Blowing Up the N-body Problem I

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Goal: Describe a qualitative, geometrical approach to understanding the Newtonian N-body problem with emphasis on existence proofs for interesting phenomena in low-dimensional cases where we can draw pictures.

Results:

- Qualitative approach to some classical results
- Existence proofs for interesting collision orbits and periodic orbits
- Construction of chaotic invariant sets
- Properties of parabolic motions near infinity

Methods:

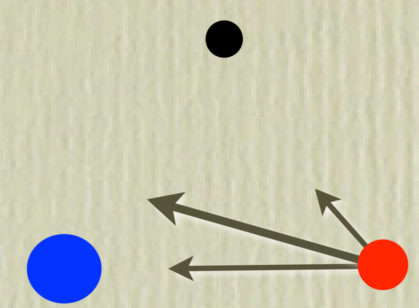
- McGehee blow-up of total collision
- Stable and unstable manifolds and their intersections
- Poincaré maps
- Symbolic dynamics

N body Problem in R^{d}

Masses: $m_i > 0$

Positions: $q = (q_1, \ldots, q_n) \in \mathbb{R}^{nd}$

Velocities: $v = (v_1, \ldots, v_n) \in \mathbb{R}^{nd}$



Lagrangian: $L = K + U$

$$K = \frac{1}{2}\sum_i m_i |v_i|^2 \qquad U = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \qquad r_{ij} = |q_i - q_j|$$

Euler-Lagrange equations:

$$\begin{aligned} \dot{q} &= v \\ \dot{v} &= M^{-1}\nabla U(q) \end{aligned} \qquad M = \mathrm{diag}(m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots)$$

Center of Mass:

$$\begin{aligned} m_1 q_1 + \ldots + m_n q_n &= 0 \\ m_1 v_1 + \ldots + m_n v_n &= 0 \end{aligned} \qquad \longrightarrow \qquad \text{Subspace } X \subset \mathbb{R}^{nd}$$

Phase Space: $T(X - \Delta) = (X - \Delta) \times X$ of dimension $2(n-1)d$ where

$$\Delta = \text{Collision Set}\{q : q_i = q_j \text{ for some } i \neq j\}$$

Energy, Symmetry, Angular Momentum

Energy: $$H(q,v) = \frac{1}{2}K(v) - U(q)$$

Rotational Symmetry:

$$R \in \mathbb{SO}(d) \qquad Rq = (Rq_1, \ldots, Rq_n) \qquad Rv = (Rv_1, \ldots, Rv_n)$$

Angular momentum bivector: $\lambda = \sum_i m_i q_i \wedge v_i$

λ has $\frac{d(d-1)}{2}$ components:

$$\lambda_{kl} = \sum_i m_i (q_{ik} v_{il} - q_{il} v_{ik}) \qquad 1 \leq k < l \leq d$$

Integral Manifolds:

$$\mathcal{M}(h,\lambda) = \{(q,v) : H(q,v) = h, \lambda(q,v) = \lambda\} \subset T(X - \Delta)$$

$$\tilde{\mathcal{M}}(h,\lambda) = \mathcal{M}(h,\lambda)/SO(d)$$

$$\mathcal{M}(h) = \{(q,v) : H(q,v) = h\}$$

$$\tilde{\mathcal{M}}(h) = \mathcal{M}(h)/SO(d)$$

McGehee Blowup of Total Collision

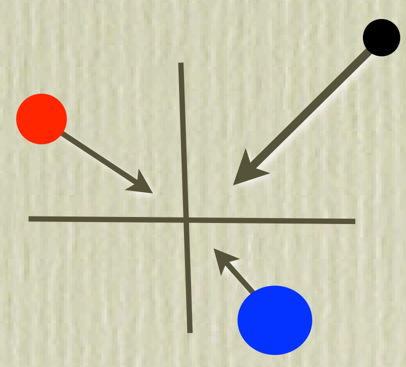
Mass metric and norm on R^{nd}:

$$\langle\langle v, w\rangle\rangle = V^T M W = \sum_i m_i v \cdot w$$

$$\|v\|^2 = \langle\langle v, v\rangle\rangle$$

Then $K(v) = \frac{1}{2}\|v\|^2$ and $\|q\|^2 =$ moment of inertia around the origin

Total collision at the origin.: $r = \|q\| = 0$



New coordinates: (r, s, z)

$$r = \|q\|$$

$$s = \frac{q}{r} = \text{ normalized configuration} \qquad \|s\| = 1$$

$$z = \sqrt{r}v$$

New timescale: $\dfrac{dt}{d\tau} = r^{\frac{3}{2}} \qquad {}' = r^{\frac{3}{2}}\cdot$

$$r' = vr$$

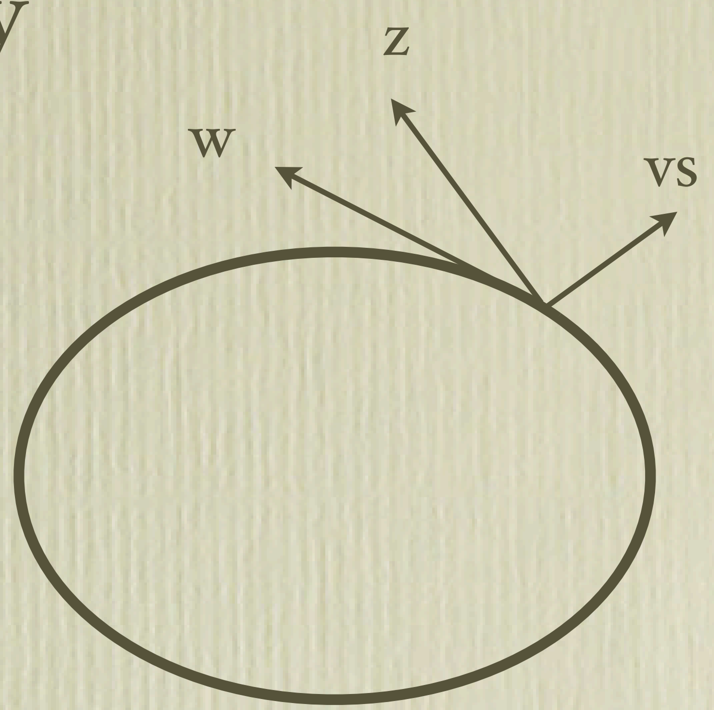
$$* \qquad s' = z - vs$$

$$z' = M^{-1}\nabla U(s) + \frac{1}{2}vz$$

New $v = \langle\langle s, z\rangle\rangle$

- r=0 invariant Total Collision Manifold
- s, z equations independent of r

Radial and tangential velocity



The normalized configuration vector s lies in an ellipsoid

$$\mathcal{E} = \{s \in X : \|s\| = 1\}$$

The rescaled velocity can also be split as

$$z = vs + w \qquad \langle\langle s, w \rangle\rangle = 0$$

New ODEs:

$$r' = vr$$

** $$v' = \frac{1}{2}v^2 + \|w\|^2 - U(s)$$

$$s' = w$$

$$z' = M^{-1}\nabla U(s) + U(s)s - \frac{1}{2}vw - \|w\|^2 s = \tilde{\nabla}U(s) - \frac{1}{2}vw - \|w\|^2 s$$

Tangential gradient:

$$\tilde{\nabla}U(s) = M^{-1}\nabla U(s) + U(s)s$$

$$\|s\| = 1 \qquad \langle\langle s, w \rangle\rangle = 0$$

$$\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - U(s) = rh$$

$$\lambda(q, v) = \sqrt{r}\,\lambda(s, w)$$

Lyapunov Function

The radial velocity variable v plays a role analogous to the classical Lagrange-Jacobi identity:

$$\frac{1}{2}\ddot{I} = U(q) + 2h$$

where $I = r^2$ is the moment of inertia. Since $r' = vr$, v' is related to this second derivative. Using the energy equation one finds:

$$v' = \frac{1}{2}v^2 + \|w\|^2 - U(s) = \frac{1}{2}\|w\|^2 + rh$$

If $r = 0$ or $h = 0$, then $v'(\tau) \geq 0$. For the flows on the total collision manifold and the zero energy manifold the function v is a nondecreasing Lyapunov function.

Using this one can give nice proofs of some classical results about total collision orbits and orbits tending parabolically to infinity.

- Sundman
- Chazy
- Siegel

Total Collision Orbits

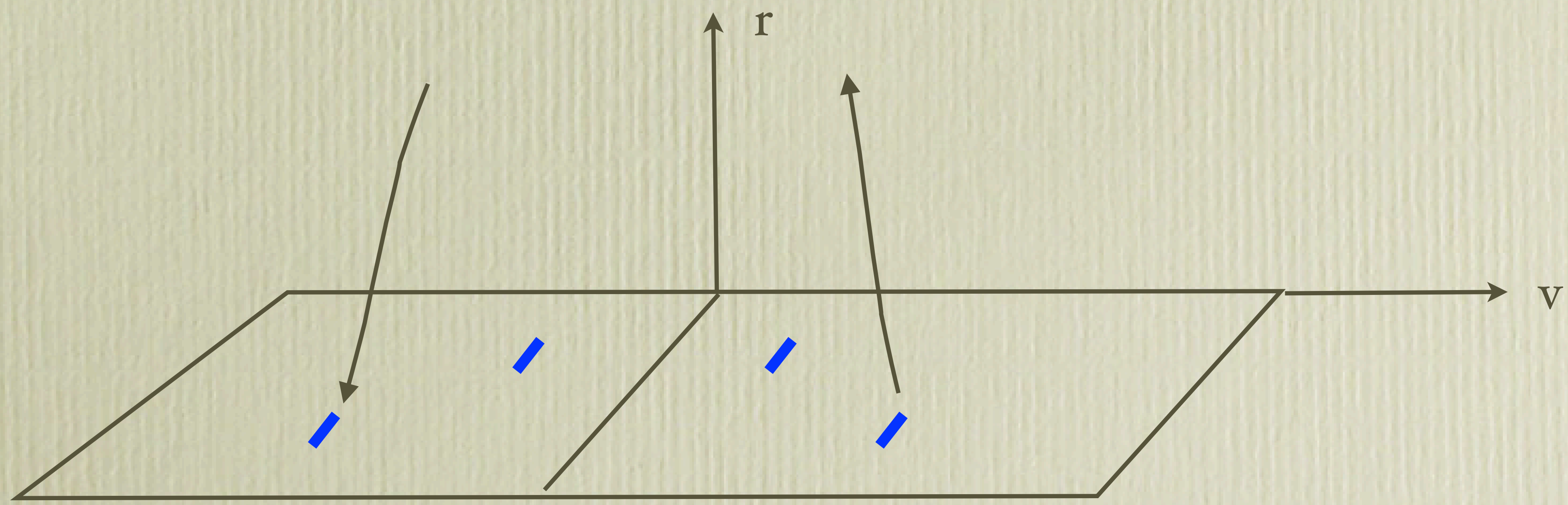
Consider a solution $q(t)$ defined for $t \in [a, t_0)$ such that $r(t) \to 0$ as $t \to t_0$ and let

$$\gamma(\tau) = (r(\tau), v(\tau), s(\tau), z(\tau))$$

be the corresponding solution in McGehee coordinates. Then one can show:

