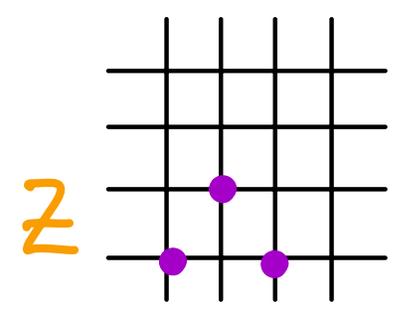
Z Z' $Z+Z'$ = Minkowski sum

By construction, $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \ V_{Z_{nm}P}^n$

has $\quad \text{Hilb}(\mathcal{H}_P, t, q) = \sum_{m=0}^{\infty} t^m \ i_P(m; q) = \mathcal{E}_P(t, q)$

(Classical) THEOREMS	CONJECTURES (R.-Rhoades 2024)
• $\Bbbk[\Lambda_P]$ is Noetherian (Gordan 1873)	• $\mathcal{H}_P$ is Noetherian
• $\Bbbk[\Lambda_P]$ is Cohen-Macaulay (Hochster 1972)	• $\mathcal{H}_P$ is Cohen-Macaulay
• $\Omega \Bbbk[\Lambda_P] \cong \Bbbk[\Lambda_{\text{interior}(P)}]$ (Danilov 1978)	• $\Omega \mathcal{H}_P \cong \mathcal{H}_{\text{interior}(P)}$

CONJECTURES on $\mathcal{H}_P$ $\Rightarrow$ CONJECTURES on $E_P(t,q)$

- $\mathcal{H}_P$ is Noetherian $\Rightarrow$

  (RATIONALITY)
  - $E_P(t,q) = \dfrac{N_P(t,q)}{\prod\limits_{i=1}^{\nu}(1-q^{a_i}t^{b_i})}$ with $N_P(t,q)$ in $\mathbb{Z}[t,q]$ and $\nu \geq d+1$

- $\mathcal{H}_P$ is Cohen-Macaulay $\overset{(almost)}{\Longrightarrow}$

  (SIMPLEX NONNEGATIVITY)
  - For lattice simplices with $\nu = d+1$, $N_P(t,q)$ lies in $\mathbb{N}[t,q]$

- $\Omega \mathcal{H}_P \cong \mathcal{H}_{interior(P)}$ $\Rightarrow$

  (RECIPROCITY)
  - $E_P\left(\frac{1}{t}, \frac{1}{q}\right) = (-1)^{d+1} q^d \, \bar{E}_P(t, q)$

Even without CONJECTURES on $\mathcal{H}_P$, various classical Ehrhart theorems have q-analogues that are explained by $\mathcal{H}_P$.

4. Ehrhart tidbits ⟿ q-tidbits

THEOREM (Stanley 1986) For a poset Q on $\{1,2,...,n\}$

its order polytope $\quad \mathcal{O}_Q := \{ \underline{x} \in [0,1]^n : \quad x_i < x_j \text{ if } i <_P j \}$

chain polytope $\quad \mathcal{C}_Q := \{ \underline{x} \in [0,1]^n : \quad \sum_{i \in C} x_i \leq 1 \; \forall \text{ chains } C < Q\}$

have the same Ehrhart function / series:

$$i_{\mathcal{O}_Q}(m) = i_{\mathcal{C}_Q}(m) \quad \text{or} \quad E_{\mathcal{O}_Q}(t) = E_{\mathcal{C}_Q}(t)$$

EXAMPLE

order polytope

$Q = $
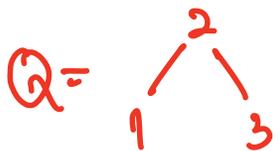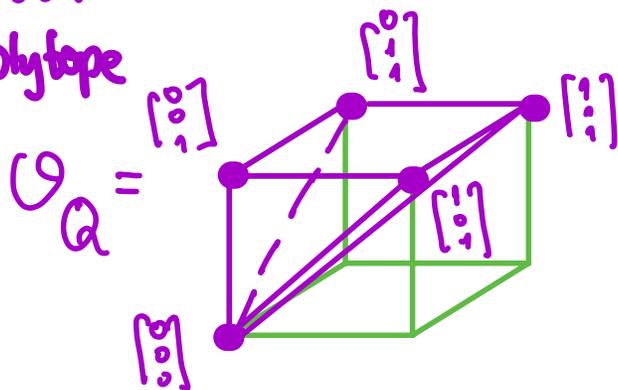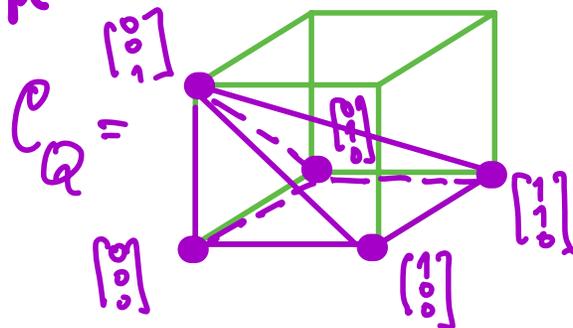
chain polytope

