Recent progress in the topology of simplicial complexes

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Outline

I. Extremal Betti numbers, algebraic shifting

II. Combinatorial Laplacians

Leitmotifs

- Alexander duality
- Shifted complexes

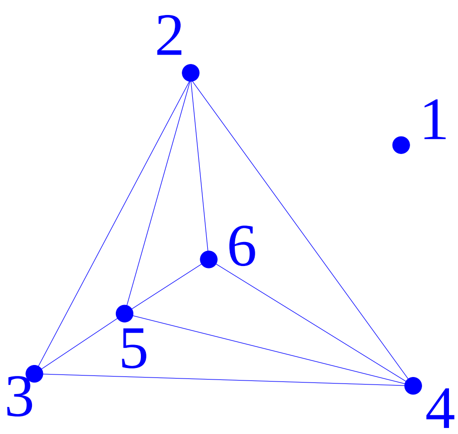
I. Extremal Betti numbers, algebraic shifting

An (abstract) simplicial complex Δ on vertex set $[n] := \{1, 2, \ldots, n\}$ is a collection of subsets $F \subset [n]$ which is closed under inclusion, i.e.

$$F \in \Delta \text{ and } F' \subset F \Rightarrow F' \in \Delta.$$

Example:

$$\begin{aligned}\Delta = \{&\emptyset,\\ &1, 2, 3, 4, 5, 6,\\ &23, 24, 25, 26, 34, 35, 45, 46, 56\\ &235, 246, 256, 345, 456\}\end{aligned}$$



What sorts of numerical (isomorphism) invariants have been associated with Δ, and have people tried to characterize?

Some are combinatorial, e.g.

- the dimension

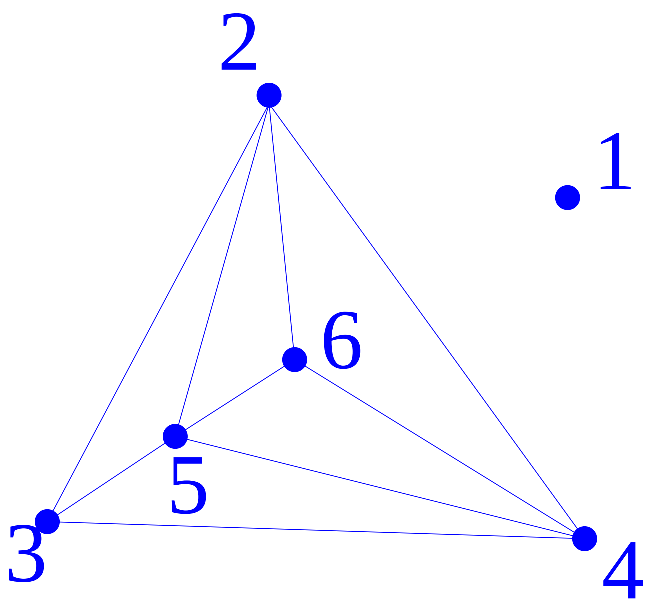
$$\mathrm{dim}(\Delta) := \max\{|F| : F \in \Delta\} - 1$$

- the f-vector

$$f(\Delta) := (f_{-1}, f_0, f_1, \ldots, f_{\mathrm{dim}(\Delta)})$$

 where f_i is the number of faces F in Δ of dimension i (i.e. $|F| = i + 1$).

E.g. Δ from before



has

- $\dim(\Delta) = 2$
- $f(\Delta) = (f_{-1}, f_0, f_1, f_2) = (1, 6, 9, 5).$

Some are topological
(homeomorphism, homotopy-type invariants), e.g.

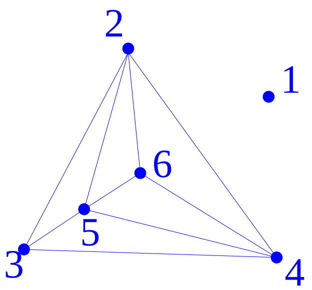
- the dimension (again)

- the (topological, reduced) Betti numbers or β-vector over some fixed field k:

$$\beta_k(\Delta) := (\beta_{-1}, \beta_0, \beta_1, \ldots, \beta_{\dim(\Delta)})$$

  where $\beta_i := \dim_k \tilde{H}_i(\Delta; k)$

E.g. Δ from before



has

$$\beta_k(\Delta) = (\beta_{-1}, \beta_0, \beta_1, \beta_2) = (0, 1, 0, 0).$$

Some are algebraic invariants associated with the Stanley-Reisner ring

$$k[\Delta] := A/I_\Delta$$
$$A := k[x_1, \ldots, x_n]$$
$$I_\Delta := (x_{i_1} \cdots x_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta)$$

as a graded k-algebra and graded A-module:

- the Krull dimension
  $(= \dim(\Delta) + 1$, again!)

