Math 8502 — Homework II

due Friday, March 28. Write up any 4 of these 5 problems.

1. The Lagrange points of the planar, circular, restricted three-body problem were shown to be critical points of the function

$$V(x,y) = \tfrac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_{13}} + \frac{\mu}{r_{23}}$$

where $r_{13}^2 = (x+\mu)^2+y^2, r_{23}^2 = (x+\mu-1)^2+y^2$. Let the collinear Lagrange points be denoted $L_i = (x_i, 0)$, $i = 1,2,3$, where $x_1 < -\mu < x_2 < 1-\mu < x_3$.

The second partial derivatives of the potential, V, played a crucial role in analyzing the dynamics. Show that $V_{xx}(x,0) > 0$ and $V_{xy}(x,0) = 0$ for all x and that $V_{yy}(x_i,0) < 0$ at the Lagrange points.

2. Suppose $\phi_t(x)$ is a flow on $\mathbf{R}^n$ and x_0 is a periodic point of minimal period $T > 0$, i.e., $\phi_T(x_0) = x_0$ but $\phi_t(x_0) \neq x_0, 0 < t < T$.

a. Show that $\phi_{t+T}(x_0) = \phi_t(x_0)$ for all $t \in \mathbf{R}$. Show that every point $x_1 = \phi_{t_1}(x_0)$ on the orbit of x_0 also has minimal period T.

b. Let $x_1 = \phi_{t_1}(x_0)$. Show that the monodromy matrices $D\phi_T(x_0)$ and $D\phi_T(x_1)$ are similar.

c. Let Σ_0 be a Poincaré section through x_0. Let $x_1 = \phi_{t_1}(x_0)$ and Σ_1 a Poincaré section through x_1. Then for $i = 0,1$, there are neighborhoods $\mathcal{U}_i$ of x_i, smooth return-time functions $\tau_i : \mathcal{U}_i \to \mathbf{R}$, $\tau_i(x_i) = T$ and Poincaré maps $\psi_i : \mathcal{U}_i \cap \Sigma_i \to \Sigma_i$ where $\psi_i(x) = \phi_{\tau_i(x)}(x)$. Show that these two Poincaré maps are locally conjugate, i.e., there are neighborhoods $\mathcal{V}_i$ and a homeomorphism $h : \mathcal{V}_0 \cap \Sigma_0 \to \mathcal{V}_1 \cap \Sigma_1$ such that $h \circ \psi_0 = \psi_1 \circ h$. Hint: use the flow to define h.

3. Here is an example which shows that the hypothesis about integer multiples of eigenvalues in the Lyapunov center theorem is necessary. Consider the system of ODEs:

$$\begin{aligned} \dot{x}_1 &= y_1 + x_1x_2 - y_1y_2 & \qquad \dot{x}_2 &= -2y_2 + \tfrac{1}{2}(x_1^2 - y_1^2) \\ \dot{y}_1 &= -x_1 - y_1x_2 - x_1y_2 & \qquad \dot{y}_2 &= 2x_2 - x_1y_2 \end{aligned}$$

a. Verify that the system is Hamiltonian with

$$H(x_1,x_2,y_1,y_2) = \tfrac{1}{2}(x_1^2+y_1^2) - (x_2^2+y_2^2) + x_1y_1x_2 + \tfrac{1}{2}(x_1^2-y_1^2)y_2.$$

Show that the origin is a nondegenerate critical point of H with eigenvalues $\pm i, \pm 2i$. Thus the linearized flow has periodic orbits with frequencies $\omega_1 = 1$ and $\omega_2 = 2$.

b. Show that the two-dimensional plane $x_1 = y_1 = 0$ is filled with periodic orbits of the nonlinear system with frequency ω_2. (This is the family guaranteed by the Lyapunov center theorem applied to the imaginary eigenvalues $\pm 2i$)

c. On the other hand, you will now show that there are no other periodic solutions. Note that outside the plane in part b, the squared radius $\alpha = x_1^2 + y_1^2 > 0$. Show that $\ddot{\alpha} > 0$ for any such solution (this means that $\alpha(t)$ is a strictly convex function of time and so cannot be periodic). Hint: The computation is a bit messy. One way to organize it is to begin by showing

$$\dot{\alpha} = 2x_2\gamma - 2y_2\beta \qquad \dot{\beta} = -2\gamma - 2y_2\alpha \qquad \dot{\gamma} = 2\beta + 2x_2\alpha$$

where $\alpha = x_1^2 + y_1^2, \beta = 2x_1y_1, \gamma = x_1^2 - y_1^2$.

4. Suppose that the ODE $\dot{z} = f(z)$ admits k independent first integrals $H_i(z), i = 1\ldots,k$ (this means that the derivatives $DH_i(z_0)$ are linearly independent). Let z_0 be a periodic point of minimal period $T > 0$. Show that the monodromy matrix $D\phi_T(z_0)$ has $\mu = 1$ as an eigenvalue of multiplicity at least $k+1$. Let the other $n-k-1$ eigenvalues be denoted $\mu_1, \ldots, \mu_{n-k-1}$.

Suppose the periodic orbit lies on the level set $\mathcal{M}(h_1,\ldots,h_k)$ where $H_i(z) = h_i, i = 1,\ldots,k$. ($\mathcal{M}(h_1,\ldots,h_k)$ is a submanifold of $\mathbf{R}^n$ of dimension $n-k$, at least locally near z_0.) If Σ is a Poincaré section through z_0 we can restrict the Poincaré map to get a map $\psi : \Sigma \cap \mathcal{M}(h_1,\ldots,h_k) \to \Sigma \cap \mathcal{M}(h_1,\ldots,h_k)$. Show that the eigenvalues of $D\psi(x_0)$ are $\mu_1,\ldots,\mu_{n-k-1}$.

5. Hill's problem is a simplified version of the restriced three-body problem given by the following system of differential equations

$$\begin{aligned} \dot{x} &= u & \dot{u} &= 2v + V_x \\ \dot{y} &= v & \dot{v} &= -2u + V_y \end{aligned}$$

where $V(x,y) = \frac{3}{2}x^2 + \frac{1}{\sqrt{x^2+y^2}}$.

a. Show that this is a Lagrangian system. Do a Legendre transform to find the corresponding Hamiltonian system. Express the energy integral in terms of (x,y,u,v).

b. Find the equilibrium points and their eigenvalues. Show that there are families of periodic orbits nearby and use the linearized ODE to find how they move in the (x,y)-plane.

c. Consider the related system:

$$\begin{aligned} \dot{X} &= U & \dot{U} &= 2\epsilon^3 V + W_X \\ \dot{Y} &= V & \dot{V} &= -2\epsilon^3 U + W_Y \end{aligned}$$

where $W(X,Y) = \frac{3}{2}\epsilon^6 X^2 + \frac{1}{\sqrt{X^2+Y^2}}$ and ϵ is a small parameter. When $\epsilon = 0$ this system describes a two-body problem. Show that the simple, circular periodic solutions of this problem can be continued for $\epsilon \neq 0$ sufficiently small.

If $(X(t),Y(t))$ is a periodic solution of this system for $\epsilon \neq 0$ (such as those just obtained by continuation), show that

$$(x(t),y(t)) = \epsilon^2(X(t/\epsilon^3), Y(t/\epsilon^3))$$

is a periodic solution of Hill's equation.