Math 8502 — Homework I

due Friday, February 22. Write up any 4 of these 5 problems.

1. Consider a scalar, autonomous ODE $\dot{x} = f(x)$, $x \in \mathbf{R}^1$, where $f(x)$ is a polynomial of degree at least 2. Show that there is at least one maximal solution $x(t)$ which is not defined for all $t \in \mathbf{R}$.

2. Let $\phi_t(x), \psi_t(y)$ be two flows on $\mathbf{R}^n$. They are called *linearly conjugate* if there is an invertible linear map $y = Qx$ such that

$$Q\phi_t(x) = \psi_t(Qx) \quad \text{for all } t \in \mathbf{R}, x \in \mathbf{R}^n.$$

They are *topologically conjugate* if there is a homeomorphism $y = h(x)$, $h : \mathbf{R}^n \to \mathbf{R}^n$, such that

$$h(\phi_t(x)) = \psi_t(h(x)) \quad \text{for all } t \in \mathbf{R}, x \in \mathbf{R}^n.$$

a. Let A, B be two $n \times n$ real matrices. The corresponding linear flows are given by $\phi_t(x) = e^{tA}x, \psi_t(y) = e^{tB}y$. Show that they are linearly conjugate if and only if the two matrices A, B are similar.

b. Show that the linear flows determined by the matrices below are topologically conjugate but not *linearly conjugate*. Here a, b are any two positive numbers not both equal to 1. Hint: Try a map, h, of the form $y = (y_1, y_2) = (\operatorname{sgn}(x_1)|x_1|^\alpha, \operatorname{sgn}(x_2)|x_2|^\beta)$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}.$$

c. Same as part b. but for the matrices below, where $a > 0, b \neq 0$. Hint: It is possible to find an explicit formula for $h(x)$. One approach uses the fact that he distance to the origin $r(t)$ is decreasing for both flows. Let $t_1(x)$ be the time when $\phi_t(x)$ crosses the unit circle (find a formula for it) and consider $h(x) = e^{-t_1(x)B}e^{t_1(x)A}x$.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}.$$

d. Show that the linear flows determined by the matrices below are not topologically conjugate.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Let $\phi_t(x)$ be a flow on phase space $\mathcal{D}$. Suppose $\phi_t(x_0)$ exists for all $t \geq 0$. Define the *omega limit set* to be the set of limit points of the forward orbit:

$$\omega(x_0) = \{y \in \mathcal{D} : \exists t_n \to \infty, \phi_{t_n}(x_0) \to y\}.$$

Suppose that there is a compact subset $K \subset \mathcal{D}$ such that $\phi_t(x_0) \in K$ for all $t \geq 0$. Show that $\omega(x_0)$ is a non-empty, compact subset of K. Also show that $\omega(x_0)$ is an invariant set and that orbits in $\omega(x_0)$ exist for all $t \in \mathbf{R}$, i.e., show that if $y \in \omega(x_0)$ then for all $t \in \mathbf{R}$, $\phi_t(y)$ exists and $\phi_t(y) \in \omega(x_0)$.

4. The Lorenz Equation. Consider the following ODE in $\mathbf{R}^3$:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

where $\sigma > 0, b > 0, r > 0$ are parameters.

a. Find all the equilibrium points. For which values of the parameters are they non-degenerate? For which values of the parameters are they hyperbolic and what are the dimensions of the stable and unstable manifolds?

b. Show that the z-axis is an invariant set which is contained in the stable manifold of the origin: $W^s(0)$.

c. Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map $L(x, y, z) = (-x, -y, z)$. Geometrically, L is rotation around the z-axis by 180 degrees. Show that L is a symmetry of the flow of the Lorenz equation, i.e., if $(x(t), y(t), z(t))$ is a solution, so is $L(x(t), y(t), z(t))$. Show that L leaves the stable and unstable manifolds $W^s(0)$ and $W^u(0)$ invariant.

d. Show that if $r < 1$ then the Lorenz flow is gradient-like with respect to the function $g(x, y, z) = \frac{1}{2}(x^2/\sigma + y^2 + z^2)$, i.e., this function is strictly decreasing except at the restpoints. Use this to show that, in this case, $W^s(0) = \mathbf{R}^3$, i.e., every solution converges to 0.

5. (Linearized Hamiltonian Systems) Let $q \in \mathbf{R}^n$ and $p \in \mathbf{R}^n$ and let $z = (q, p) \in \mathbf{R}^{2n}$. Consider a *Hamiltonian system of ODEs*:

$$\begin{aligned}\dot{q} &= H_p\\ \dot{p} &= -H_q\end{aligned}$$

or

$$\dot{z} = J\nabla H \qquad\qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

a. A $2n \times 2n$ matrix, A, is called *Hamiltonian* if JA is symmetric. If $\bar{z}$ is an equilbrium point of a Hamiltonian system, show that the linearized ODE is of the form $\dot{v} = Av$ where A is a Hamiltonian matrix.

b. B is called *symplectic* if $B^T JB = J$ where B^T is the transpose of B. Show that A is Hamiltonian if and only if $B = e^{tA}$ is symplectic for all t. Hint: differentiate the expression $(e^{tA})^T J e^{tA}$.

c. If A Hamiltonian matrix, show that the characteristic polynomial $p(\lambda) = |A - \lambda I|$ is an even function, i.e., $p(-\lambda) = p(\lambda)$. If B is symplectic show that $\lambda^{2n} p(1/\lambda) = p(\lambda)$. Hint: Start by multiplying $|A - \lambda I|$ on the left by $|J| = 1$.