Ehrhart theory
and a q-analogue

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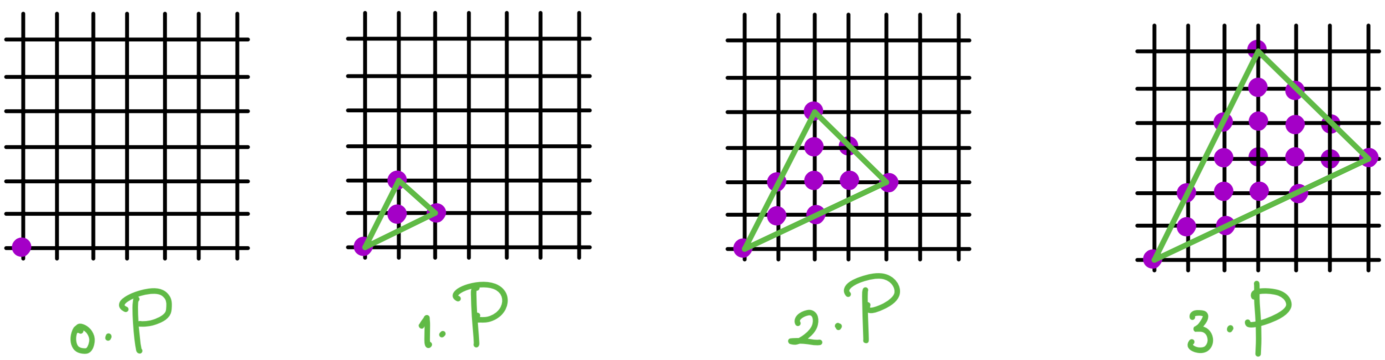
(arXiv: 2407.06511)



Texas A&M Math Colloquium April 11, 2025

OUTLINE

1. Ehrhart theory review,
   including Greatest Hits

2. q-analogues and CONJECTURE

3. Harmonic algebra CONJECTURE

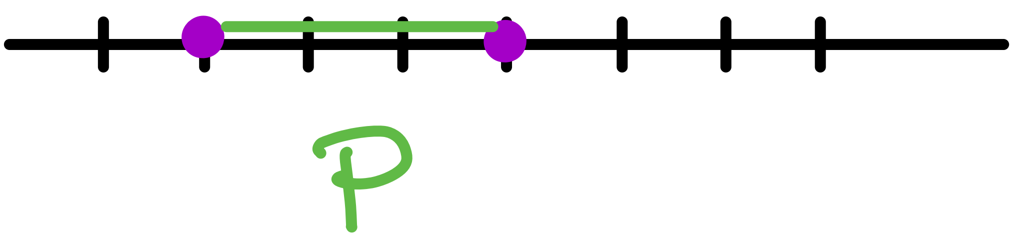
If time!

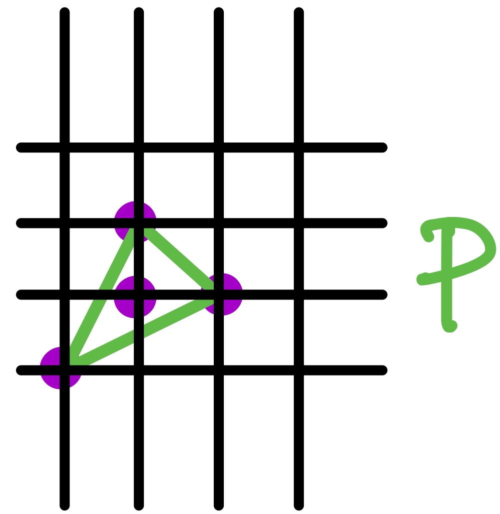
4. Ehrhart tidbits $\rightsquigarrow$ q-tidbits

1. Ehrhart theory review

vertices in $\mathbb{Z}^n$

$P \subset \mathbb{R}^n$ a lattice polytope





$\rightsquigarrow$ Ehrhart function

$$i_P(m) := \# \mathbb{Z}^n \cap mP$$

for $m = 0, 1, 2, \ldots$

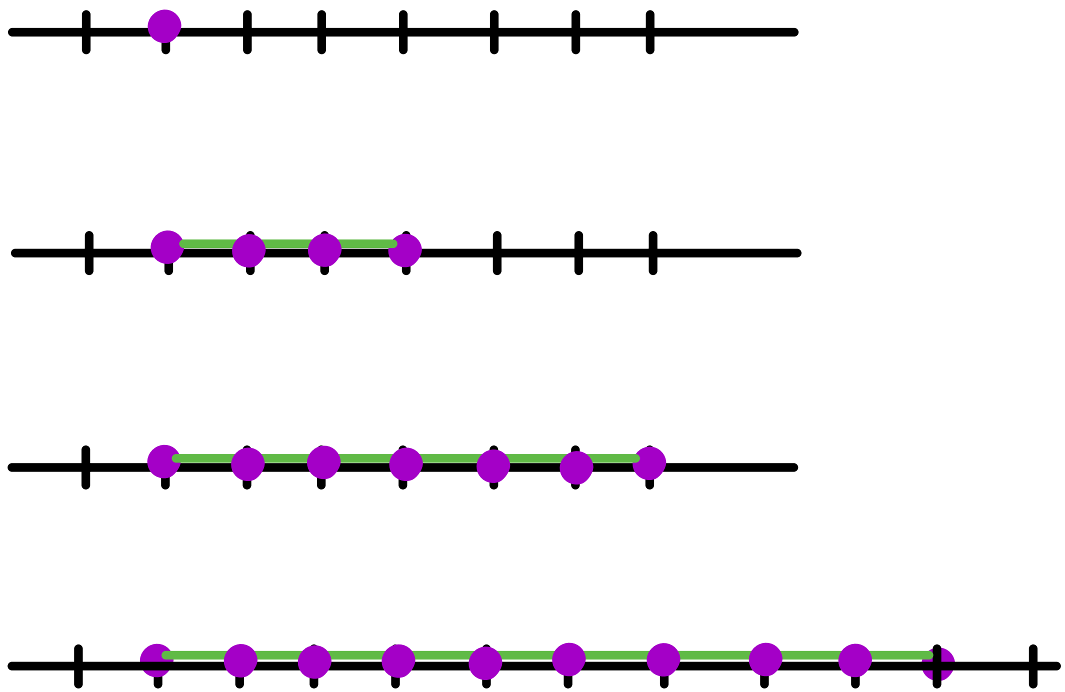
EXAMPLE

$0 \cdot P$ $\quad i_P(0) = 1$

$1 \cdot P$ $\quad i_P(1) = 4$

$2 \cdot P$ $\quad i_P(2) = 7$

$3 \cdot P$ $\quad i_P(3) = 10$

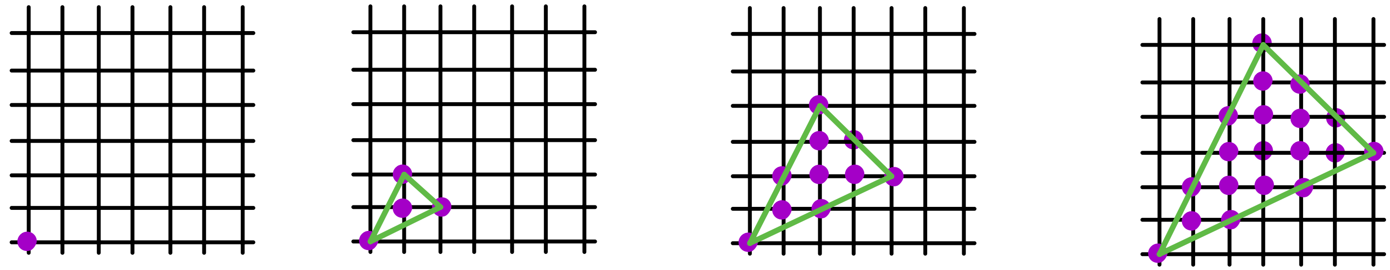


$$i_P(m) = 1 + 3m$$

1-dimensional (Lebesgue) volume of P

EXAMPLE

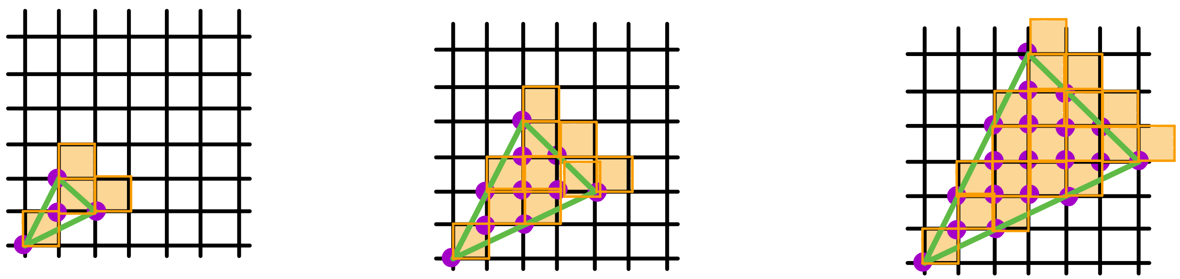
$0\cdot P$	$1\cdot P$	$2\cdot P$	$3\cdot P$



1	4	10	19
$i_P(0)$	$i_P(1)$	$i_P(2)$	$i_P(3)$

Guess $i_P(m) \approx$ area of $mP = m^2 \cdot (\text{area of } P)$

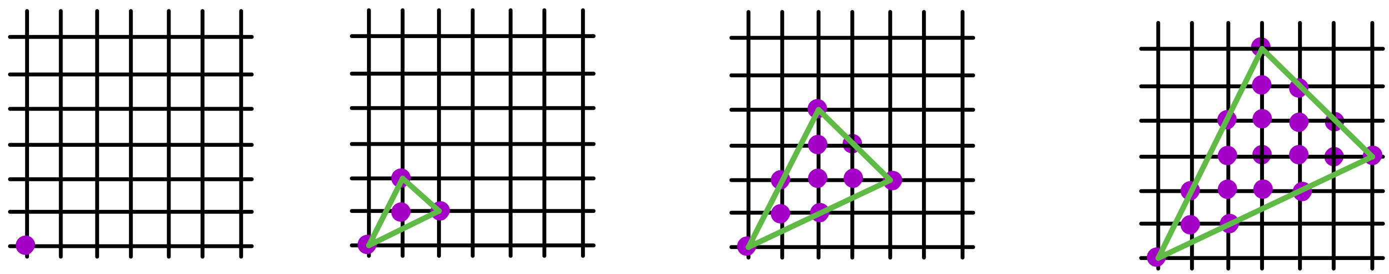
$\approx$ quadratic in m



SURPRISE:

It's on the nose quadratic for polygons.