$$E_{\mathcal{O}_Q}(t) = E_{\mathcal{C}_Q}(t) = \frac{1+t}{(1-t)^4}$$

THEOREM (Rhoades-R. 24) For a poset Q on $\{1,2,\dots,n\}$

its order polytope $\mathcal{O}_Q$

and chain polytope $\mathcal{C}_Q$

have the same q-Ehrhart function/series

$$i_{\mathcal{O}_Q}(m,q) = i_{\mathcal{C}_Q}(m,q) \qquad \text{or} \qquad E_{\mathcal{O}_Q}(t,q) = E_{\mathcal{C}_Q}(t,q)$$

and, in fact, same harmonic algebras:

$$\mathcal{H}_{\mathcal{O}_Q} = \mathcal{H}_{\mathcal{C}_Q} \quad \text{inside} \quad \mathbb{R}[y_0, y_1, \dots, y_n].$$

PROPOSITION: For lattices polytopes $P_1, P_2 \subset \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$

their (free) join $P_1 * P_2 :=$ convex hull of

$$\left(P_1 \times \{\underline{0}_{n_2}\} \times \{0\}\right) \cup \left(\{\underline{0}_{n_1}\} \times P_2 \times \{1\}\right)$$

P_1 P_2

$P_1 * P_2$

$2\, P_1 * P_2$

$3\, P_1 * P_2$
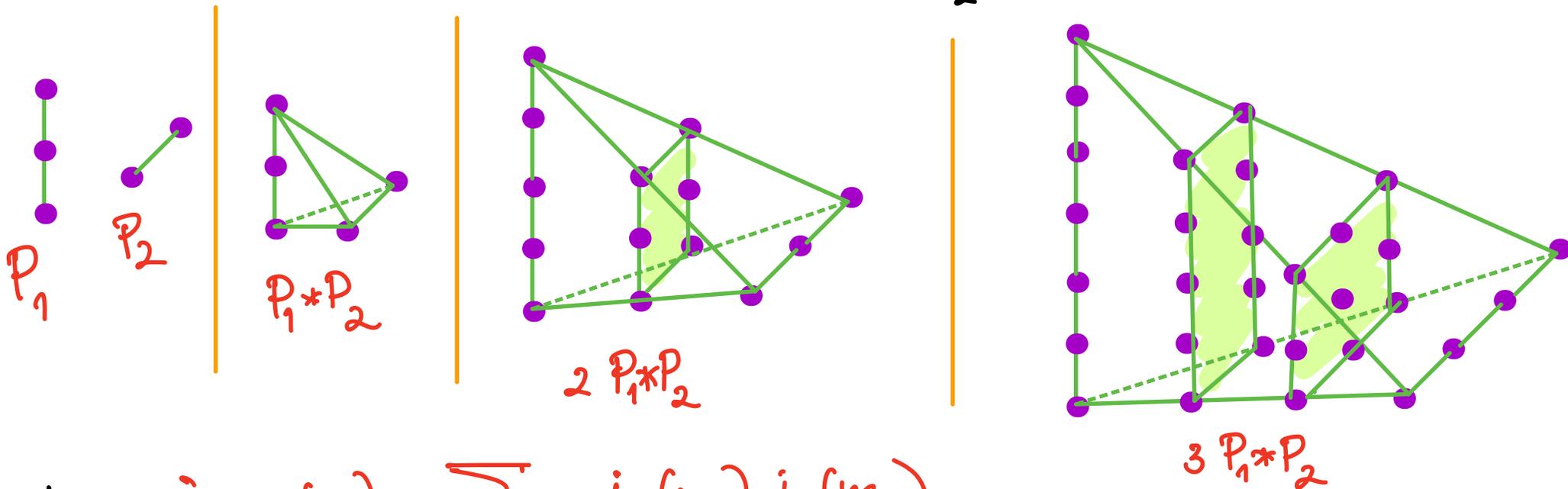
has $\quad i_{P_1 * P_2}(m) = \sum\limits_{\substack{(m_1, m_2): \\ m_1 + m_2 = m}} i_{P_1}(m_1) \cdot i_{P_2}(m_2)$

or equivalently $\boxed{E_{P_1 * P_2}(t) = E_{P_1}(t)\, E_{P_2}(t)}$

For lattices polytopes P_1, P_2

$$E_{P_1 * P_2}(t,q) = \frac{1-t}{1-qt} \; E_{P_1}(t,q) \; E_{P_2}(t,q)$$

$$\left(\; \overset{q=1}{\rightsquigarrow} \; \boxed{E_{P_1 * P_2}(t) = E_{P_1}(t) \, E_{P_2}(t)} \; \right)$$

---

proof idea:

$$(1-qt) \cdot E_{P_1 * P_2}(t,q) = (1-t) \cdot E_{P_1}(t,q) \cdot E_{P_2}(t,q)$$

follows from a ring isomorphism

$$\mathcal{H}_{P_1 * P_2} \, / \, \left(y_{P_1 * P_2} \cdot y_0 \right) \; \cong \; \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2} \, / \, \left(y_{P_1} \otimes 1 - 1 \otimes y_{P_2} \right)$$

a nonzero-divisor tracked by qt

a nonzero-divisor tracked by t

Thanks

for your

attention !