- the Hilbert function, Hilbert series
  (equivalent to f-vector again)

- the depth $\mathrm{depth}_A(k[\Delta])$

- the homological dimension $\mathrm{hd}_A(k[\Delta])$

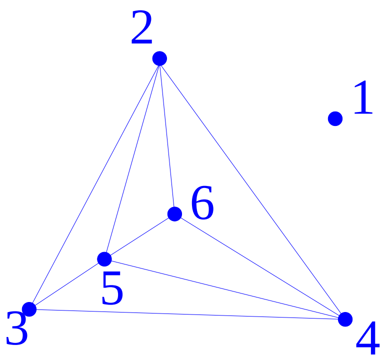
- the (Castelnuovo-Mumford) regularity $\mathrm{reg}_A(k[\Delta]$

and perhaps most importantly ...

the (algebraic) Betti numbers $\beta_{ij}(\Delta)$ from the (finite) minimal free resolution of $k[\Delta]$ (or I_Δ) as an A-module

$$
\begin{aligned}
0 \to \bigoplus_j A(-j)^{\beta_{\mathrm{hd}_A(k(\Delta)),j}} &\to \\
\cdots &\to \\
\bigoplus_j A(-j)^{\beta_{2j}} &\to \\
\bigoplus_j A(-j)^{\beta_{1j}} &\to \\
A^1 \to \underbrace{A/I_\Delta}_{=k[\Delta]} &\to 0
\end{aligned}
$$

which capture all of the previous numerical invariants...




```
i1 :  A = QQ[x1,x2,x3,x4,x5,x6];
i2 :  Idelta = ideal(
x1*x2, x1*x3, x1*x4,x1*x5, x1*x6,
x3*x6,
x2*x3*x4, x2*x3*x6 );

i3 :  MFR = resolution( Idelta );
```

o3 = $A^1 \leftarrow A^8 \leftarrow A^{15} \leftarrow A^{12} \leftarrow A^5 \leftarrow A^1$

```
i4 :  betti( MFR )
o4 =
```

total	:	1	8	15	12	5	1
0	:	1	.	.	.	.	.
1	:	.	6	11	10	5	1
2	:	.	2	4	2	.	.

Why do the $\beta_{ij}(\Delta)$ capture the rest?

$$\begin{aligned}\text{HilbertSeries}(k[\Delta], t) &:= \sum_{j \geq 0} \dim_k k[\Delta]_j t^j \\ &= \sum_{d \geq 0} \frac{f_{d-1} t^d}{(1-t)^d} \\ &= \sum_{i,j \geq 0} (-1)^j \frac{\beta_{ij} t^j}{(1-t)^n}\end{aligned}$$

$$\begin{aligned}\text{hd}_A(k[\Delta]) &:= \max\{i : \beta_{ij} \neq 0 \text{ for some } j\} \\ \text{reg}_A(k[\Delta]) &:= \max\{r : \beta_{i,i+r} \neq 0 \text{ for some } i\}\end{aligned}$$

$\text{depth}_A(k[\Delta]) = n - \text{hd}_A(k[\Delta])$
(via Auslander-Buchsbaum formula)

capturing
Hilbert series,
f-vector,
dimension (= order of pole).
$\text{hd}_A(k[\Delta])$,
$\text{reg}_A(k[\Delta])$,
$\text{depth}_A(k[\Delta])$

How do the $\beta_{ij}(\Delta)$ capture the topological Betti numbers, i.e. the β-vector?

The $\beta_{ij}(\Delta)$ are actually a mixture of topological/combinatorial invariants (in disguise):

THEOREM(Hochster 1977)

$$\beta_{ij}(\Delta) = \sum_{V \subset [n] : |V| = j} \dim_k \tilde{H}_{j-i-1}(\Delta|_V ; k).$$

In particular, setting $j = n$, we have

$$\underbrace{\beta_{i,n}}_{resolution} = \dim_k \tilde{H}_{n-i-1}(\Delta|_V ; k) = \underbrace{\beta_{n-i-1}}_{topological} .$$

Some selected characterizations of numerical invariants of simplicial complexes:

f-vectors:
Schützenberger, Kruskal, Katona, Harper, Lindström 1959

f-vectors when depth $=$ dim $+1$
(i.e. Δ Cohen-Macaulay):
Stanley 1981

(f, β) pairs: Bjorner & Kalai 1985

(f, β, depth) triples: Bjorner 1996

(Likely too hard a ...)
PROBLEM: Characterize all possible resolution Betti numbers $\beta_{ij}(\Delta)$ for simplicial complexes Δ.