- $\gamma(\tau)$ exists for $\tau \in [0, \infty)$ and $r(\tau) \to 0$ as $\tau \to \infty$
- $v(\tau) \to -v_0 < 0$ and so $r(\tau) \to 0$ exponentially fast
- $\omega(\gamma)$ is a nonempty compact subset of the set of restpoints in $\{r = 0\}$

angular momentum is $\lambda(q, v) = 0$

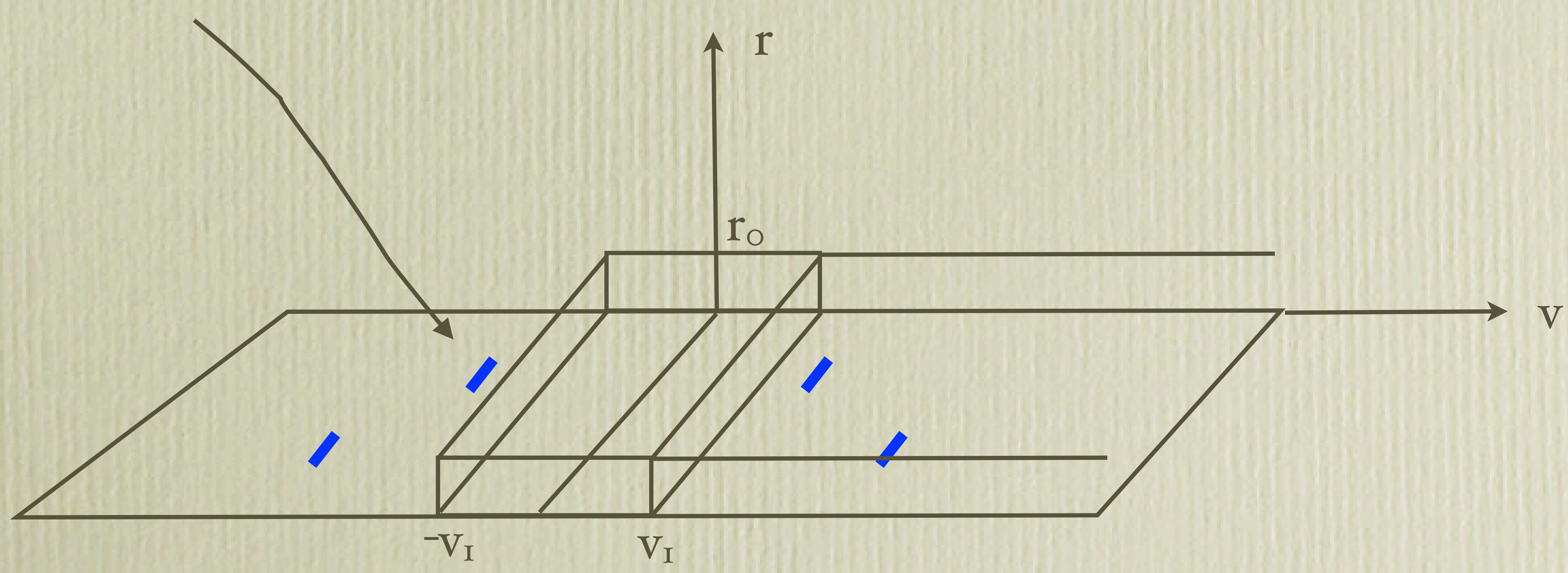


Proof sketch: Using the formula for v' one can find positive constants r_0, v_1, c such that $v' \geq c$ when $r \leq r_0$ and $-v_1 \leq v \leq v_1$. Since $r(\tau)$ eventually has $r(\tau) \leq r_0$ it follows that $v(\tau) \leq -v_1$ holds. Otherwise we would eventually have $v(\tau) \geq v_1$ and $r' \geq v_1 r$ so $r(\tau) \nrightarrow 0$.

Once we have $v(\tau) \leq -v_1 < 0$ we get exponential decay of $r(\tau)$ which together with the fact that $v' \geq 0$ on the collision manifold, forces $v(\tau)$ to converge to a limit $-v_0 \leq -v_1$. This gives a lower bound for $v(\tau)$ which implies it must take infinite τ-time for $r(\tau)$ to reach 0.

Next, near the singular set, $v(\tau)$ increases rapidly and in order to have a limit for $v(\tau)$ there must eventually be an upper bound $U(s(\tau)) \leq U_0$. Energy gives an upper bound $|w| \leq w_0$. These estimates confine $\gamma(\tau)$ to a compact set so we get a nonempty, compact limit set with $r = 0, v = -v_0$. The only solutions with $v(\tau)$ constant are equilibrium points.

Finally, the angular momentum is $\lambda(q, v) = \sqrt{r}\lambda(s, w)$ and the estimates above show $\lambda(q, v) \to 0$. Since $\lambda(q, v)$ is constant it must be 0.



Restpoints and Central Configurations

The restpoints of the ODE ** satisfy $r = 0, w = 0, v^2 = 2U(s)$ and

$$\tilde{\nabla} U(s) = 0 \qquad \text{(CC)}$$

The last equation states that s should be a normalized central configuration.

Normalized CCs are characterized as critical points of $U|_{\mathcal{E}}$. If s_c is such a critical point, there are two associated restpoints on the collision manifold

$$(r, v, s, w) = (0, \pm\sqrt{2U(s_c)}, s_c, 0)$$

The points with $v < 0$ are possible limits of forward time collision orbits while the points with $v > 0$ are possible limits of backward time collision orbits.

Classical Asymptotics

In the classical theory of Sundman and Chazy it is shown that

$$r(t) \simeq (t - t_0)^{\frac{2}{3}} \qquad \frac{q(t)}{(t - t_0)^{\frac{2}{3}}} \to \{\text{normalized CCs}\}$$

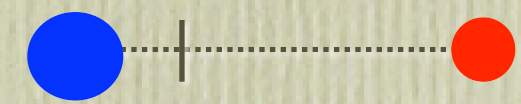
Here we have $r(\tau) \simeq e^{-v_0 \tau}$ $\quad \frac{dt}{d\tau} = r(\tau)^{\frac{3}{2}} \simeq e^{-\frac{3}{2} v_0 \tau}$ $\quad \frac{q(\tau)}{r(\tau)} \to \{\text{normalized CCs}\}$

To relate these, note that by L'Hospital's rule and the equation $r' = vr$

$$\lim_{\tau \to \infty} \frac{r(\tau)^{\frac{3}{2}}}{(t(\tau) - t_0)} = \lim_{\tau \to \infty} \frac{\frac{3}{2} r(\tau)^{\frac{1}{2}} r'(\tau)}{r(\tau)^{\frac{3}{2}}} == \lim_{\tau \to \infty} \frac{3}{2} v(\tau) = -\frac{3}{2} v_0$$

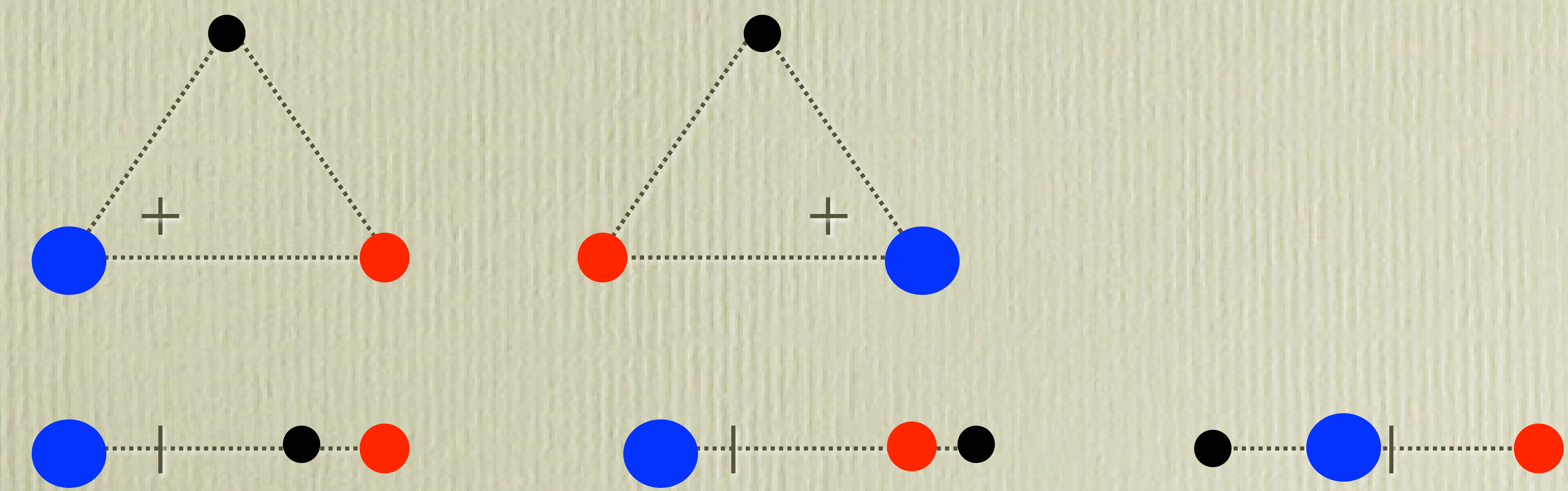
N=2

Up to symmetry, there is only one possible configuration -- a line segment. It is a central configuration.