$0 \cdot P$ $1 \cdot P$ $2 \cdot P$ $3 \cdot P$



1	4	10	19
$i_P(0)$	$i_P(1)$	$i_P(2)$	$i_P(3)$

$$i_P(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$$

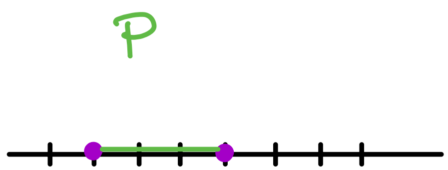
In general, $i_P(m) = 1 + am + Am^2$

area of P

It is also convenient to encode $i_P(m)$ via the **Ehrhart series**

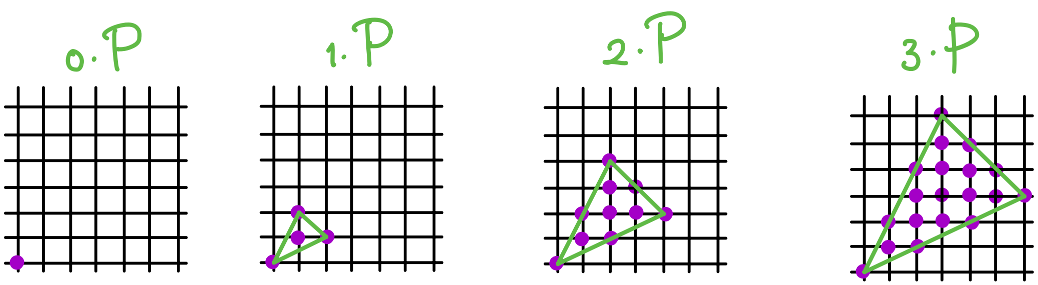
$$E_P(t) := \sum_{m=0}^{\infty} t^m \cdot i_P(m) = \sum_{m=0}^{\infty} t^m \cdot \#(\mathbb{Z}^n \cap mP)$$

EXAMPLES



$i_P(m) = 1+3m$

$$E_P(t) = \sum_{m=0}^{\infty} t^m (1+3m) = \frac{1+3t}{(1-t)^2}$$



$$E_P(t) = 1 + 4t^1 + 10t^2 + 19t^3 + \dots = \sum_{m=0}^{\infty} t^m \cdot \left(1 + \tfrac{3}{2}m + \tfrac{3}{2}m^2\right)$$

$$= \frac{1+t+t^2}{(1-t)^3}$$

Classical Ehrhart Theory's Greatest Hits :

THEOREM
(Ehrhart 1962)

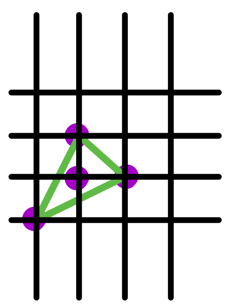
$i_P(m) := \# \mathbb{Z}^n \cap mP$ is a polynomial in m, of degree $d := \dim(P)$

or equivalently,

$$E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m) = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}}$$

EXAMPLE

P



$d=2$

$$i_P(m) = 1 + \tfrac{3}{2}m + \tfrac{3}{2}m^2$$

$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

THEOREM (Stanley 1980)

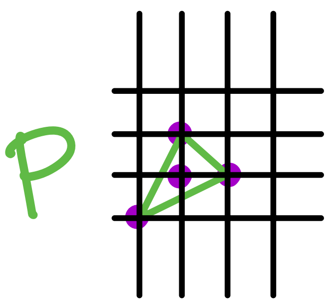
In the numerator of

$$E_P(t) = \sum_{m=0}^{\infty} t^m i_P(m) = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}},$$

the h^*-vector entries $(h_0^*, h_1^*, \dots, h_d^*)$

are nonnegative.

EXAMPLE



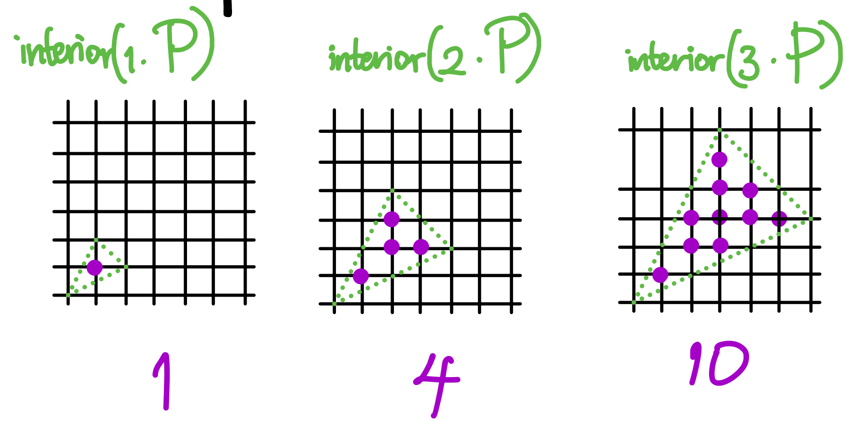
$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

$$(h_0^*, h_1^*, h_2^*) = (1,1,1)$$

One can also count interior lattice points...

$$\overline{i}_P(m) := \#\mathbb{Z}^n \cap \operatorname{interior}(mP)$$

$$\overline{E}_P(t) := \sum_{m=1}^{\infty} t^m \cdot \overline{i}_P(m)$$



THEOREM (Ehrhart-Macdonald reciprocity)

CONJECTURE 1959 PROOF 1971

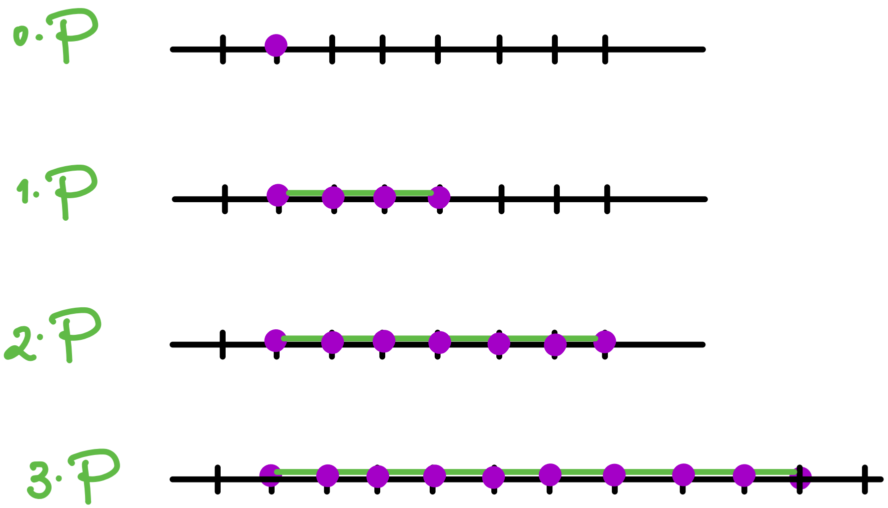
I.G. Macdonald
1928-2023

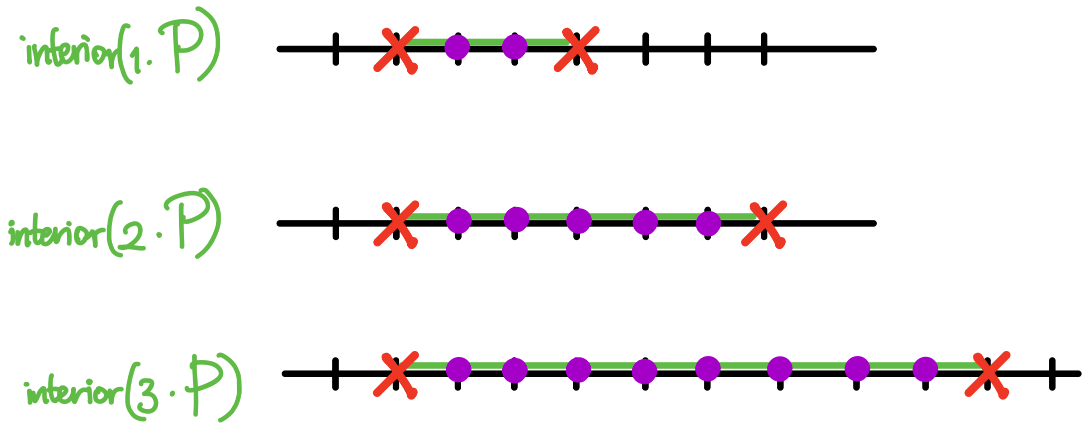
$$\overline{i}_P(m) = (-1)^d \, i_P(-m) \quad \text{for } m = 1,2,3,\ldots$$