Bayer-Charalambous-Popescu 1999: Some of the β_{ij} are more important than others.

Say (i, j) is extremal for Δ if $\beta_{ij} \neq 0$, but $\beta_{i'j'} = 0$ whenever

- $i' \geq i$, and
- $j' - i' \geq j - i$, and
- $(i', j') \neq (i, j)$.

In other words, (i, j) is extremal if β_{ij} is a southeast corner of the non-vanishing entries in the Macaulay diagram.

total	:	1	8	15	12	5	1
0	:	1	.	.	.	.	.
1	:	.	6	11	10	5	1
2	:	.	2	4	2	.	.

Extremal Betti numbers capture all previous invariants except f-vector/Hilbert series:

- ◁ The (non-zero) topological Betti numbers all show up among the extremal Betti numbers, lying on an antidiagonal in the Macaulay diagram (since $\beta_{n-i-1} = \beta_{i,n}$).

- ◁ Location of the rightmost extremal Betti number in the Macaulay diagram determines homological dimension (and hence also depth).

- ◁ Location of the bottommost extremal Betti number in the Macaulay diagram controls regularity.

(A more reasonable ...)

PROBLEM: Characterize all possible "pairs"

(f-vector, locations (i, j) and values $\beta_{i,j}$ of extremal Betti numbers)

for simplicial complexes Δ.

CONCERN:

How do we know extremal Betti numbers are an important/natural invariant?

Important?

BCP showed that for homogeneous ideals I in A (like I_Δ), the extremal Betti numbers are unchanged when one replaces I by its

$$\text{generic initial ideal } \mathrm{Gin}_{grevlex}(I)$$

Using this and polarization, characterizing pairs

(*f*-vector, extremal Betti number data)
for simplicial complexes

would characterize the possible pairs

(Hilbert series, extremal Betti number data)
for homogeneous ideals in A.

NB: BCP note that extremal Betti data for homogeneous ideals can be arbitrary (just as Betti numbers of simplicial complexes can be arbitary).

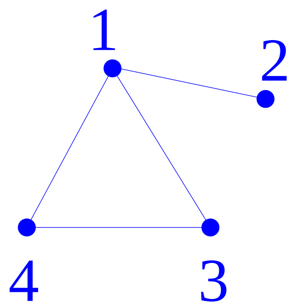
Natural?

They interact beautifully with two important constructions:

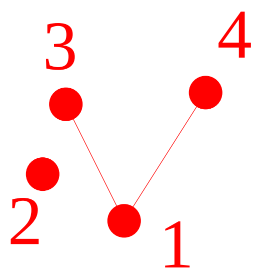
1. The canonical Alexander dual $\Delta^{\vee}$

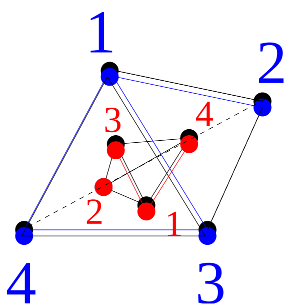
$$\Delta^{\vee} := \{F \subset [n] : [n] - F \notin \Delta\}$$

Δ



$\Delta^{\vee}$





THEOREM (Alexander duality)

For any field k,

$$\beta_i(\Delta^{\vee}) = \beta_{n-3-i}(\Delta).$$

THEOREM(Eagon-R. 1998):
$I_{\Delta^\vee}$ has a linear resolution
$\Leftrightarrow k[\Delta]$ is Cohen-Macaulay.

$\Leftarrow$ THEOREM(Terai 1998):
$\text{reg}_A(I_{\Delta^\vee}) = \text{hd}_A(S/I_\Delta)$.

$\Leftarrow$ THEOREM(BCP 1999):
• (i, j) is extremal for $\Delta^\vee$ if and only if $(j - i - 1, j)$ is extremal for Δ.
• The corresponding extremal Betti numbers are **equal**:

$$\beta_{i,j}(\Delta^\vee) = \beta_{j-i-1,j}(\Delta) \text{ if extremal.}$$

(i.e. the extremal parts of the Macaulay diagrams for $\Delta^\vee, \Delta$ are "flips" of each other).

$\Leftarrow$ THEOREM(E. Miller 2000):
A slight generalization of this.