N=3

There are always five central configurations— two equilateral triangles and three collinear shapes (one for each ordering).

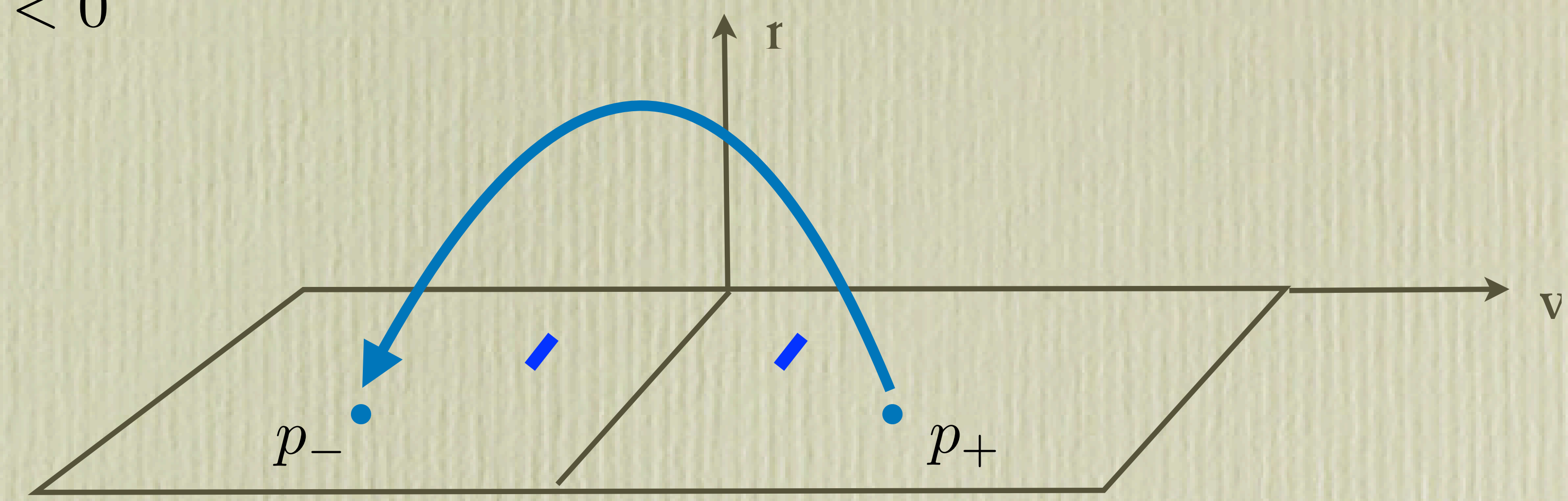


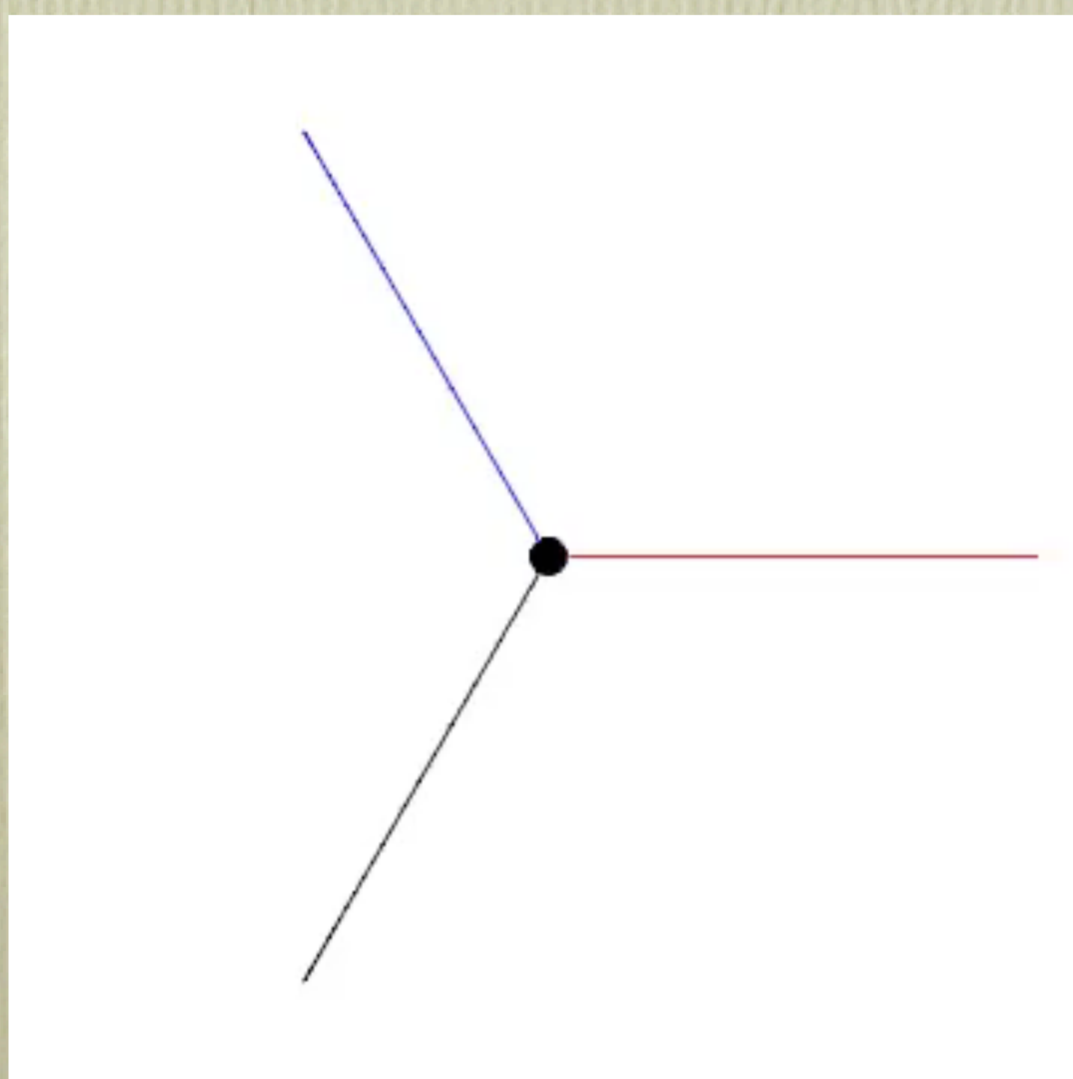
N > 3

Many interesting examples and lots of open problems

Homothetic Orbits

Every CC s_0 determines a homothetic orbit which provides a restpoint connection $p_+ \to p_-$ between the corresponding restpoints where p_+, p_- is restpoint with $v > 0, v < 0$





There is a homothetic connecting orbit on each energy level with h<0. These will play an important role later.

Later: existence of many more solutions which begin and end at collision.

Parabolic Orbits

A solution $q(t)$ is parabolic if it exists for $t \in [0, \infty)$ and tends to infinity with zero asymptotic velocity vector:

$$r(t) \to \infty \qquad \dot{q}(t) \to 0 \qquad\qquad t \to \infty$$

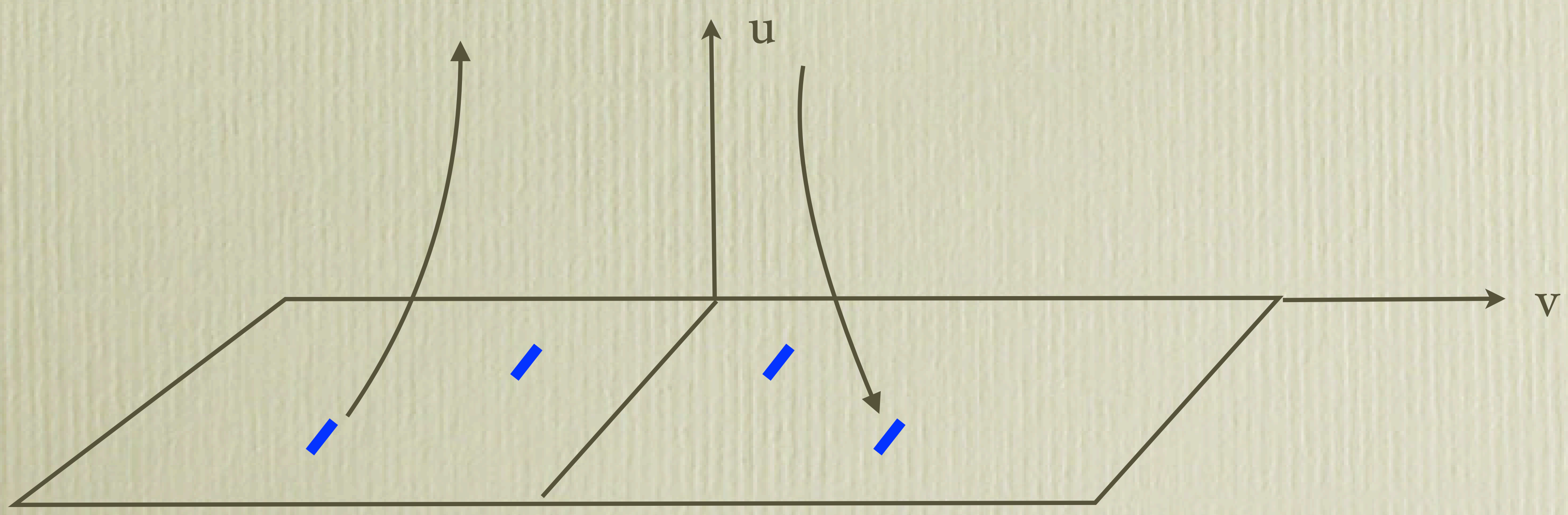
Parabolic solution have energy $h = 0$. Let $u = 1/r$. Then

$$u' = -vu$$

while the ODEs for the other McGehee variables are unchanged. The energy equation is

$$u(\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - U(s)) = h = 0$$

The behavior of parabolic solutions is similar to that of total collision solutions except that the roles of the restpoints with $v > 0$ and $v < 0$ are reversed. But still we have $v' = \frac{1}{2}\|w\|^2 \geq 0$ on the whole energy manifold $h = 0$.