or equivalently,

$$\overline{E}_P(t) = (-1)^{d+1} E_P(1/t)$$

EXAMPLE





$$i_P(m) = 1+3m$$

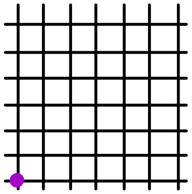
$$\overline{i}_P(m) = -1+3m = (-1)^1(1-3m) = (-1)^1\, i_P(-m)$$

$$E_P(t) = \frac{1+3t}{(1-t)^2}$$

$$\overline{E}_P(t) = \frac{3t+t^2}{(1-t)^2} = (-1)^2\, E_P(1/t)$$

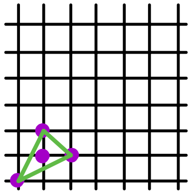
EXAMPLE

$i_P(0) = 1$



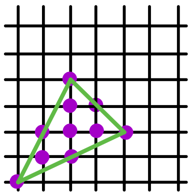
$0 \cdot P$

$i_P(1) = 4$



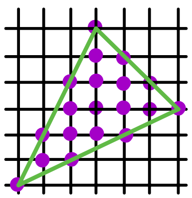
$1 \cdot P$

$i_P(2) = 10$



$2 \cdot P$

$i_P(3) = 19$

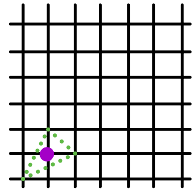


$3 \cdot P$

$$i_P(m) = 1 + \frac{3}{2}m + \frac{3}{2}m^2$$

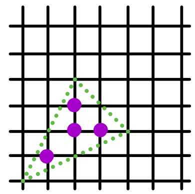
$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

$\mathrm{interior}(1 \cdot P)$



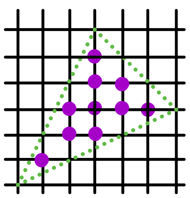
$\bar{i}_P(1) = 1$

$\mathrm{interior}(2 \cdot P)$



$\bar{i}_P(2) = 4$

$\mathrm{interior}(3 \cdot P)$

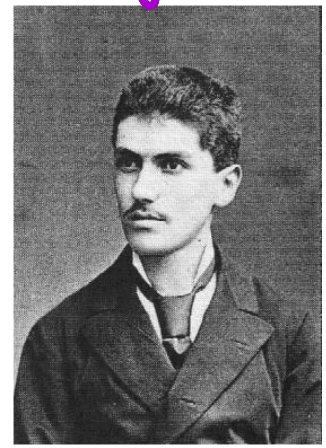


$\bar{i}_P(3) = 10$

$$\bar{i}_P(m) = 1 - \frac{3}{2}m + \frac{3}{2}m^2 = (-1)^2\, i_P(-m)$$

$$\bar{E}_P(t) = \frac{t+t^2+t^3}{(1-t)^3} = (-1)^3\, E_P(1/t)$$

Georg Pick



(A traditional ...) COROLLARY

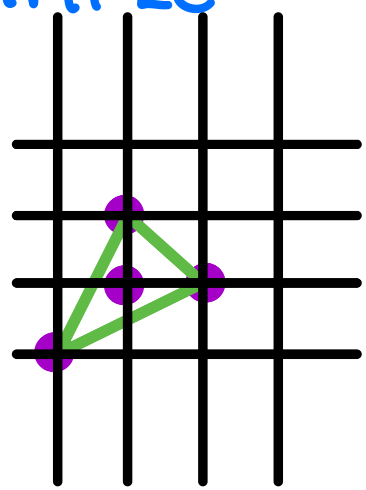
(Pick's Theorem 1899)

Any lattice (2-dimensional) polygon $\mathcal{P}$ has

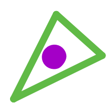
$$\underset{A}{\text{area}} = \#\text{interior points} + \#\text{boundary points}/2 - 1$$

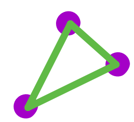
EXAMPLE

P



$$\underset{A}{\frac{3}{2}} = 1 + 3/2 - 1$$





PROOF:

Want $\underset{\text{area}}{A} \overset{?}{\underset{\text{Pick}}{=}} \underbrace{\#\text{interior points}}_{\bar{i}_P(1)} + \underbrace{\#\text{boundary points}}_{i_P(1) - \bar{i}_P(1)} / 2 - 1$

Ehrhart-Macdonald reciprocity

$$i_P(m) = 1 + am + Am^2$$

$$\bar{i}_P(m) = (-1)^2 \, i_P(-m) = 1 - am + Am^2$$

set $m=1$

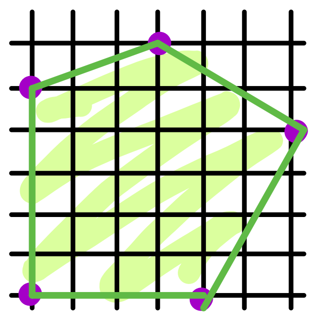
$$\left.\begin{aligned} i_P(1) &= 1+a+A \\ \bar{i}_P(1) &= 1-a+A \end{aligned}\right\} \xrightarrow{\text{solve for } a, A} \left\{\begin{aligned} 2A &= i_P(1) + \bar{i}_P(1) - 2 \\ 2a &= i_P(1) - \bar{i}_P(1) \end{aligned}\right.$$

Pick's formula, re-written

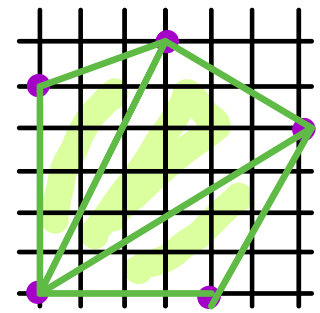
Two proof methods (for the Greatest Hits)

Method 1 (Ehrhart Macdonald Stanley): Reduce to simplices via triangulations

P

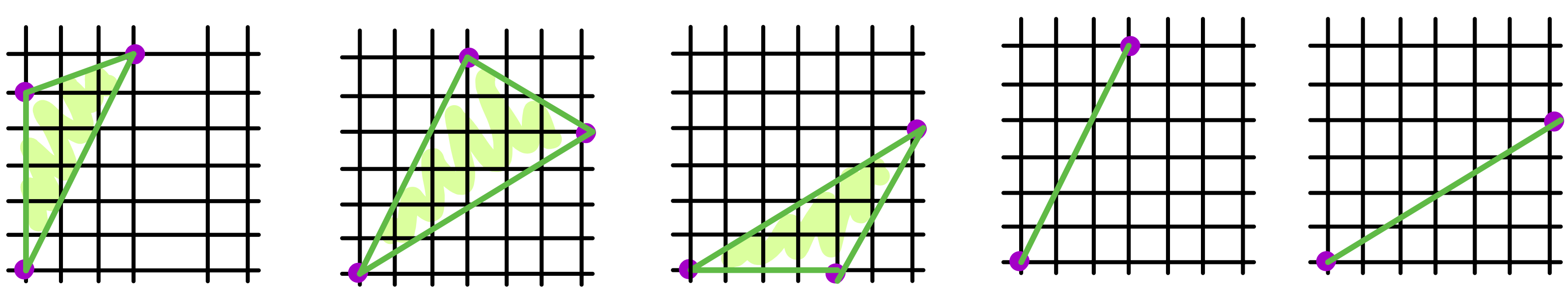


triangulate



$E_P(t)$ is a valuative function of P:

$$E_P(t) = E_{P_1}(t) + E_{P_2}(t) + E_{P_3}(t) - E_{P_1 \cap P_2}(t) - E_{P_2 \cap P_3}(t)$$



P_1 $\quad$ P_2 $\quad$ P_3 $\quad$ $P_1 \cap P_2$ $\quad$ $P_2 \cap P_3$

... and simplices have very explicit formulas :

PROPOSITION: For a lattice d-simplex $P \subset \mathbb{R}^n$ with vertices $v^{(1)}, v^{(2)}, \dots, v^{(d+1)}$

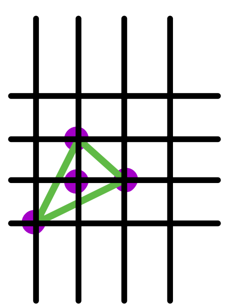
$$E_P(t) = \frac{\sum_{i=0}^{d} h_i^* \, t^i}{(1-t)^{d+1}} \quad \text{where } h_i^* = \#\left(\left(\mathbb{Z}^n \times \{i\}\right) \cap \Pi\right)$$

semi-open parallelepiped

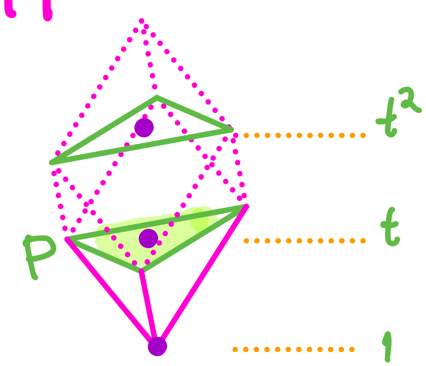
$$\Pi := \sum_j [0,1) \cdot \begin{bmatrix} v^{(j)} \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^{n+1}$$

