Blowing Up the N-Body Problem II
The Three-Body Problem

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Review:

- McGehee blow up for the N-body problem
- Total collision manifold and parabolic infinity manifold
- Lyapunov function
- Restpoints and Central Configurations
- Example: the planar 2-body problem, viewed from this perspective

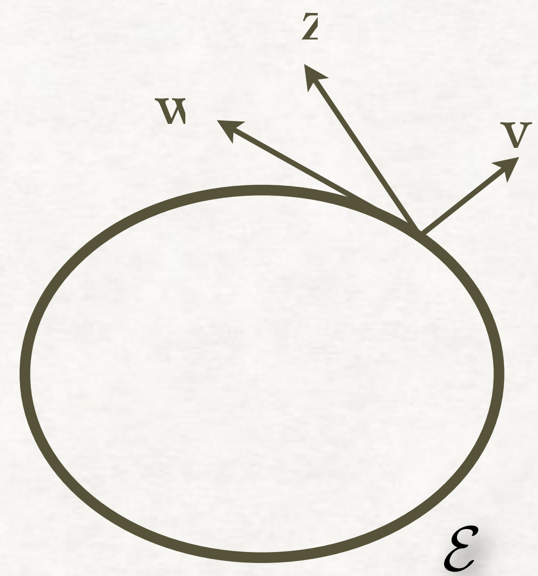
Next:

- Planar, collinear and isosceles 3-body problems
- Shape sphere
- Stable and unstable manifolds at triple collision
- Spiraling, existence of heteroclinic connections

Review for the N-body problem

Blown-up ODEs:

$$
\begin{aligned}
r' &= vr \\
v' &= \frac{1}{2}v^2 + \|w\|^2 - U(s) = \frac{1}{2}\|W\|^2 + rh \\
s' &= w \\
z' &= \tilde{\nabla}U(s) - \frac{1}{2}vw - \|w\|^2 s
\end{aligned}
$$



$$\|s\| = 1 \qquad \langle\langle s, w\rangle\rangle = 0$$

$$\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - U(s) = rh$$

$$\lambda(q, v) = \sqrt{r}\,\lambda(s, w)$$

Tangential gradient of Newtonian potential

$$\tilde{\nabla}U(s) = M^{-1}\nabla U(s) + U(s)s$$

Central Configurations: Critical points $\tilde{\nabla}U(s_c) = 0$

Restpoints: Two restpoints on collision manifold $p_\pm$, $(r, s, v, w) = (0, s_c, \pm\sqrt{2U(s_c)}, 0)$ and two restpoints at parabolic infinity with $u = 1/r = 0$

Next: Look at the linearized flow near the restpoints

Eigenvalues at the Restpoints

Let $p = (r, v, s, w) = (0, v_0, s_0, 0)$ be a restpoint on the collision manifold. Then

$$s_0 = \text{normalized CC} \qquad v_0 = \pm\sqrt{2U(s_0)}$$

For forward time collision orbits, we are interested in the stable manifolds of the restpoints with $v_0 < 0$. The linearized ODEs are

$$\begin{bmatrix} \delta r \\ \delta v \\ \delta s \\ \delta w \end{bmatrix}' = \begin{bmatrix} v_0 & 0 & 0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & D\tilde{\nabla}U(s_0) & -\frac{1}{2}v_0 I \end{bmatrix} \begin{bmatrix} \delta r \\ \delta v \\ \delta s \\ \delta w \end{bmatrix}$$

Here $D\tilde{\nabla}U(s_0)$ involves second derivatives of $U|_{\mathcal{E}}$ at the critical point s_0

There are two eigenvalues v_0, v_0 with eigenvectors of the form $(\delta r, \delta v, 0, 0)$. And if α is an eigenvalue of $D\tilde{\nabla}U(s_0)$ with eigenvector δs we get two eigenvalues and eigenvectors of the restpoint:

$$\lambda_{\pm} = \frac{-v_0 \pm \sqrt{v_0^2 + 16\alpha}}{4} \qquad (0, 0, \delta s, \lambda_{\pm}\delta s)$$

We are only interested in vectors tangent to the phase space $T(X - \Delta)$ of dimension $2(n-1)d$ or perhaps to a submanifold of constant energy and/or angular momentum.

Nondegeneracy

Recall that $\mathbb{SO}(d)$ acts as a symmetry group. Hence central configurations and restpoints are not isolated and instead comprise $\mathbb{SO}(d)$ orbits. Let $O(p)$ denote the orbit of a restpoint p. Then $\alpha = 0$ occurs as an eigenvalue of $D\tilde{\nabla}U(s_0)$ at least $\dim O(p)$ times.