Δ from before had Macaulay diagram

total	:	1	8	15	12	5	1
0	:	1	.	.	.	.	.
1	:	.	6	11	10	5	1
2	:	.	2	4	2	.	.

$\Delta^\vee$ has Macaulay diagram

total	:	1	6	7	2
0	:	1	.	.	.
1	:	.	.	.	.
2	:	.	5	6	2
3	:	.	.	.	.
4	:	.	1	1	.

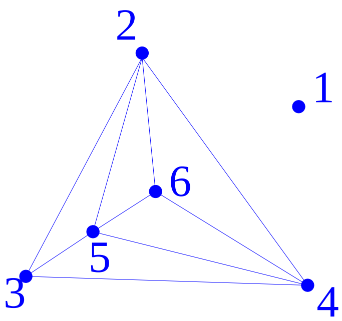
(cf. the statement of Alexander duality)

2. The algebraic shift Δ^s

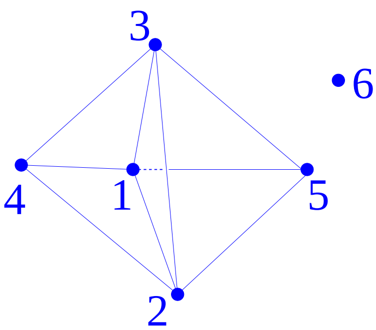
$$E := \text{ exterior algebra on } e_1, \ldots, e_n$$
$$J_\Delta := (e_{i_1} \wedge \cdots \wedge e_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta)$$
$$k\{\Delta\} := E/J_\Delta = \text{exterior face ring}$$

$$J_{\Delta^s} := \text{Gin}(J_\Delta) \text{ with respect to } grevlex$$

e.g. Δ has facets $\{1, 235, 246, 256, 345, 356\}$
$\rightsquigarrow$ Δ^s has facets $\{6, 123, 124, 125, 134, 135\}$:



Δ



Δ^s

Δ^s enjoys the property of being shifted:
if $i < j \in F \in \Delta^s$, and $i \notin F$,
then $F - \{i\} \cup \{j\} \in \Delta^s$.

THEOREM(Björner and Kalai 1985):
Δ and Δ^s have same f-vector and β-vector over k.
(proof: not hard)

THEOREM(Kalai 1993):
Δ is Cohen-Macaulay if and only if Δ^s is.
(proof: hard)

THEOREM(Duval and Rose 1995):
$\text{depth}_A k[\Delta] = \text{depth}_A k[\Delta^s]$.
(proof: apply previous result to skeleta)

THEOREM(Herzog and Terai 1999):
$\text{reg}_A k[\Delta] = \text{reg}_A k[\Delta^s]$.
(proof: note that algebraic shifting and Alexander duality commute, i.e. $(\Delta^s)^\vee = (\Delta^\vee)^s$)

And more recently,

THEOREM(Aramova and Herzog 2000):
Δ, Δ^s have same extremal Betti numbers!
(proof: Haven't absorbed it, but brings in the data associated with MFR of $k\{\Delta\} := E/J_\Delta$ as E-module)

Thanks to this, one really only needs to solve...

PROBLEM: Characterize all possible pairs

(f-vector, extremal Betti number data)

for shifted simplicial complexes Δ.

II. Combinatorial Laplacians

The set-up:

$$C_{i+1}(\Delta;\mathbb{R}) \underset{\partial_{i+1}^T}{\overset{\partial_{i+1}}{\rightleftharpoons}} C_i(\Delta;\mathbb{R}) \underset{\partial_i^T}{\overset{\partial_i}{\rightleftharpoons}} C_{i-1}(\Delta;\mathbb{R})$$

Define the (combinatorial) Laplacian

$$L_i := \underbrace{\partial_i^T \partial_i}_{L_i^{down}} + \underbrace{\partial_{i+1}\partial_{i+1}^T}_{L_i^{up}} : C_i(\Delta;\mathbb{R}) \to C_i(\Delta;\mathbb{R})$$

PROPOSITION(Hodge 1941, Eckmann 1945):

$$H_i(\Delta;\mathbb{R}) \cong \ker L_i \quad (= \text{harmonic } i\text{-chains})$$

So the 0-eigenspace of L is giving us homology. What about the rest of the spectrum of the self-adjoint positive-semidefinite operator L?

Is this spectral data attached to a simplicial complex Δ worth studying?

In particular, how does it interact with our old friend, the canonical Alexander dual?