EXAMPLE

$P \subset \mathbb{R}^2$



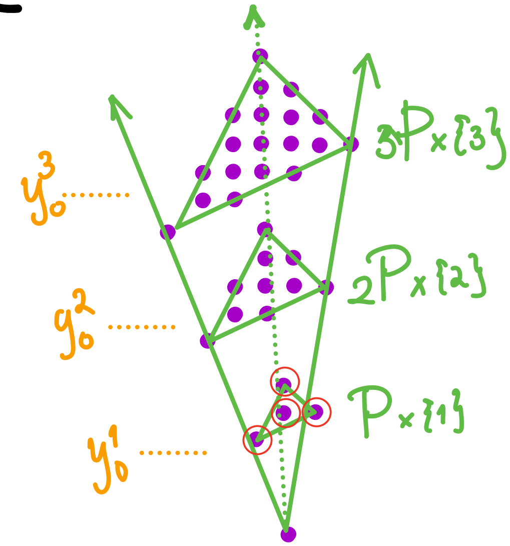
$\Pi \subset \mathbb{R}^3$



$$E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

Method 2 (Stanley): Commutative algebra

of the affine semigroup ring



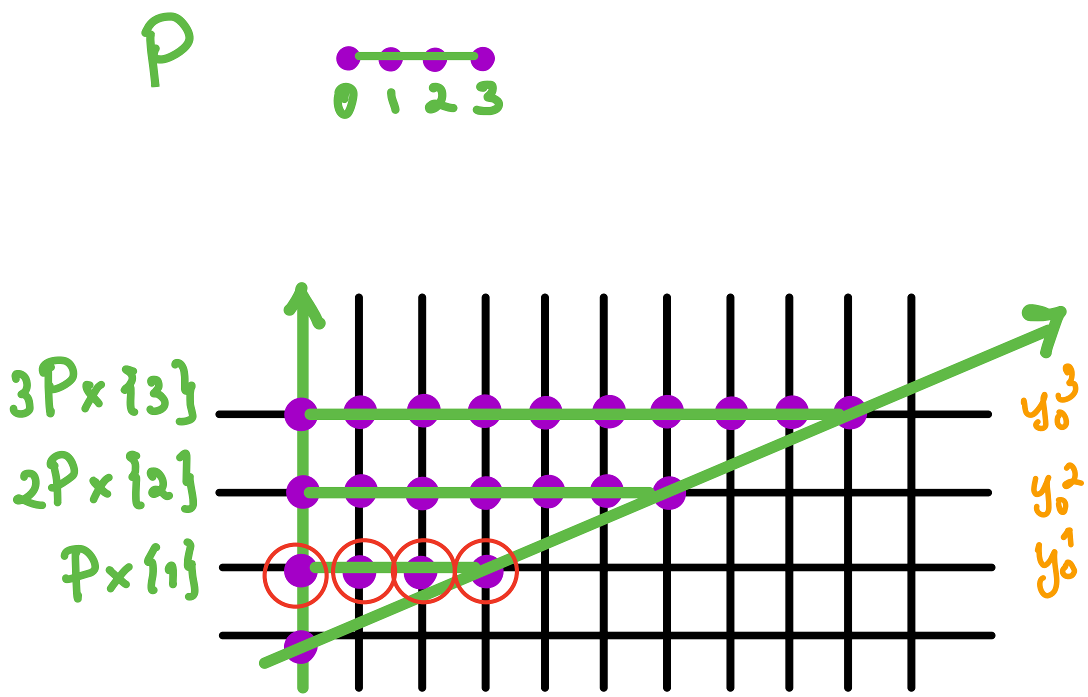
$$\Bbbk[\Lambda_P] := \operatorname{span}_{\Bbbk}\{ y_0^m y_1^{a_1} \cdots y_n^{a_n} : \underline{a} \in \mathbb{Z}^n \cap mP, \; m = 0,1,2,\ldots \}$$

$$\subset \Bbbk[y_0, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

which has Hilbert series equal to the Ehrhart series:

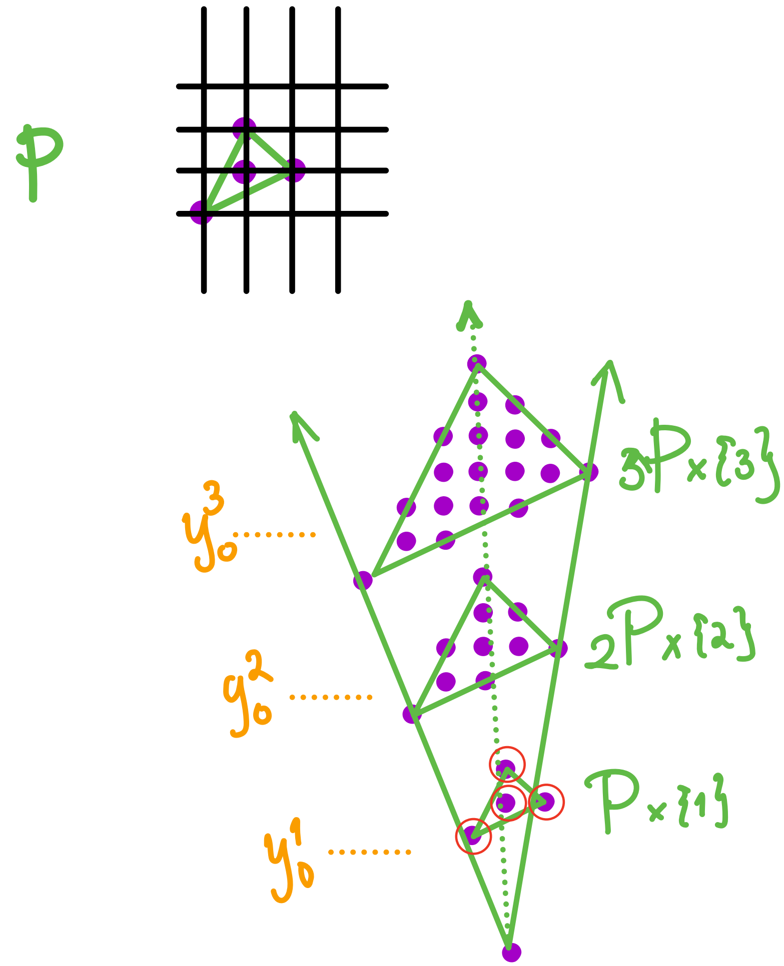
$$\mathrm{Hilb}(\Bbbk[\Lambda_P], t) := \sum_{m=0}^{\infty} t^m \cdot \dim_{\Bbbk} \Bbbk[\Lambda_P]_m = E_P(t)$$

EXAMPLES



$$\Bbbk[\Lambda_P] = \Bbbk[y_0,\ y_0y_1,\ y_0y_1^2,\ y_0y_1^3]$$

$$\subset \Bbbk[y_0, y_1^{\pm 1}]$$



$$\Bbbk[\Lambda_P] = \Bbbk[y_0,\ y_0y_1y_2,\ y_0y_1^2y_2,\ y_0y_1y_2^2]$$

$$\subset \Bbbk[y_0, y_1^{\pm 1}, y_2^{\pm 1}]$$

- $\Bbbk[\Lambda_P]$ is Noetherian (Gordan 1873)

- $\Bbbk[\Lambda_P]$ has a linear system of parameters (Noether 1926)

$$\Rightarrow E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

- $\Bbbk[\Lambda_P]$ is Cohen-Macaulay (Hochster 1972)

$\Rightarrow h_i^* \geq 0$ for $i = 1, 2, \dots, d$

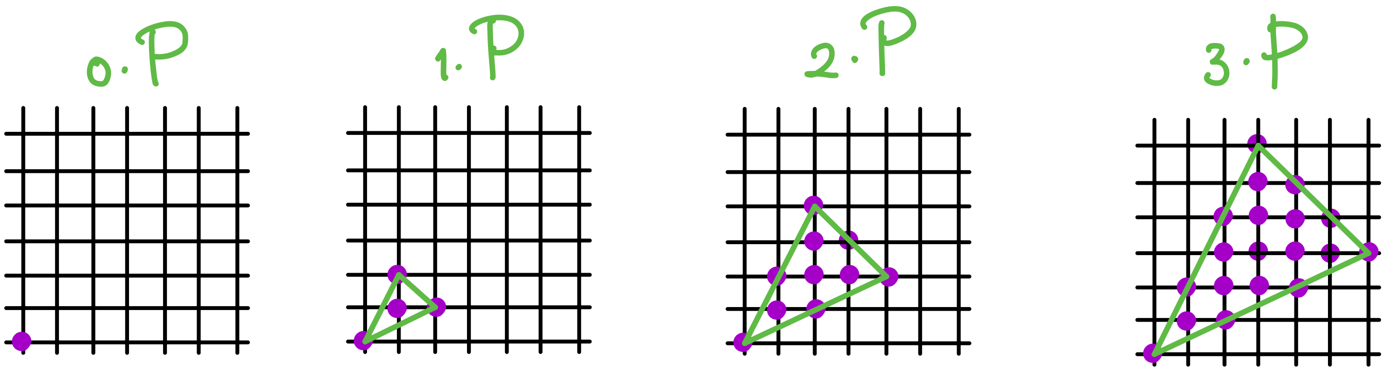
- $\Omega \, \Bbbk[\Lambda_P] \cong \Bbbk[\Lambda_{\text{interior}(P)}]$

canonical module

(Danilov 1978)

$$\Rightarrow E_P(t) = (-1)^{d+1} E_P(1/t)$$

2. q-analogues



$$1 + 4t^1 + 10t^2 + 19t^3 + \dots = E_P(t)$$

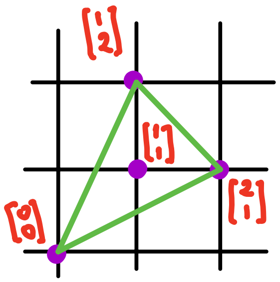
$$= \frac{1+t+t^2}{(1-t)^3}$$

$\Big\downarrow$ q-analogue

$$1 + (1+2q+q^2)t^1 + \begin{pmatrix} 1+2q+3q^2 \\ +3q^3+q^4 \end{pmatrix} t^2 + \begin{pmatrix} 1+2q+3q^2+4q^3 \\ +5q^4+3q^5+q^6 \end{pmatrix} t^3 + \dots = E_P(t,q)$$

$$= \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

Those q-analogues come from using the point orbit method to deform the coordinate ring of $Z := \mathbb{Z}^n \cap mP \subset \mathbb{R}^n$