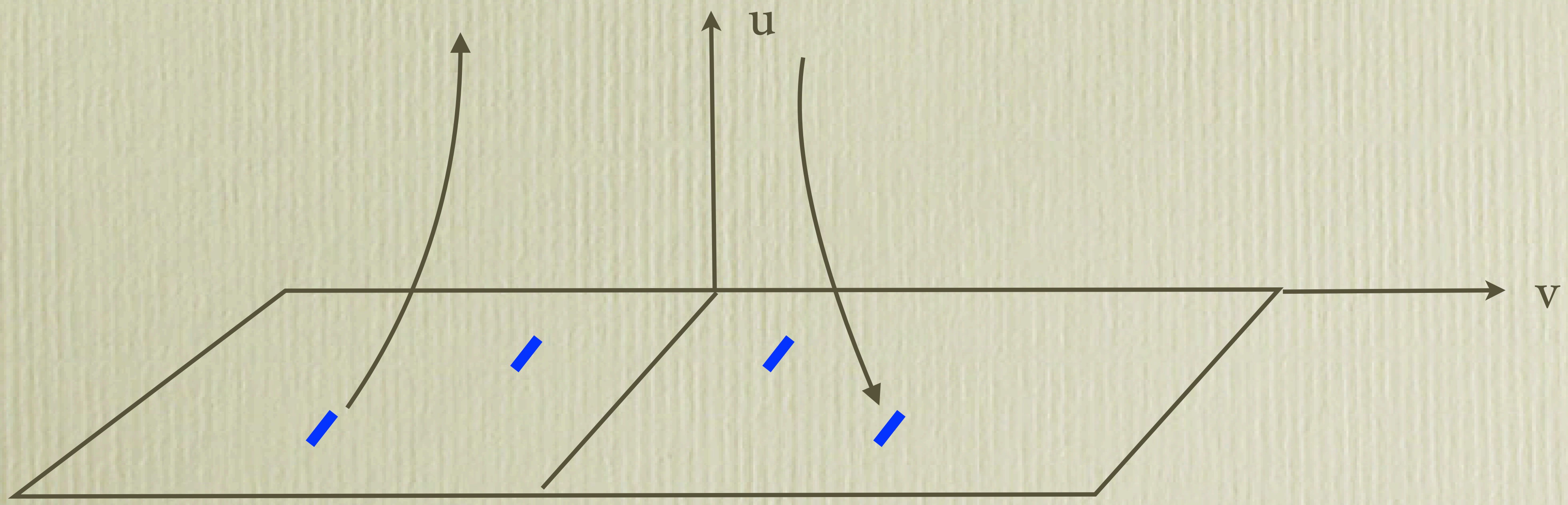


More precisely, if $q(t)$ is a forward time parabolic orbit and

$$\gamma(\tau) = (u(\tau), v(\tau), s(\tau), z(\tau))$$

is the corresponding solution in McGehee coordinates, then

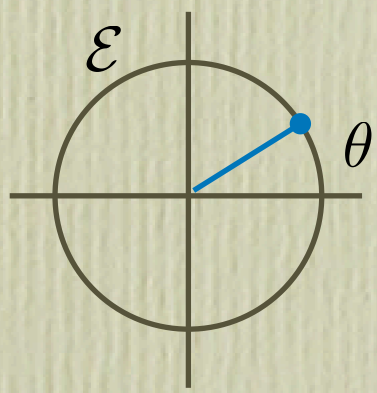
- $\gamma(\tau)$ exists for $\tau \in [0, \infty)$ and $u(\tau) \to 0$ as $\tau \to \infty$
- $v(\tau) \to v_0 > 0$ so $u(\tau) \to 0$ exponentially fast
- $\omega(\gamma)$ is a nonempty compact subset of the set of equilibrium points in $\{u = 0\}$ the energy $h = 0$



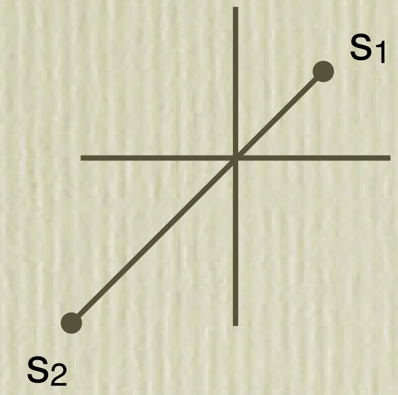
Low Dimensional Examples — 2BP

To take a break from all the formulas and get some intuition about the flows on the triple collision and parabolic infinity manifolds, we will consider an example where the flow on the usual phase space is well understood, the planar two-body problem.

Planar 2BP (N=2, d=2). This leads to some nice pictures and already contains some of the features which will be important later. Replace $q = (q_1, q_2)$ by $r = |q|$ and a normalized configuration $s = (s_1, s_2)$. Using $s = s_2 - s_1 \in \mathbb{R}^2$ to parametrize the zero center of mass subspace we have



$$\|s\|^2 = \frac{m_1 m_2}{m_1 + m_2}|s|^2 = \mu|s|^2$$



The ellipsoid $\mathcal{E}$ is the circle $|s| = 1/\sqrt{\mu}$ and the potential $U_{\mathcal{E}} = U_0 = \text{const}$. Replace the variable s by an angle θ and the tangential velocity w by $\omega = \theta'$. Then

$$\begin{aligned} r' &= vr \\ v' &= \frac{1}{2}v^2 + \frac{1}{2}\omega^2 - U_0 \\ \theta' &= w \\ \omega' &= -\frac{1}{2}v\omega \end{aligned}$$

$$\frac{1}{2}v^2 + \frac{1}{2}\omega^2 - U_0 = rh$$

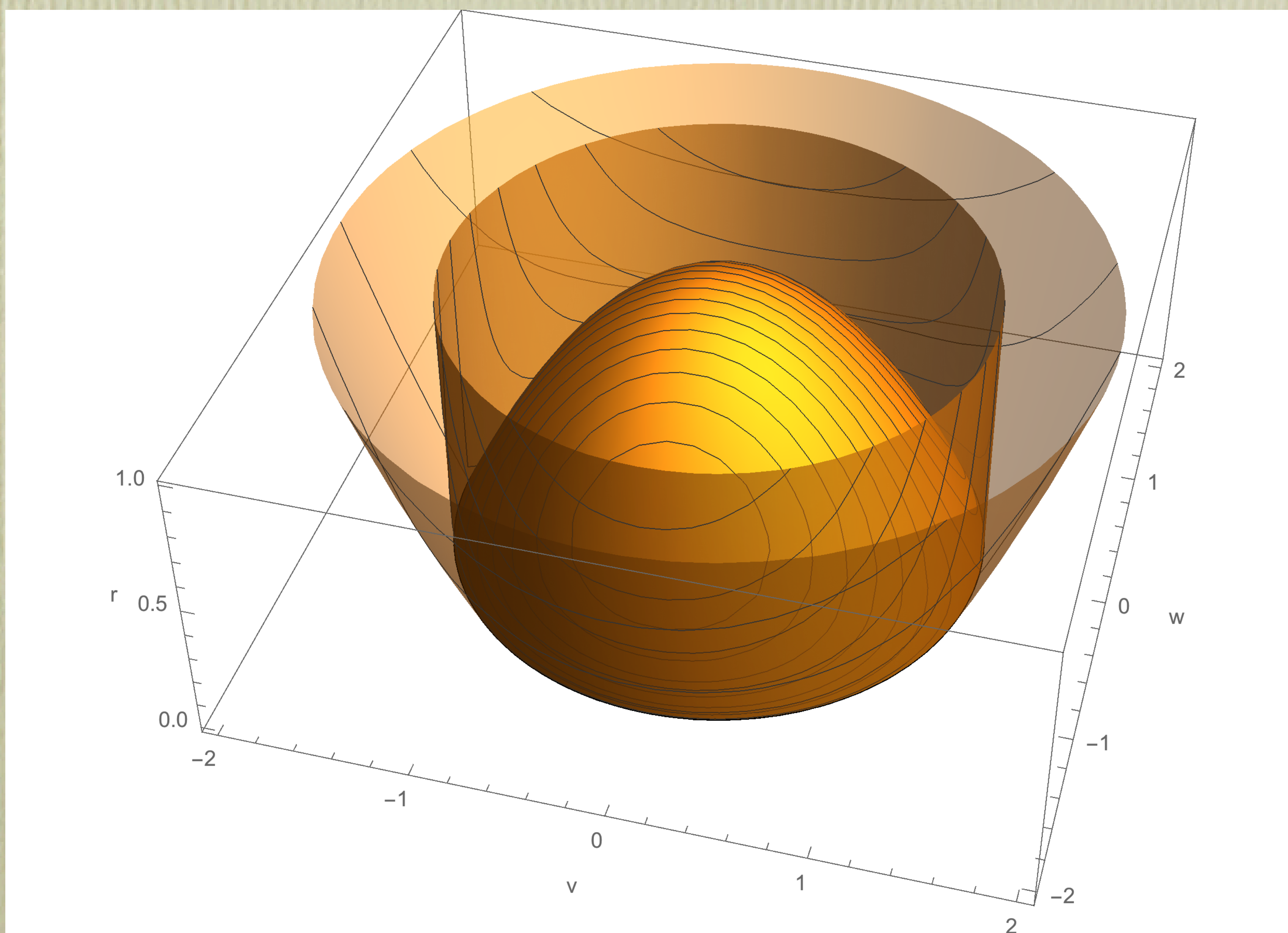
$$\lambda = \sqrt{r}\omega$$

Still 4D — not simple enough …..

