Math 8502 — Homework III

due Friday, May 2

1. (Shift Maps). Let $l \geq 2$ and define Σ_l to be the set of all bi-infinite sequences on l symbols, i.e., sequences $\epsilon = \ldots \epsilon_{-1}\underline{\epsilon_0}\epsilon_1 \ldots$ with $\epsilon_k = 0, 1, \ldots, l-1$. Let $\sigma : \Sigma_l \to \Sigma_l$ be the shift map, $\sigma(\epsilon)_k = \epsilon_{k+1}$. Finally, define the distance between two sequences to be

$$d(\epsilon, \epsilon') = \sum_{k=-\infty}^{\infty} \frac{|\epsilon_k - \epsilon'_k|}{l^{|k|}}.$$

a. Show that Σ_l is a compact metric space. Hint: for compactness it suffices to show that every sequence in Σ_l has a convergent subsequence.

b. Show that the shift map is a homeomorphism.

c. We showed in class that when $l = 2$ there is a sequences whose forward orbit under σ is dense in Σ_2. Let $D \subset \Sigma_l$ be the set of all sequences with dense forward orbits. Show that D is a *residual set*, i.e., it is an intersection of countably many sets each of which is open and dense in Σ_l. (Since Σ_l is a "Baire space", it follows that D is dense in Σ_l; see a book on topology for more about Baire category). Hint: Consider the set of all sequences ϵ containing a given finite sequence as a subsequence of their forward halves.

d. Let $\Lambda \subset \mathbf{R}^2$ be the invariant set of orbits of the Smale horseshoe map which remain in the unit square. We constructed a 1-1 correspondence $h : \Sigma_2 \to \Lambda$ by using itineraries with respect to the horizontal boxes H_0, H_1. Show that h is a homeomorphism.

2. (Subshifts of finite type). Let G be any directed graph with vertices labelled $0, 1, \ldots, l-1$. In other words, G consists of the vertices together with arrows connecting some pairs of vertices. Associated with G is a *transition matrix* M such that $M_{ij} = 1$ if there is a directed edge $j \to i$ and $M_{ij} = 0$ otherwise. Also, there is an associated subset $\Sigma' \subset \Sigma_l$ consisting of all sequences $\epsilon \in \Sigma_l$ such that for every k, the vertices with labels ϵ_k and ϵ_{k+1} are connected by a directed edge in G (thus G is a graphical representation of some rules about which symbols may be adjacent in ϵ).

a. Show that Σ' is a compact subset of Σ_l which is invariant under the shift map.

b. Consider the powers M^n of the transition matrix M. Prove by induction that M^n_{ij} is the number of distinct paths of length n connecting vertex j to vertex i. Let $T(n)$ be the number of periodic sequences in Σ' of (not necessarily minimal) period n. Show that $T(n) = \operatorname{trace} M^n$.

c. Consider the subshift $\Sigma' \subset \Sigma_2$ with transition matrix $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Prove that the associated subshift is chaotic, i.e., periodic points are dense, sensitive dependence on initial conditions and existence of a dense orbit. Finally, find an explicit formula for the number $T(n)$ of periodic points of period n.

3. Apply the Melnikov integral method to show that the periodically forced Duffing equation (written below) has a periodic orbit with a transverse homoclinic orbit for ϵ sufficiently small. The unperturbed problem with $\epsilon = 0$ is Hamiltonian. You should sketch the phase portrait, find the unperturbed homoclinic orbit and then compute the Melnikov integral. (You may assume that the theory applies in this case even though the perturbation does not vanish along the unperturbed periodic orbit).

$$\ddot{q} - q + q^3 = \epsilon \sin \omega t.$$