$Z := \mathbb{Z}^n \cap mP$

$\#Z = 4$

$\rightsquigarrow$

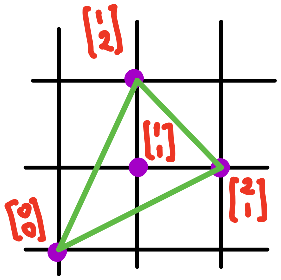
affine coordinate ring of Z as 0-dimensional variety

$$\mathbb{R}[Z] = \mathbb{R}[x,y] / \underbrace{I(Z)}_{\text{polynomials } f(x,y) \text{ vanishing on } Z}$$

$$= \mathbb{R}[x,y] / \underset{\begin{bmatrix}0\\0\end{bmatrix}}{(x-0, y-0)} \cap \underset{\begin{bmatrix}1\\1\end{bmatrix}}{(x-1, y-1)} \cap \underset{\begin{bmatrix}1\\2\end{bmatrix}}{(x-1, y-2)} \cap \underset{\begin{bmatrix}2\\1\end{bmatrix}}{(x-2, y-1)}$$

$$= \mathbb{R}[x,y] / \left(\underline{2xy - y^2} - 2x + y,\ \underline{x^2 - y^2} - 3x + y,\ \underline{y^3} - 3y^2 + 2y \right)$$

Macaulay2 computation



$Z := \mathbb{Z}^n \cap mP$

$\# Z = 4$

affine coordinate ring

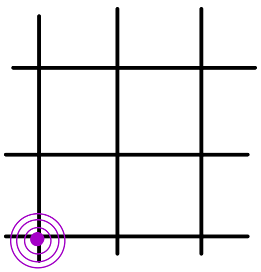
$$\mathbb{R}[Z] = \mathbb{R}[x,y] \big/ I(Z)$$
$$= \mathbb{R}[x,y] \big/ \left(\underline{2xy - y^2} - 2x + y,\ \underline{x^2 - y^2} - 3x + y,\ \underline{y^3} - 3y^2 + 2y \right)$$

DEFORM!

(= by taking top degree components of all $f(x,y)$ in $I(Z)$)

associated graded ring

$$\operatorname{gr} \mathbb{R}[Z] = \mathbb{R}[x,y] \big/ \operatorname{gr} I(Z)$$
$$= \left(\underline{2xy - y^2},\ \underline{x^2} - y^2,\ \underline{y^3} \right)$$



fat point at $\begin{bmatrix}0\\0\end{bmatrix}$ of multiplicity 4

$$\operatorname{Hilb}(\operatorname{gr} \mathbb{R}[Z], q) = 1 + 2q + q^2 \overset{q=1}{\rightsquigarrow} 4 = \# Z$$

= the q-analogue of $\# Z$

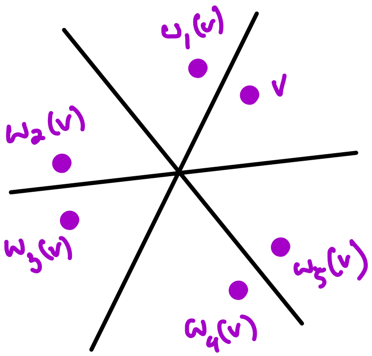
MOTIVATING EXAMPLE (Kostant 1963):

$Z :=$ regular orbit of a vector v under a Weyl group W

$$\mathbb{R}[Z] = \mathbb{R}[x_1,\dots,x_n] \,/\, (f_1(\underline{x}) - f_1(v), \dots, f_n(\underline{x}) - f_n(v))$$

where $\mathbb{R}[x_1,\dots,x_n]^W = \mathbb{R}[f_1(\underline{x}),\dots,f_n(\underline{x})]$

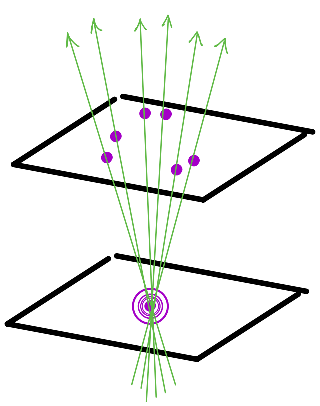
degrees: $d_1, \dots, d_n$



deform

$$\mathrm{gr}\,\mathbb{R}[Z] = \mathbb{R}[x_1,\dots,x_n]/(f_1(\underline{x}),\dots,f_n(\underline{x})) \cong H^\bullet(G/B)$$

cohomology of the flag manifold



$$\mathrm{Hilb}(\mathrm{gr}\,\mathbb{R}[Z], q) = [d_1]_q [d_2]_q \cdots [d_n]_q \,,$$

a q-analogue of $\#W = d_1 d_2 \cdots d_n$

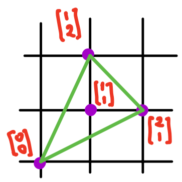
where $[d]_q := 1 + q + q^2 + \dots + q^{d-1}$

REMARK: Instead of Hilbert series for the fat point coordinate ring $\mathbb{R}[x_1,\dots,x_n]/\operatorname{gr} I(Z)$, would have used its Macaulay inverse system:

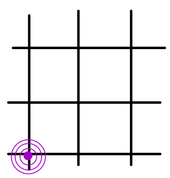
$$V_Z := \left\{ g(\underline{y}) \in \mathbb{R}[y_1,\dots,y_n] : f\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\right) g(\underline{y}) = 0 \quad \forall f(\underline{x}) \in \operatorname{gr} I(Z) \right\}$$

since $\quad \operatorname{Hilb}(V_Z, q) = \operatorname{Hilb}\left(\mathbb{R}[x_1,\dots,x_n]/\operatorname{gr} I(Z),\, q\right)$

EXAMPLE

Z 

$$\mathbb{R}[Z] = \mathbb{R}[x_1,x_2] \Big/ \left(\underline{2x_1x_2 - x_2^2} - 2x_1 + y_1,\ \underline{x_1^2 - x_2^2} - 3x_1 + x_2,\ \underline{x_2^3} - 3x_2^2 + 2x_2\right)$$



$$\operatorname{gr} \mathbb{R}[Z] = \mathbb{R}[x_1,x_2] \Big/ \left(2x_1x_2 - x_2^2,\ x_1^2 - x_2^2,\ x_2^3\right)$$

$$V_Z = \operatorname{span}_{\mathbb{R}}\left\{ 1,\ y_1,\ y_2,\ y_1^2 + y_1y_2 + y_2^2 \right\} \subset \mathbb{R}[y_1,y_2]$$

$$\operatorname{Hilb}(V_Z, q) = 1 + 2q + q^2 \quad \overset{q=1}{\rightsquigarrow} \quad 4 = \#Z$$

MAIN DEFINITION:

For a lattice polytope $P \subset \mathbb{R}^n$, define its q-Ehrhart function

$$i_P(m;q) = \mathrm{Hilb}(\mathrm{gr}\,\mathbb{R}[\mathbb{Z}^n \cap mP], q)$$
$$= \mathrm{Hilb}(V_{\mathbb{Z}^n \cap mP}, q)$$

$$\xrightarrow{q=1} \quad i_P(m) = \#\mathbb{Z}^n \cap mP$$

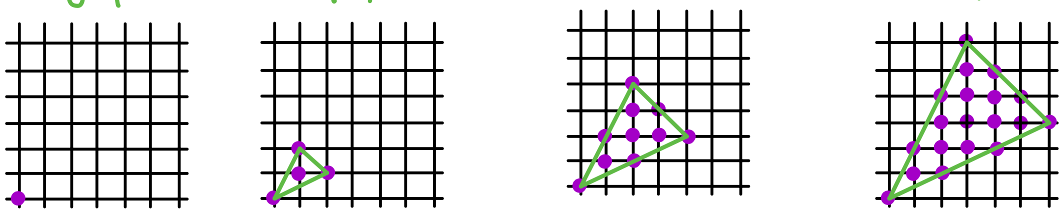
and then its q-Ehrhart series in $\mathbb{R}[q][[t]]$

$$\boxed{E_P(t,q) := \sum_{m=0}^{\infty} t^m \cdot i_P(m;q)}$$

$$\xrightarrow{q=1} \quad E_P(t) = \sum_{m=0}^{\infty} t^m \, i_P(m)$$

EXAMPLE

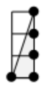
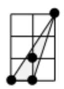
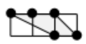
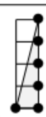
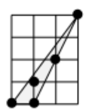
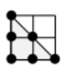
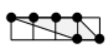
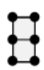
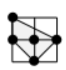
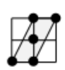
$0 \cdot P$ $\quad 1 \cdot P$ $\quad 2 \cdot P$ $\quad 3 \cdot P$