Some 3D Pictures

First ignore θ or equivalently, consider the quotient space under the $\mathbb{SO}(2)$ action. The energy manifolds $\mathcal{M}(h)$ are paraboloids which are foliated by curves of constant angular momentum $\mathcal{M}(h,\lambda)$.

They all intersect the triple collision manifold along the circle $\frac{1}{2}v^2+\frac{1}{2}\omega^2 = U_0$

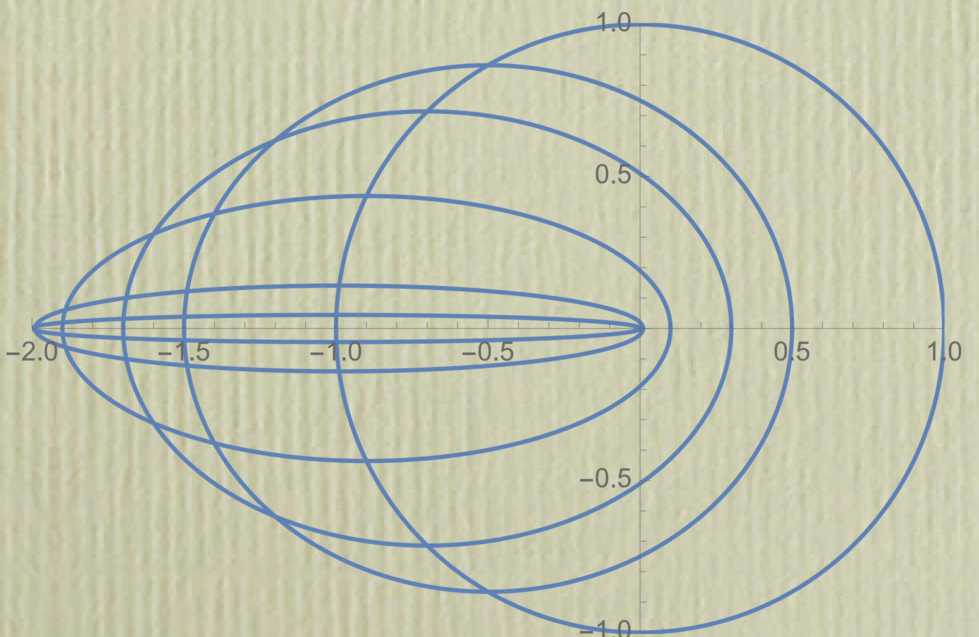


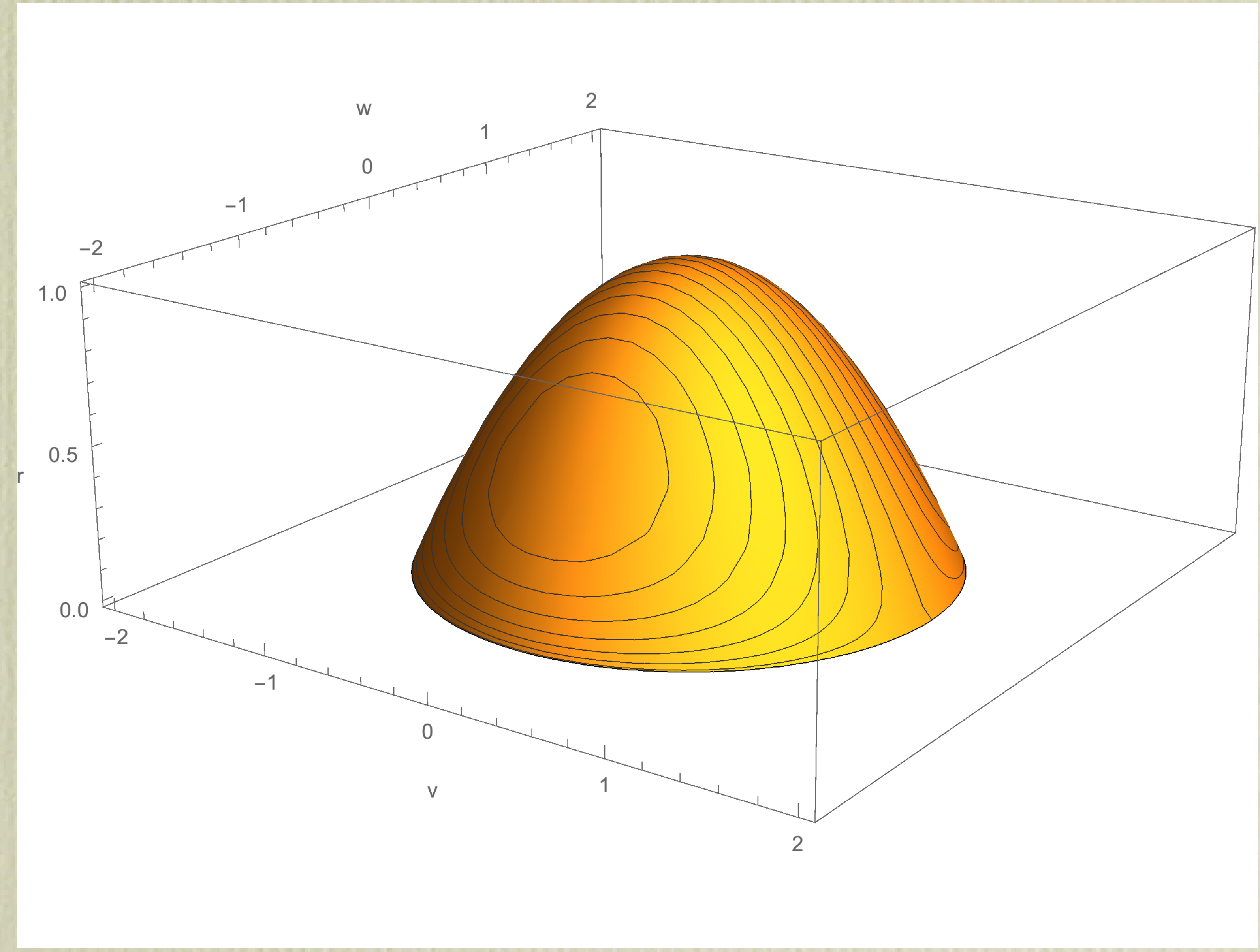
$$\frac{1}{2}v^2 + \frac{1}{2}\omega^2 - U_0 = rh$$

$$\lambda = \sqrt{r}\omega$$

Negative Energy, h<0

For $h < 0$ we have the elliptical orbits, both clockwise and counterclockwise, as well as a collinear, homothetic collision orbit. The collision orbit connects two restpoints at $r = 0$. It emerges from the one with $v_0 > 0$ and approaches the one with $v_0 < 0$.





The Homothetic Orbit and the Flow on the Collision Manifold

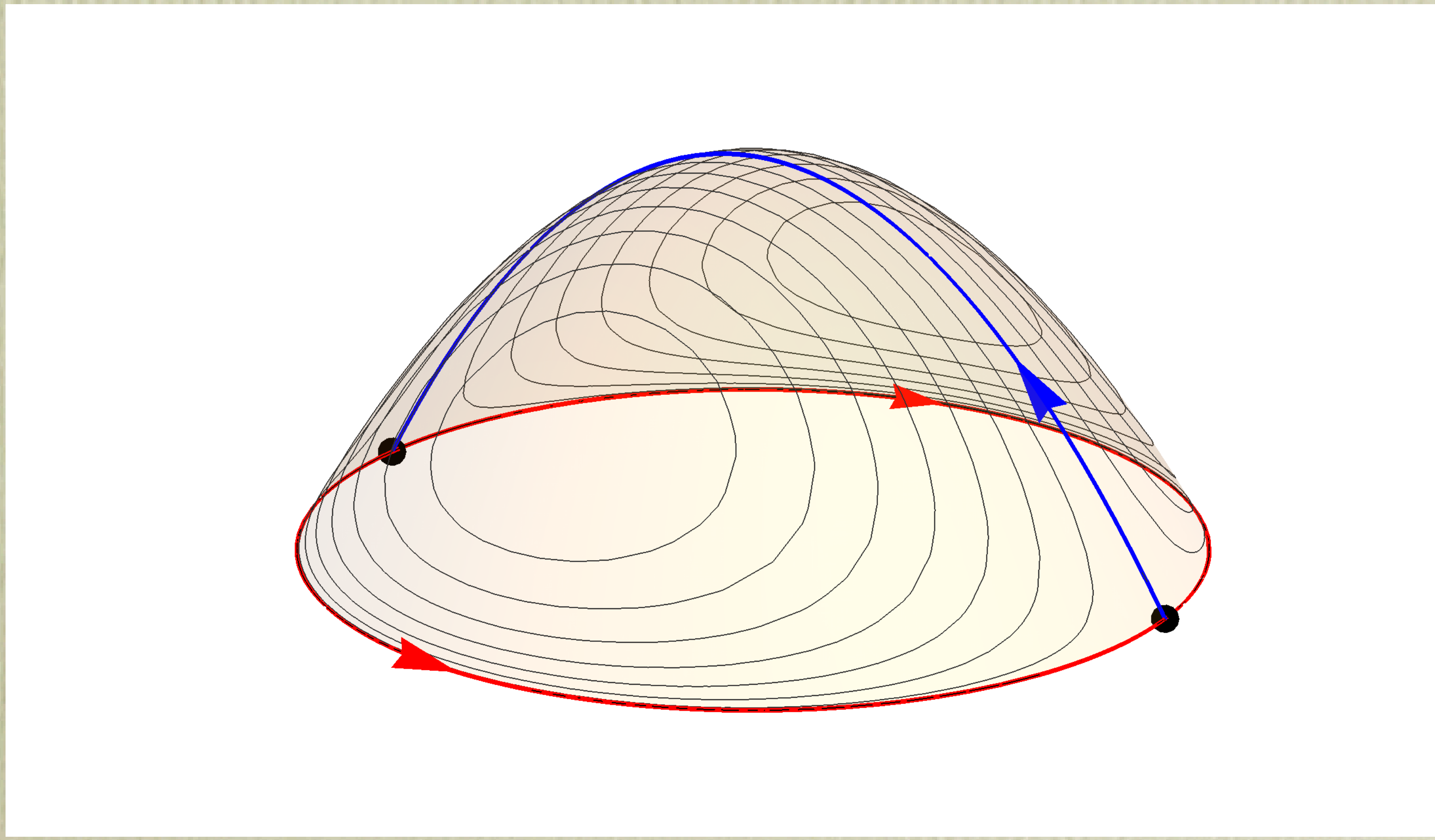
$$r = 0 \qquad \Rightarrow \qquad \frac{1}{2}v^2 + \frac{1}{2}\omega^2 = U_0$$

