

Math 4242 – More problems on the SVD

1. Let $A = U\Sigma V^T$ be the SVD of matrix A . Let σ_i be the i -th (nonzero) singular value and let U_i, V_i be the i -th columns of U and V . Show that these satisfy the following equations:

$$A^T A V_i = \sigma_i^2 V_i \quad \sigma_i U_i = A V_i.$$

2. The matrix two-norm $\|A\|_2$ is the same thing as the largest singular value σ_1 . The squares of the σ_i are precisely the numbers λ such that the matrix $A^T A - \lambda I$ is singular (compare problem 1). For each of the following matrices, A , find all the values of λ with this property and compute the singular values and, in particular, the matrix norm $\sigma_1 = \|A\|_2 = \sqrt{\lambda_{\max}}$. Hint: A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if and only if $ad - bc = 0$. Part d involves a 4×4 matrix – try Gauss elimination or a four-dimensional determinant to find the λ 's.

a. $\begin{bmatrix} 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ e. $\begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$ f. $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

3. For each matrices A in problem 2 find SVD matrices U, V, Σ such that $A = U\Sigma V^T$. Here U, V should be orthogonal matrices of size $m \times m$ and $n \times n$ respectively and Σ is the $m \times n$ matrix containing the singular values. You can use the equations from problem 1 to find singular vectors associated to the nonzero singular values. The other columns of U, V are bases for the null spaces of A^T, A . Hint: you already found a URV^T factorization for some of these matrices in a previous homework assignment.

4. Let A be the matrix below. What is the rank of A ? Find the best rank-1 approximation to A . According to the theory, this is the matrix $\sigma_1 u_1 v_1^T$ where σ_1 is the first singular value and u_1, v_1 are the first columns of U, V from the SVD. Hint: the first singular value is $\sigma_1 = \sqrt{10}$. Use the results of problem 1 to find U_1, V_1 .

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$