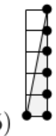
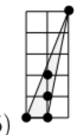
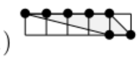
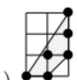
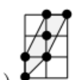


$$1 + 4t^1 + 10t^2 + 19t^3 + \dots = E_P(t) = \frac{1+t+t^2}{(1-t)^3}$$

$$1 + (1+2q+q^2)t^1 + \begin{pmatrix} 1+2q+3q^2 \\ +3q^3+q^4 \end{pmatrix} t^2 + \begin{pmatrix} 1+2q+3q^2+4q^3 \\ +5q^4+3q^5+q^6 \end{pmatrix} t^3 + \dots = E_P(t,q) = \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-tq^3)}$$

MORE EXAMPLES of $E_P(t,q)$ for lattice polygons

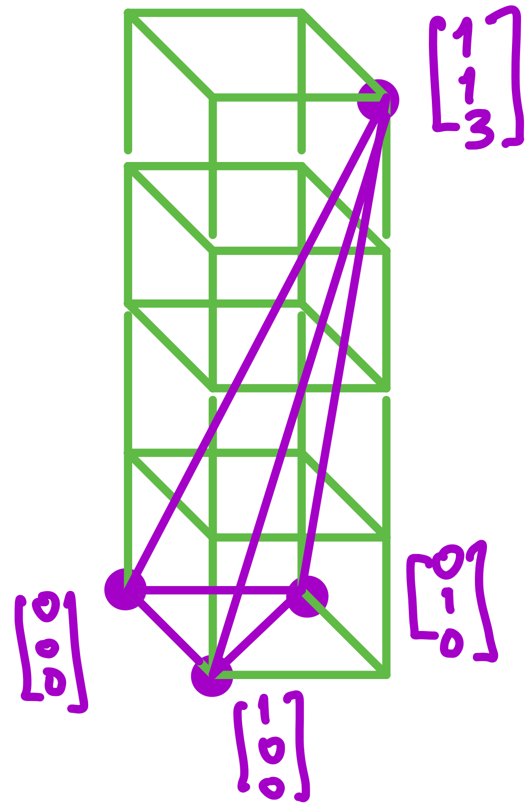
normalized area of P	vertices of P	$h^*_P(t) = 1 + h^*_1 t + h^*_2 t^2$	$E_P(t,q)$
1	$(0,0),(1,0),(0,1)$ 	1	$\frac{1}{(1-t)(1-qt)^2}$
2	$(0,0),(1,0),(1,2)$ 	$1+t$	$\frac{1+qt}{(1-t)(1-qt)(1-q^2t)}$
2	$(0,0),(1,0),(0,1),(1,1)$ 	$1+t$	$\frac{1+qt}{(1-t)(1-qt)(1-q^2t)}$
3	$(0,0),(1,0),(1,3)$ 	$1+2t$	$\frac{1+qt+q^2t}{(1-t)(1-qt)(1-q^3t)}$
3	$(0,0),(1,0),(2,3)$ 	$1+t+t^2$	$\frac{(1+qt)(1+qt+q^2t^2)}{(1-t)(1-q^2t)(1-q^3t^2)}$
3	$(0,0),(1,0),(0,1),(-2,1)$ 	$1+2t$	$\frac{1+qt-q^2t^2-q^3t^2}{(1-t)(1-qt)(1-q^2t)^2}$
4	$(0,0),(1,0),(1,4)$ 	$1+3t$	$\frac{1+t(q+q^2+q^3)}{(1-t)(1-qt)(1-q^4t)}$
4	$(0,0),(1,0),(3,4)$ 	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2}$
4	$(0,0),(2,0),(0,2)$ 	$1+3t$	$\frac{1+2qt+q^2t}{(1-t)(1-q^2t)^2}$
4	$(0,0),(1,0),(0,1),(-3,1)$ 	$1+3t$	$\frac{1+qt+q^2t-q^2t^2-q^3t^2-q^4t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}$
4	$(0,0),(1,0),(0,2),(1,2)$ 	$1+3t$	$\frac{1+qt+q^2t-q^2t^2-q^3t^2-q^4t^2}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}$
4	$(0,0),(2,0),(0,1),(1,-1)$ 	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2}$
4	$(0,0),(1,0),(1,2),(2,2)$ 	$(1+t)^2$	$\frac{(1+qt)^2}{(1-t)(1-q^2t)^2)}$

vertices of P	$h^*_P(t) = 1 + h^*_1 t + h^*_2 t^2$	$E_P(t,q)$
$(0,0),(1,0),(1,5)$ 	$1+4t$	$\frac{1+t(q+q^2+q^3+q^4)}{(1-t)(1-qt)(1-q^5t)}$
$(0,0),(1,0),(2,5)$ 	$1+2t+2t^2$	$\frac{(1+qt)(1+t(q+q^2)+t^2(q^3+q^4))}{(1-t)(1-q^2t)(1-q^5t^2)}$
$(0,0),(1,0),(0,1),(-4,1)$ 	$1+4t$	$\frac{1+qt+q^2t+q^3t-q^2t^2-q^3t^2-q^4t^2-q^5t^2}{(1-t)(1-qt)(1-q^2t)(1-q^4t)}$
$(0,0),(2,0),(0,1),(-3,1)$ 	$1+4t$	$\frac{1+qt+2q^2t-q^2t^2-q^3t^2-q^4t^2-q^5t^2}{(1-t)(1-qt)(1-q^3t)^2}$
$(0,0),(1,0),(2,3),(2,1)$ 	$1+3t+t^2$	$\frac{1+2qt+2q^2t+2q^3t^2+2q^4t^2+q^5t^3}{(1-t)(1-q^2t)(1-q^5t^2)}$
$(0,0),(1,0),(1,2),(2,2),(0,-1)$ 	$1+3t+t^2$	$\frac{1+2qt+2q^2t+2q^3t^2+2q^4t^2+q^5t^3}{(1-t)(1-q^2t)(1-q^5t^2)}$

Since $E_P(t,q)$ is an $\mathrm{Aff}(\mathbb{Z}^n)$-invariant of P, can use database of lattice polytopes by Balletti 2021

EXAMPLE (V. Kurylenko)

Reeve tetrahedron P of volume 3 seems to have



$$E_P(t,q) \overset{?}{=} \frac{(1-q^5t^4)\cdot(1+qt)\left(1+qt+(2q^2+q^3)t^2+2q^4t^3+(q^5+q^6)t^4\right)}{(1-t)(1-qt)(1-q^3t^2)(1-q^5t^3)(1-q^6t^4)}$$

q-Ehrhart theory CONJECTURE

First, recall the

CLASSICAL Ehrhart Theorems: For d-dimensional lattice polytopes P,

- $$E_P(t) = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$ (RATIONALITY)

- $$E_P\left(\frac{1}{t}\right) = (-1)^{d+1} \bar{E}_P(t)$$ (RECIPROCITY)

- For lattice simplices,
$$h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$$ (SIMPLEX NONNEGATIVITY)

- $$h_i^* \geq 0 \quad \text{for } i = 1, 2, \ldots, d$$ (GENERAL NONNEGATIVITY)

Classical Ehrhart THEOREMS

CONJECTURES (R.-Rhoades 2024)

RATIONALITY

- $E_P(t) = \dfrac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$

- $E_P(t,q) = \dfrac{N_P(t,q)}{\prod_{i=1}^{\nu}(1-q^{a_i}t^{b_i})}$ with $N_P(t,q)$ in $\mathbb{Z}[t,q]$ and $\nu \geq d+1$

RECIPROCITY

- $E_P\left(\frac{1}{t}\right) = (-1)^{d+1}\,\overline{E}_P(t)$

- $E_P\left(\frac{1}{t},\frac{1}{q}\right) = (-1)^{d+1}\,q^{d}\,\overline{E}_P(t,q)$

SIMPLEX NONNEGATIVITY

- For lattice simplices,
  $h_i^* = \#\left(\mathbb{Z}^n \times \{i\} \cap \Pi\right)$

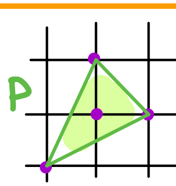
- For lattice d-simplices with $\nu = d+1$, $N_P(t,q)$ lies in $\mathbb{N}[t,q]$

GENERAL NONNEGATIVITY

- $h_i^* \geq 0$ for $i=1,2,\dots,d$

- ??