$$\lambda = \sqrt{r}\omega = 0 \qquad \Rightarrow \omega = 0 \text{ or } r = 0$$

$$\frac{1}{2}v^2 + \frac{1}{2}\omega^2 - U_0 = rh$$

$$\lambda = \sqrt{r}\omega$$

Collision Manifold Flow: Two red curves connecting the restpoints $p_- \to p_+$
Zero angular momentum flow: Blue curve connecting $p+ \to p_-$

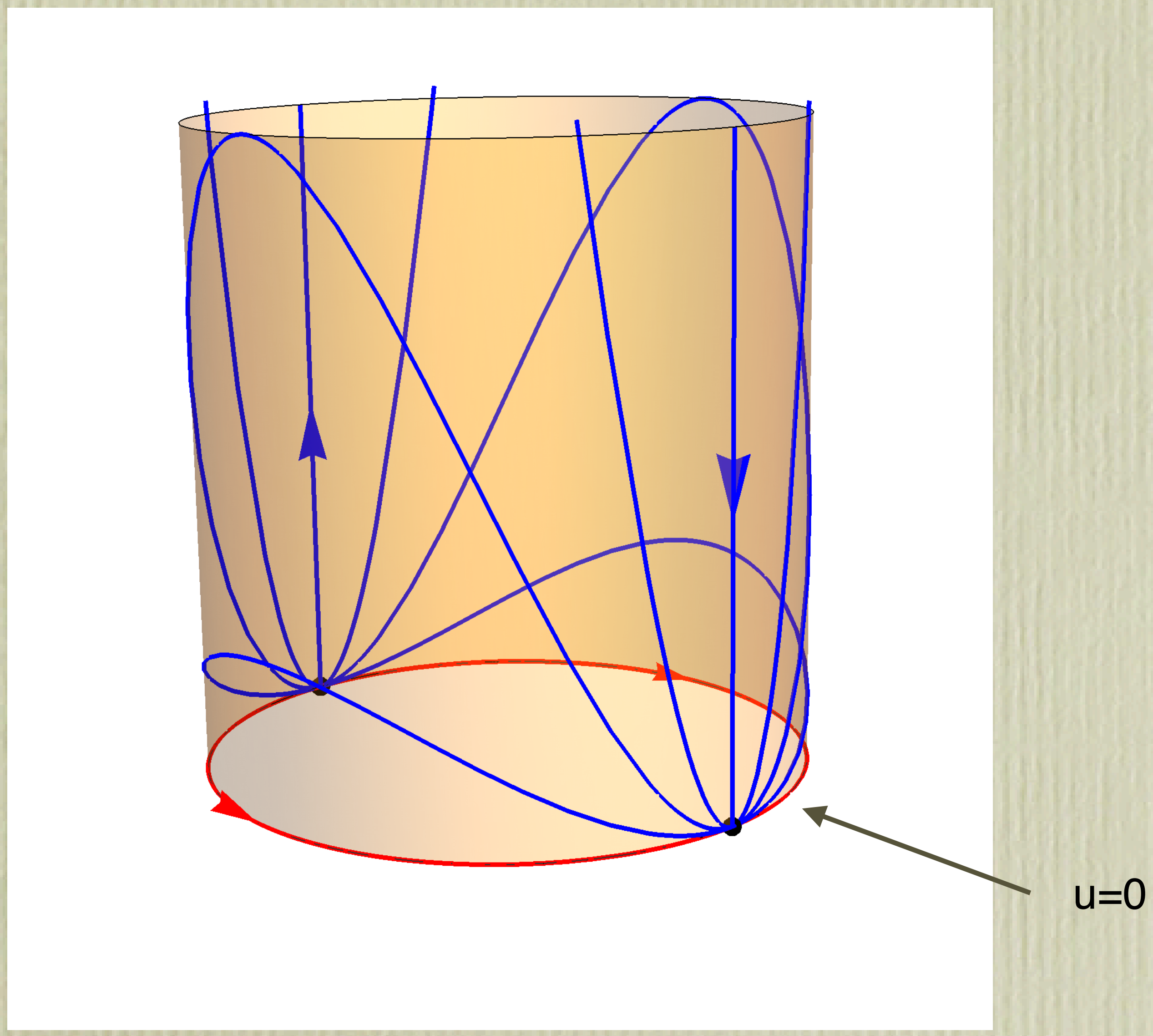


The flows on $\mathcal{M}(h, \lambda)$ converge to the restpoint cycles as $\lambda \to 0_+, \lambda \to 0_-$

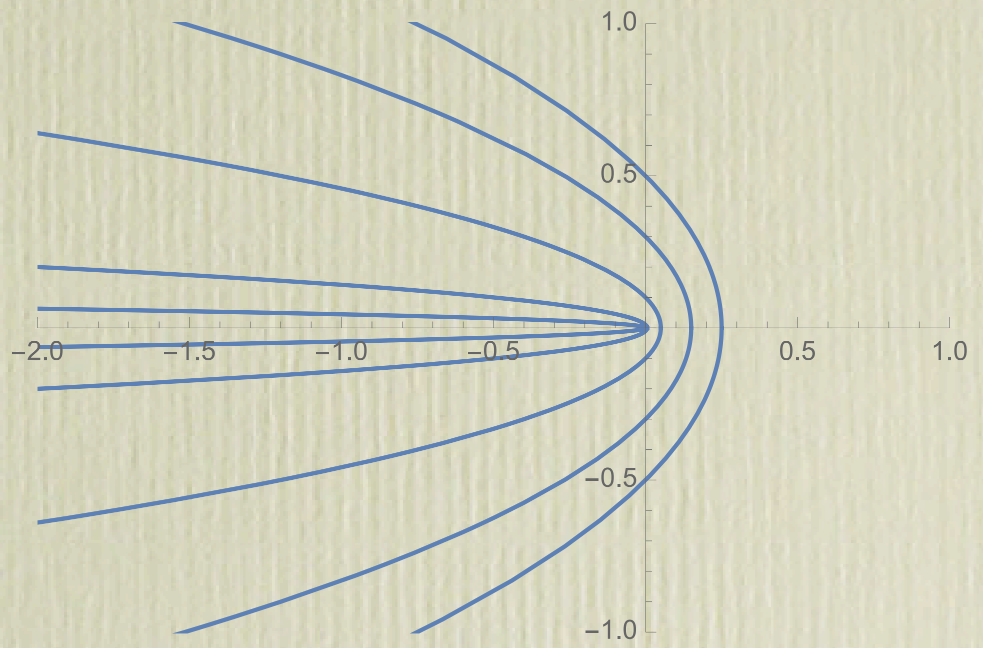
Blue curve = homothetic orbit beginning and ending at double collision
Red curves = limits of behavior of eccentric ellipses near collision, i.e., the two bodies spin around by angle 2π "at collision"

Energy h=0, Parabolic Orbits

The cylinder $h = 0$ contain entirely of parabolic orbits (in two senses of the word). The flow on the parabolic infinity manifold $u = 0$ is identical to the flow on the collision manifold. The restpoint p_+ gets an extra stable dimension and p_- gets an extra unstable one due to the equation $u' = -v_0 u$.