PROPOSITION(Duval and R. 2000):
The eigenvalues of $L_i(\Delta)$ for any simplicial complex Δ on n vertices lie in the range $[0, n]$, and

$$L_i(\Delta), L_{n-i-3}(\Delta^\vee)$$

have the same spectra except for the multiplicity of the top eigenvalue n.

Disturbing that spectra of L_i are (nonnegative) real numbers, not integers?

There are some interesting families discovered recently where they are all integers:

Two cases where symmetry/representation theory play a role:

- Chessboard complexes (Friedman and Hanlon 1998)
- Matching complexes (Dong and Wachs 2001)

Two cases with no symmetry, but with mysterious connections...

- Matroid complexes (Kook, R. and Stanton 2000)
- Shifted complexes (Duval and R. 2000)

The shifted case

K a collection of k-element subsets of $[n]$.
Δ_K:= simplicial complex with facets K.
$d(K)$:= degree sequence of the vertex set $[n]$ with respect to the set K of facets, a partition (parts in weakly decreasing order).

e.g.

$$k = 3$$
$$n = 4$$
$$K = \{123, 124, 234\}$$
has
$$\Delta_K = \{\varnothing, 1, 2, 3, 4,$$
$$12, 13, 14, 23, 24, 34,$$
$$123, 124, 234\}$$

$$d(K) = (3,2,2,2) = \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array}, \quad d(K)^t = (4,4,1)$$

THEOREM

(Duval-R. 2000, Merris 1994 for $k = 2$):

For any shifted family K of k-element sets,

$$\text{(non-zero) spectrum of } L_i^{down}(\Delta_K) = \text{the transpose partition } d(K)^t.$$

CONJECTURE:

For any family K of k-element sets,

$$\text{(non-zero) spectrum of } L_i^{down}(\Delta_K) \text{ is majorized by the transpose partition } d(K)^t$$

with equality if and only if K is shifted.

The matroid case

A matroid complex Δ_M
$:=$ independent sets of a matroid M,

Kook, R. and Stanton gave a formula for spectra of L_i, and asked whether it has a nice reformulation relating the spectra of L_i for

$$M, \quad M/e, \quad M - e.$$

THEOREM(Kook 2000)

$$\begin{aligned}\{\text{spectra for } \Delta_M\} = & \{\text{spectra for } \Delta_{M-e}\} \cup \\ & \{\text{spectra for } \Delta_{M/e}\} \cup \\ & \{\text{an explicit error term}\}.\end{aligned}$$

Note that if $\Delta := \Delta_M$, then

$$\Delta_{M-e} = \mathrm{del}_\Delta(e)$$
$$\Delta_{M/e} = \mathrm{link}_\Delta(e)$$

THEOREM(Duval 2001)
In this matroid setting, Kook's error term is the spectrum of the Laplacian for the relative simplicial pair

$$(\mathrm{del}_\Delta(e), \quad \mathrm{link}_\Delta(e))$$

AND the **same recursion** holds for shifted complexes (but **not** for all simplicial compelexes).

QUESTION: What's the deal with matroid and shifted complexes/families?

▷ Both shellable, even vertex-decomposable.
▷ Both Laplacian integral, satisfying same recursion.
▷ When $k = 2$, as families, both are determined up to isomorphism by a partition.

- For rank 2 matroids, it's their parallelism partition

- For shifted families of 2-subsets (graphs), it's their degree sequence

▷ Complicated and rich already for $k = 3$.

Matroids are better-studied, e.g.
- have more known invariants
- have good intrinsic characterizations.

What about characterizing shifted families intrinsically?

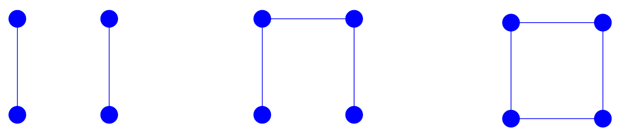
THEOREM
(Duval and Shareshian, work in progress)
For families of k-subsets, the obstructions to being isomorphic to a shifted family all have exactly $2k$ vertices, in the following precise sense:

▷ Any non-shifted k-family with **more** than $2k$ vertices has a nonshifted deletion, and

▷ any non-shifted k-family with **less** than 2k vertices has a nonshifted contraction/link.

For example, when $k = 2$, a 2-family is just a graph, and graphs isomorphic to shifted families are called threshold graphs. They have several nice intrinsic characterizations.

There are three vertex-induced subgraphs well-known to be the obstructions to being threshold, and all have $4 = 2 \cdot k$ vertices:



When $k = 3$, there are nine obstructions (up to certain symmetry operations).