EXAMPLE P

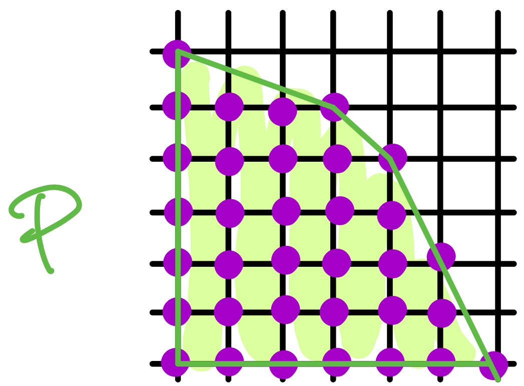


$$E_P(t,q) = \frac{(1+tq+t^2q^2)(1+tq)}{(1-t)(1-tq^2)(1-t^2q^3)}$$

EXAMPLES: These conjectures are proven for

antiblocking polytopes := polytopes P inside $(\mathbb{R}_{\geq 0})^n$ with $0 \leq z \leq z'$ and $z' \in P \Rightarrow z \in P$

componentwise comparison



PROPOSITION: For P antiblocking, every ideal $\mathrm{gr}\, I(Z)$ for $Z := \mathbb{Z}^n \cap mP$ is a monomial ideal $\mathrm{gr}\, I(Z) = \mathrm{span}_{\mathbb{R}}\{\underline{x}^{\underline{a}} : \underline{a} \notin Z\}$,

and $$E_P(t,q) = \sum_{m=0}^{\infty} t^m \sum_{\underline{a} \in \mathbb{Z}^n \cap mP} q^{a_1 + \cdots + a_n}$$

$$= \Big[\mathrm{Hilb}\big(\Bbbk[\Lambda_P], y_0, y_1, \ldots, y_n\big)\Big]_{\substack{y_0 = t \\ y_1 = y_2 = \cdots = y_n = q}}$$

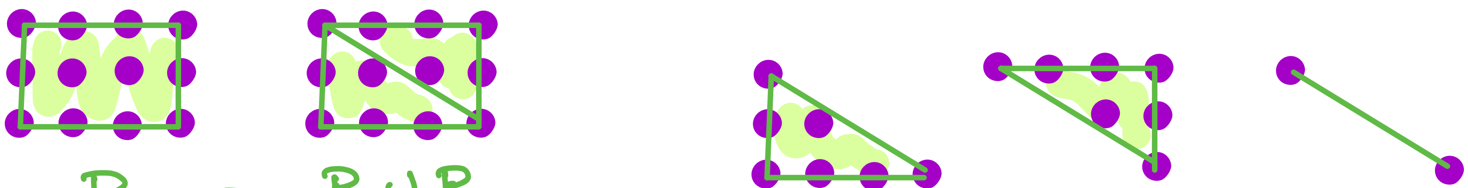
affine semigroup ring for P

3. Harmonic algebra CONJECTURE

Method 1 of Ehrhart theory (= reduce to simplices via triangulation) seems elusive, because

$i_P(m;q) = \mathrm{Hilb}(\mathrm{gr}\, R[\mathbb{Z}^n \cap mP], q)$ is not valuative as a function of P.

EXAMPLE



$$P \quad = \quad P_1 \cup P_2$$

$$i_P(1) = i_{P_1}(1) + i_{P_2}(1) - i_{P_1 \cap P_2}(1)$$

$$12 = 7 + 7 - 2$$

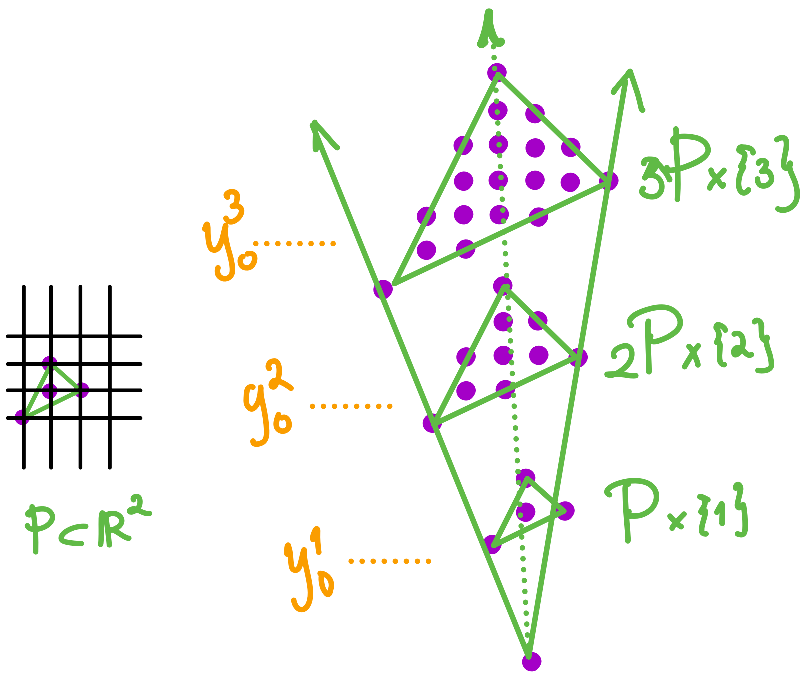
But $\quad i_P(1;q) \neq i_{P_1}(1;q) + i_{P_2}(1;q) - i_{P_1\cap P_2}(1;q)$

$= (1+q+q^2)\cdot(1+q+q^2+q^3)$

$= 1+2q+3q^2+3q^3+2q^4+q^5$

$i_{P_1}(1;q) = 1+2q+3q^2+q^3$ $\qquad$ $i_{P_2}(1;q) = 1+2q+3q^2+q^3$ $\qquad$ $i_{P_1\cap P_2}(1;q) = 1+q$

Method 2 (= commutative algebra) looks promising...



Recall

$$\Bbbk[\Lambda_P] := \bigoplus_{m=0}^{\infty} \operatorname{span}_{\Bbbk}\{y_0^m \, \underline{y}^{\underline{a}}\}_{\underline{a} \in \mathbb{Z}^n \cap mP} \subset \Bbbk[y_0, y_1, \ldots, y_n]$$

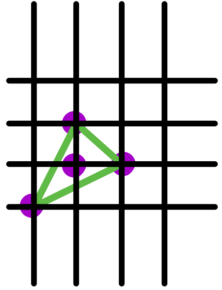
affine semigroup ring

DEFINITION:

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP} \subset \mathbb{R}[y_0, y_1, \ldots, y_n]$$

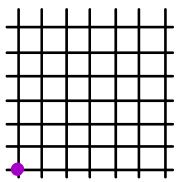
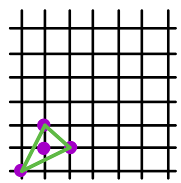
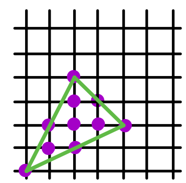
Harmonic algebra

where V_Z = harmonic space/Macaulay inverse system for $\mathbb{R}[y_1, \ldots, y_n] / \operatorname{gr} I(Z)$

EXAMPLE $P \subset \mathbb{R}^2$  has harmonic algebra

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP} \subset \mathbb{R}[y_0, y_1, y_2]$$

$$= \mathbb{R}\cdot 1 \;\oplus\; y_0 V_{\mathbb{Z}^2 \cap 1P} \;\oplus\; y_0^2 V_{\mathbb{Z}^2 \cap 2P} \;\oplus\; \cdots$$

$$= \operatorname{span}_{\mathbb{R}} \Big\{ 1, \quad y_0,\ y_0 y_1,\ y_0 y_2,\ y_0(y_1^2 + y_1 y_2 + y_2^2), \quad y_0^2,\ y_0^2 y_1,\ y_0^2 y_2,\ y_0^2 y_1^2,\ y_0^2 y_1 y_2,\ y_0^2 y_2^2,\ y_0^2 y_1^3,\ y_0^2(y_1^2 y_2 + y_1 y_2^2),\ y_0^2 y_2^3,\ y_0^2(y_1^4 + 2y_1^3 y_2 + 3y_1^2 y_2^2 + 2y_1 y_2^3 + y_2^4), \quad \ldots \Big\}$$

Why is $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP}$ even an algebra ?!

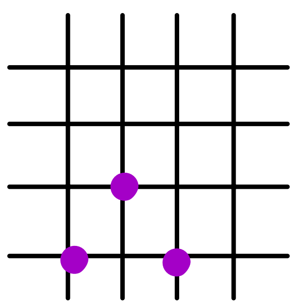
For $\Bbbk[\Lambda_P] := \bigoplus_{m=0}^{\infty} \operatorname{span}_{\Bbbk}\{y_0^m \, y^{\underline{a}}\}_{\underline{a} \in \mathbb{Z}^n \cap mP}$ it came from

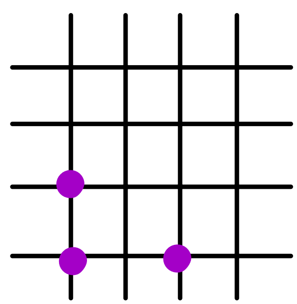
$$\mathbb{Z}^n \cap mP \; + \; \mathbb{Z}^n \cap m'P \; \subset \; \mathbb{Z}^n \cap (m+m')P$$

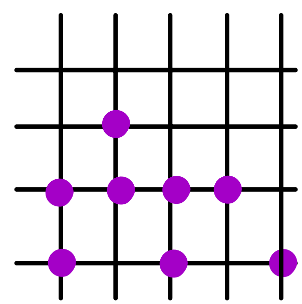
(Surprising! **THEOREM** (R. Rhoades 2024)

For finite point sets Z and $Z' \subset \Bbbk^n$,

$$V_Z \cdot V_{Z'} \subseteq V_{Z+Z'}$$

Z 

Z' 



$Z+Z'$ = Minkowski sum

By construction, $\mathcal{H}_P := \bigoplus_{m=0}^{\infty} y_0^m \, V_{\mathbb{Z}^n \cap mP}$

has $\mathrm{Hilb}(\mathcal{H}_P, t, q) = \sum_{m=0}^{\infty} t^m \, i_P(m;q) = E_P(t,q)$