All of these parabolic orbits are in $W^s(p_+)$ and in $W^u(p_-)$



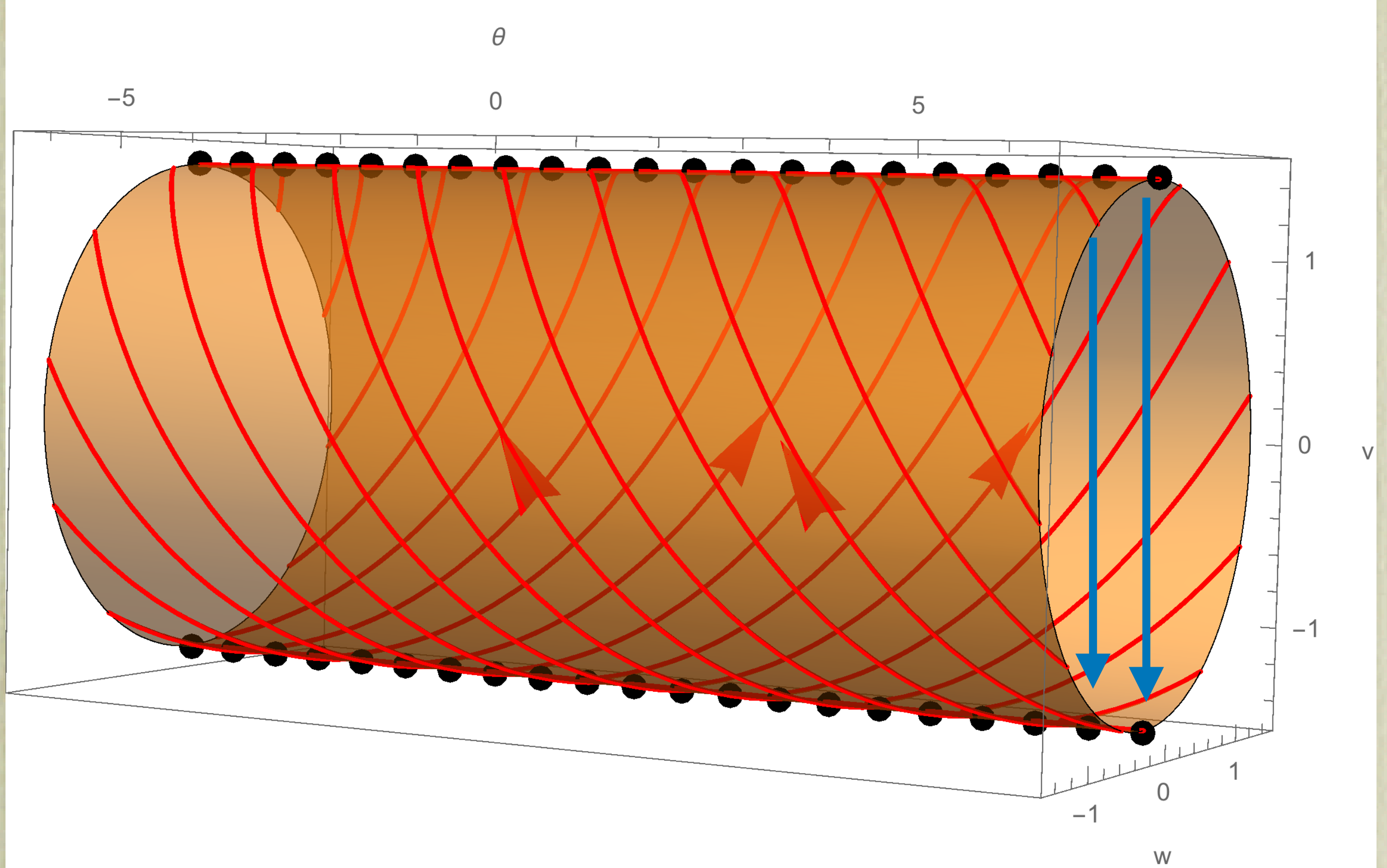
Two separate homothetic orbits. All other orbits begin and end at parabolic infinity.

What about the angle ?

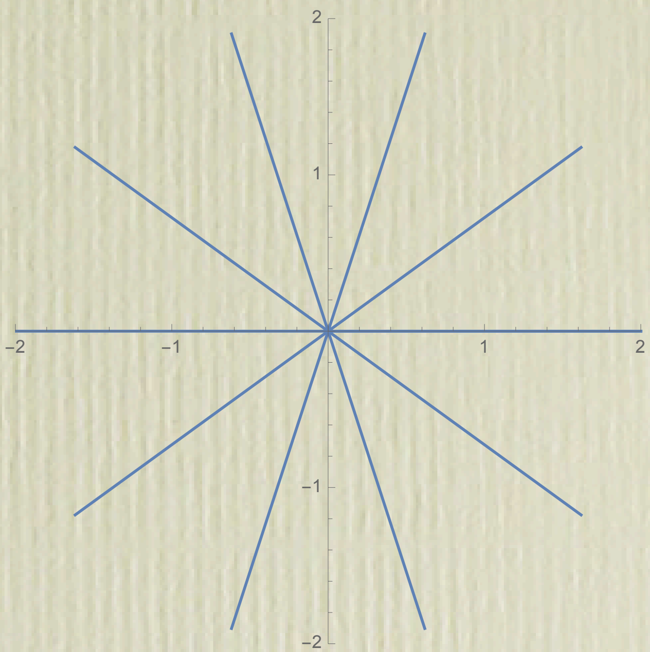
Instead of ignoring the rotation angle θ one can eliminate r by projecting an energy manifold

$$rh = \frac{1}{2}v^2 + \frac{1}{2}\omega^2 - U_0$$

to (θ, w, v) space. The collision manifold becomes a torus with two normally hyperbolic circles of restpoints. The region $r > 0$ maps to the inside of the cylinder. The restpoint connections $p_i \to p_+$ on the collision manifold become surfaces of connecting orbits connecting restpoints with $\Delta\theta = \pm 2\pi$. The collinear homothetic orbits form the stable and unstable manifolds of these circles of restpoints.



Two-body flow on a negative energy manifold $\mathcal{M}(h), h < 0$



Homothetic orbits

Blue: Homothetic orbits

Red: Limit of elliptic orbits — Spin by 2π “at collision”

Zero Energy 2BP — A Global View of Parabolic Motion

Consider the $h = 0$ two-body problem near infinity using $u = 1/r$

$$\begin{aligned} u' &= -uv \\ v' &= \frac{1}{2}v^2 + \frac{1}{2}\omega^2 - U_0 \\ \theta' &= w \\ \omega' &= -\frac{1}{2}v\omega \end{aligned}$$

$$u\left(\frac{1}{2}v^2 + \frac{1}{2}w^2 - U_0\right) = h = 0$$

Parametrize the cylinder $v^2 + w^2 = 2U_0$ by another angle α:

$$v = \sqrt{2U_0}\cos\alpha \qquad w = \sqrt{2U_0}\sin\alpha$$

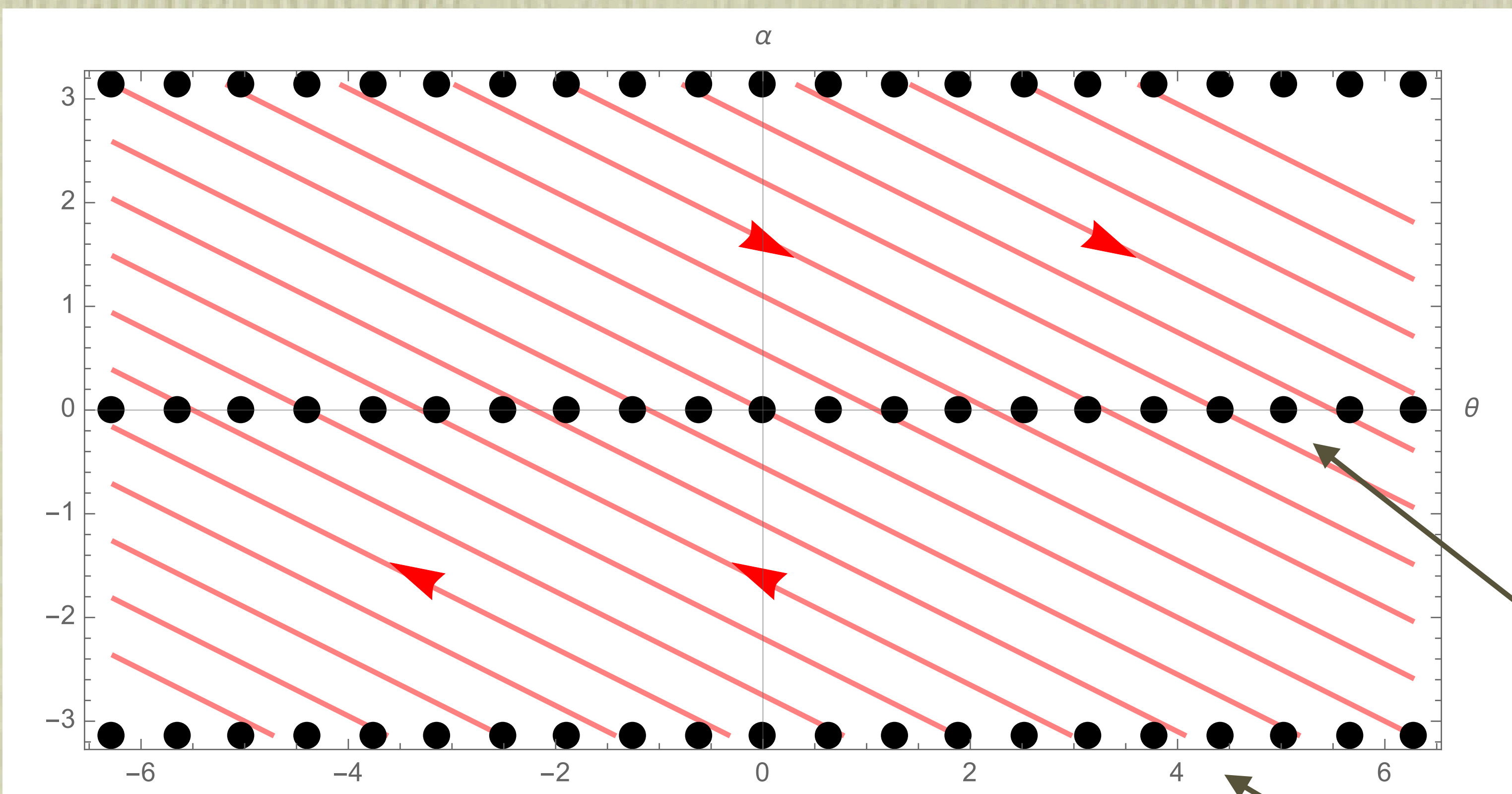
We get a simple system of ODEs on $\mathbb{R}^+ \times \mathbb{T}^2$:

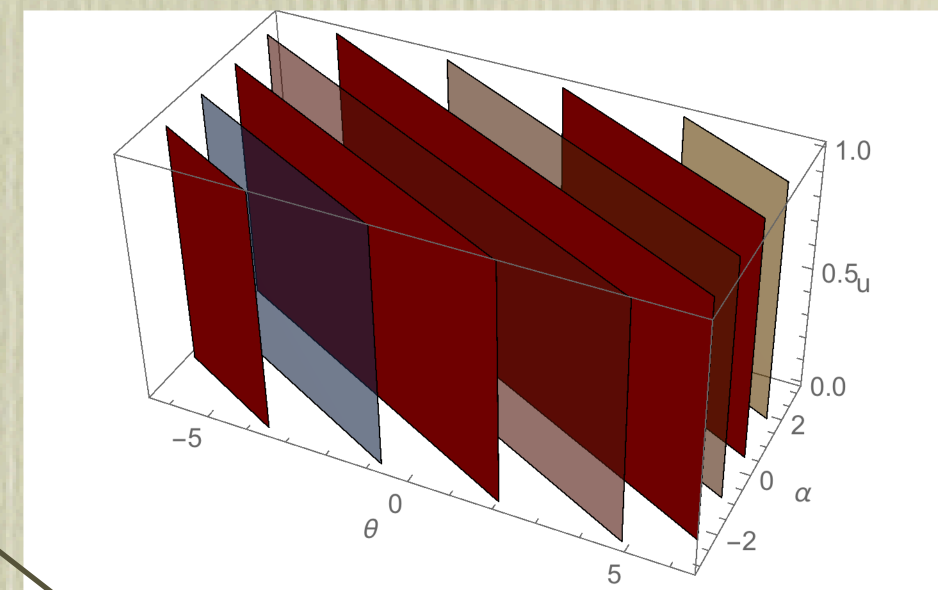
$$\begin{aligned} u' &= -\sqrt{2U_0}\cos\alpha\, u \\ \theta' &= \sqrt{2U_0}\sin\alpha \\ \alpha' &= -\frac{1}{2}\sqrt{2U_0}\sin\alpha \end{aligned}$$

Note that $\theta' + 2\alpha' = 0$

$$\theta + 2\alpha = \text{const}$$

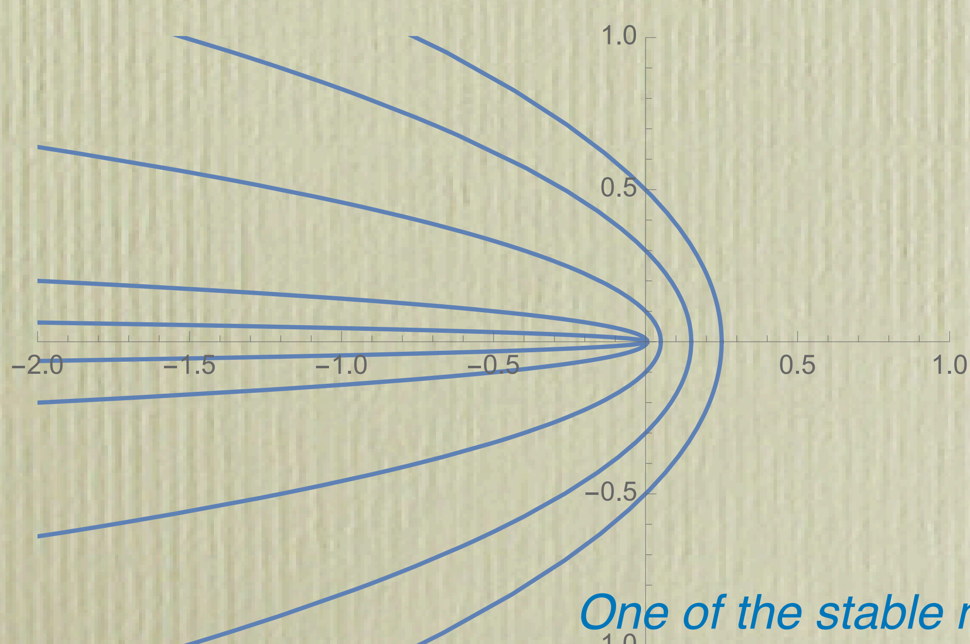
(θ, α) flow on $\mathbb{T}^2$ is independent of u. In particular it represents the flow on the parabolic infinity manifold. There are two circles of restpoints. The stable restpoints represent limits of forward time parabolic orbits.





Limits of forward-time parabolics

Limits of backward-time parabolics

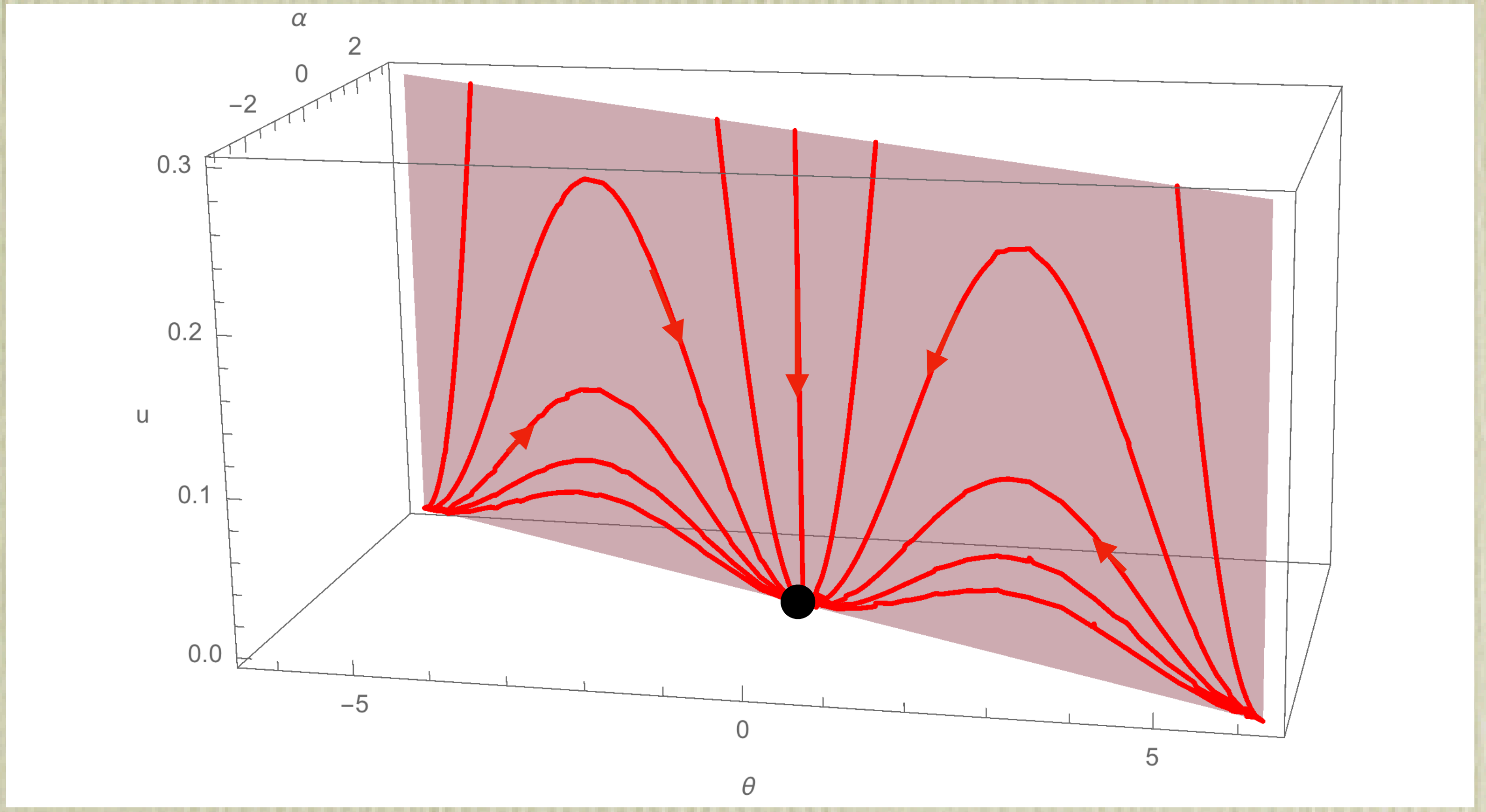


One of the stable manifolds

Each restpoint represents parabolic orbits with a fixed asymptotic angle at infinity.The whole h=0 energy manifold is foliated as a union of the stable manifolds of these restpoints.

Stable Manifolds as Lagrangian Graphs

Consider the stable manifold of one of the parabolic restpoints. It is a two-dimensional surface in our four-dimensional (u, θ, v, w) phase space. Moreover, it lies as a graph over the two-dimensional (u, θ) configuration space.



Theorem: For the planar two-body problem, the stable manifolds of the parabolic restpoints at infinity are Lagrangian submanifolds of the phase space. Moreover they are graphs over the entire configuration space.

Lagrangian graphs like this are important in the variational approach via the principle of least action

Proof sketch:

We want to show that the symplectic form on phase space is identically zero on the stable manifold of one of the restpoints. Now the blown-up coordinates are not symplectic. For the n-body problem in $\mathbb{R}^d$ in Cartesian coordinates, the symplectic structure is

$$\Omega = \sum_i m_i dq_i \wedge dv_i.$$

In blown-up coordinates (u, s, v, w) a calculation gives

$$\Omega = u^{-\frac{3}{2}} dv \wedge du + \sum_i m_i \left(u^{-\frac{1}{2}} ds_i \wedge dw_i + u^{-\frac{3}{2}} s_i \cdot dw_i \wedge du + \frac{1}{2} u^{-\frac{3}{2}} w_i \cdot ds_i \wedge du \right)$$

For $n = d = 2$ and restricting to the $h = 0$ manifold with coordinates (u, θ, α) this reduces to

$$\Omega = \sqrt{2U_0} \left(u^{-\frac{1}{2}} d\theta \wedge d\alpha - \frac{1}{2} u^{-\frac{3}{2}} (d\theta + 2d\alpha) \wedge du \right)$$

Now on the stable manifold we have $\theta + 2\alpha = \text{const}$ which gives

$$d\theta + 2d\alpha = 0 \qquad d\theta \wedge d\alpha = 0 \qquad \Longrightarrow \qquad \Omega = 0$$

In another lecture, we will give a similar result for n=3.