(Classical) THEOREMS	CONJECTURES (R.-Rhoades 2024)
• $\mathbb{k}[\Lambda_P]$ is Noetherian (Gordan 1873)	• $\mathcal{H}_P$ is Noetherian
• $\mathbb{k}[\Lambda_P]$ is Cohen-Macaulay (Hochster 1972)	• $\mathcal{H}_P$ is Cohen-Macaulay
• $\Omega\, \mathbb{k}[\Lambda_P] \cong \mathbb{k}[\Lambda_{\mathrm{interior}(P)}]$ (Danilov 1978)	• $\Omega\, \mathcal{H}_P \cong \mathcal{H}_{\mathrm{interior}(P)}$

CONJECTURES on $\mathcal{H}_P$ $\Longrightarrow$ CONJECTURES on $E_P(t,q)$

- $\mathcal{H}_P$ is Noetherian $\Longrightarrow$ (RATIONALITY)
  - $E_P(t,q) = \dfrac{N_P(t,q)}{\prod_{i=1}^{\nu}(1-q^{a_i}t^{b_i})}$ with $N_P(t,q)$ in $\mathbb{Z}[t,q]$ and $\nu \geq d+1$

- $\mathcal{H}_P$ is Cohen-Macaulay $\overset{\text{(almost)}}{\Longrightarrow}$ (SIMPLEX NONNEGATIVITY)
  - For lattice simplices with $\nu = d+1$, $N_P(t,q)$ lies in $\mathbb{N}[t,q]$

- $\Omega\mathcal{H}_P \cong \mathcal{H}_{\text{interior}(P)}$ $\Longrightarrow$ (RECIPROCITY)
  - $E_P\left(\frac{1}{t},\frac{1}{q}\right) = (-1)^{d+1} q^{d} E_P(t,q)$

Even without CONJECTURES on $\mathcal{H}_P$, various classical Ehrhart theorems have q-analogues that are explained by $\mathcal{H}_P$.

4. Ehrhart tidbits ⇝ q-tidbits

THEOREM (Stanley 1986) For a poset Q on $\{1,2,\dots,n\}$

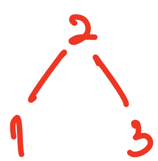
its order polytope $\mathcal{O}_Q := \{ \underline{x} \in [0,1]^n : x_i < x_j \text{ if } i <_P j \}$

chain polytope $\mathcal{C}_Q := \{ \underline{x} \in [0,1]^n : \sum_{i \in C} x_i \le 1 \ \forall \text{ chains } C \subset Q \}$

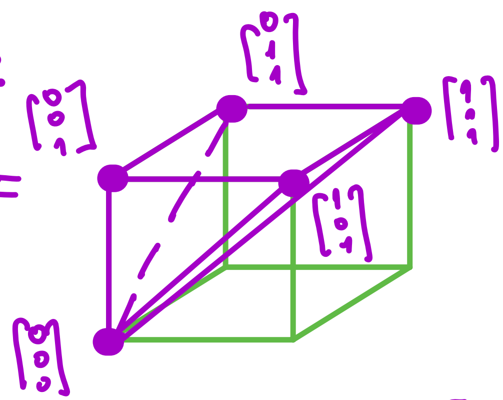
have the same Ehrhart function / series:

$$i_{\mathcal{O}_Q}(m) = i_{\mathcal{C}_Q}(m) \quad \text{or} \quad E_{\mathcal{O}_Q}(t) = E_{\mathcal{C}_Q}(t)$$

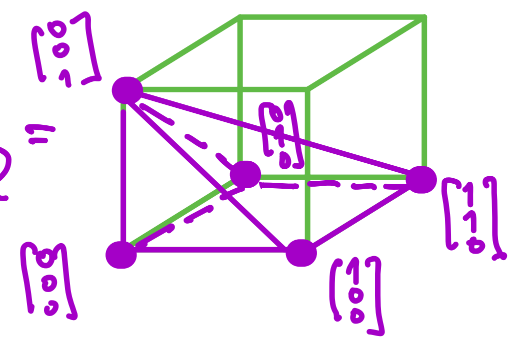
EXAMPLE

$Q =$ 

order polytope

$\mathcal{O}_Q =$ 

chain polytope

$\mathcal{C}_Q =$ 

$$E_{\mathcal{O}_Q}(t) = E_{\mathcal{C}_Q}(t) = \frac{1+t}{(1-t)^4}$$

THEOREM (Rhoades-R. 24) For a poset Q on $\{1,2,\ldots,n\}$

its order polytope $\mathcal{O}_Q$

and chain polytope $\mathcal{C}_Q$

have the same q-Ehrhart function / series

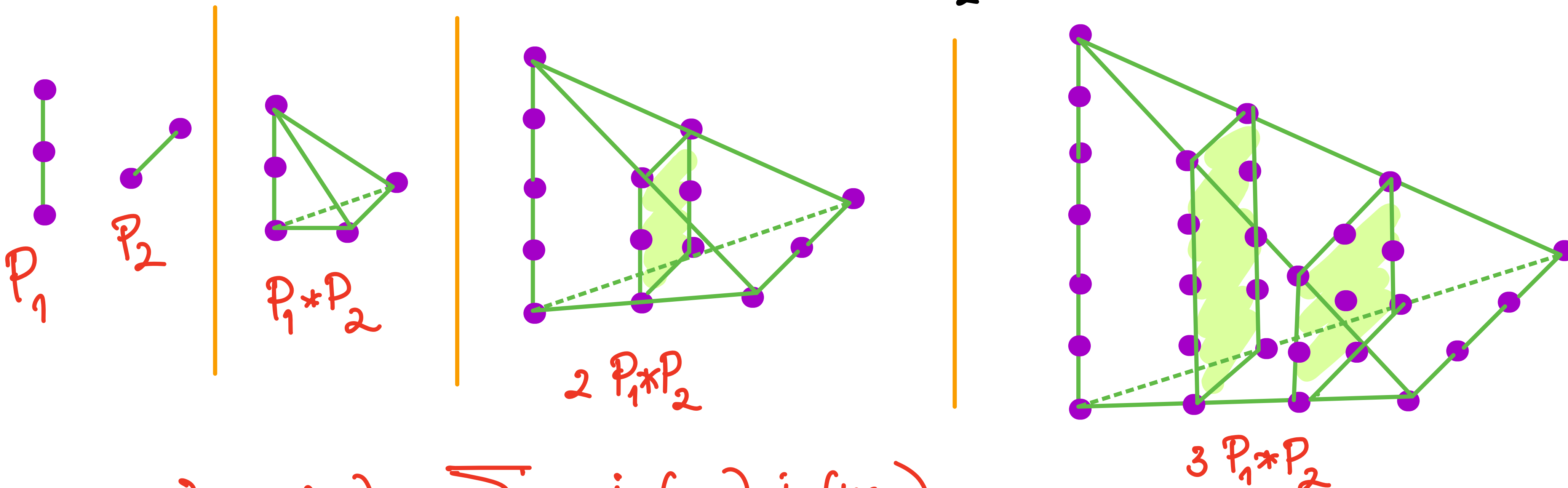
$$i_{\mathcal{O}_Q}(m,q) = i_{\mathcal{C}_Q}(m,q) \quad \text{or} \quad E_{\mathcal{O}_Q}(t,q) = E_{\mathcal{C}_Q}(t,q)$$

and, in fact, same harmonic algebras:

$$\mathcal{H}_{\mathcal{O}_Q} = \mathcal{H}_{\mathcal{C}_Q} \quad \text{inside } \mathbb{R}[y_0, y_1, \ldots, y_n].$$

PROPOSITION: For lattices polytopes $P_1, P_2 \subset \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ their (free) join $P_1 * P_2 :=$ convex hull of

$$(P_1 \times \{\underline{0}_{n_2}\} \times \{0\}) \sqcup (\{\underline{0}_{n_1}\} \times P_2 \times \{1\})$$



has $$i_{P_1 * P_2}(m) = \sum_{\substack{(m_1, m_2): \\ m_1 + m_2 = m}} i_{P_1}(m_1) \cdot i_{P_2}(m_2)$$

or equivalently $$\boxed{E_{P_1 * P_2}(t) = E_{P_1}(t)\, E_{P_2}(t)}$$

THEOREM (Rhoades - R. '24) For lattices polytopes P_1, P_2

$$E_{P_1 * P_2}(t,q) = \frac{1-t}{1-qt} E_{P_1}(t,q)\, E_{P_2}(t,q)$$

$$\left(\overset{q=1}{\rightsquigarrow} \boxed{E_{P_1 * P_2}(t) = E_{P_1}(t)\, E_{P_2}(t)} \right)$$

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proof idea:

$$(1-qt) \cdot E_{P_1 * P_2}(t,q) = (1-t) \cdot E_{P_1}(t,q) \cdot E_{P_2}(t,q)$$

follows from a ring isomorphism

$$\mathcal{H}_{P_1 * P_2} \Big/ \underbrace{(y_{P_1 * P_2} \cdot y_0)}_{\substack{\text{a nonzero-divisor} \\ \text{tracked by } qt}} \cong \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2} \Big/ \underbrace{(y_{P_1} \otimes 1 - 1 \otimes y_{P_2})}_{\substack{\text{a nonzero-divisor} \\ \text{tracked by } t}}$$

∎

Thanks

for your

attention !