Note that

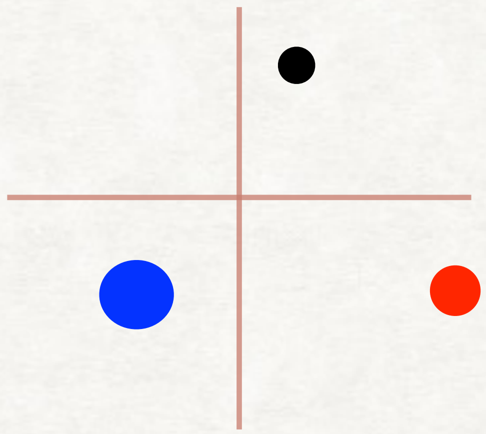
$$\begin{aligned} \alpha = 0 \qquad &\Rightarrow \qquad \lambda_\pm = 0, -v_0/2 \\ \alpha \neq 0 \qquad &\Rightarrow \qquad \operatorname{re} \lambda_\pm = \operatorname{re} \frac{-v_0 \pm \sqrt{v_0^2 + 16\alpha}}{4} \neq 0 \end{aligned}$$

Call the CC s_0 and the restpoint p nondegenerate if the multiplicity of $\alpha = 0$ as an eigenvalue of $D\tilde{\nabla}U(s_0)$ is exactly $\dim O(p)$. Then

$$\begin{aligned} s_0 \text{ nondegenerate} \qquad &\Rightarrow \qquad \text{Multiplicity of } \lambda = 0 \text{ is also } \dim O(p) \\ &\Rightarrow \qquad \operatorname{re}(\lambda) \neq 0 \text{ for all other eigenvalues of } p \\ &\Rightarrow \qquad O(p) \text{ is normally hyperbolic} \\ &\Rightarrow \qquad W^s(O(p)) = \cup_{q \in O(p)} W^s(q) \text{ (no spin)} \end{aligned}$$

Example: 2BP, SO(2) = circle orbit of CCs, Restpoints

Planar Three-Body Problem: N=3 d=2



$$s = (s_1, s_2, s_3) \in X \subset \mathbb{R}^6$$

$$\dim X = 4 \qquad \dim \mathcal{E} = 3$$

$$\mathcal{E} \simeq \mathbb{S}^3$$

$$w = (w_1, w_2, w_3) \in T_s\mathcal{E} \subset \mathbb{R}^6$$

Phase Space: $(r, v, s, w) \in \mathbb{R}^+ \times \mathbb{R} \times T(\mathcal{E} \setminus \Delta)$ $\qquad \dim = 8$

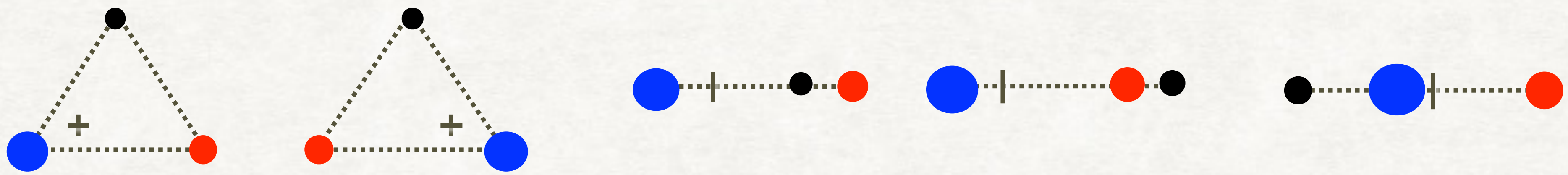
Symmetry Group: $\mathbb{SO}(2) \simeq \mathbb{S}^1$

Integral Manifolds:

$\mathcal{M}(h)$	$\dim = 7$
$\mathcal{M}(h, \lambda)$	$\dim = 6$
$\tilde{\mathcal{M}}(h) = \mathcal{M}(h)/SO(2)$	$\dim = 6$
$\tilde{\mathcal{M}}(h, \lambda) = \mathcal{M}(h, \lambda)/SO(2)$	$\dim = 5$

Planar 3BP:Central Configurations and Eigenvalues

There are always five central configurations— two equilateral triangles and three collinear shapes (one for each ordering).



All of them are nondegenerate The ellipsoid $\mathcal{E}$ has dimension 3 and $\dim O(p) = 1$. There are two nonzero eigenvalues α_1, α_2 of $D\tilde{\nabla}U(s_0)$. There are $2(N-1)d = 8$ eigenvalues of p:

$$v_0, v_0, 0, -\frac{v_0}{2}, \frac{-v_0 \pm \sqrt{v_0^2 + 16\alpha_1}}{4}, \frac{-v_0 \pm \sqrt{v_0^2 + 16\alpha_2}}{4}$$

The equilateral points are local minima of U on $\mathcal{E}$ so the Hessian eigenvalues satisfy $\alpha_1 > 0$, $\alpha_2 > 0$. The collinear points are saddles with $\alpha_1 > 0$, $\alpha_2 < 0$. More about the corresponding eigenvalues later.

The Shape Sphere

Since the planar problem has such high dimension we will mostly work with the reduced problem after quotienting by the $\mathbb{SO}(2)$ action. The configuration of the three bodies is given by $(r, s) \in \mathbb{R}^+ \times \mathcal{E}$. The reduced configuration space is

$$\mathbb{R}^+ \times \mathcal{E}/\mathbb{SO}(2) = \mathbb{R}^+ \times \mathcal{S}$$

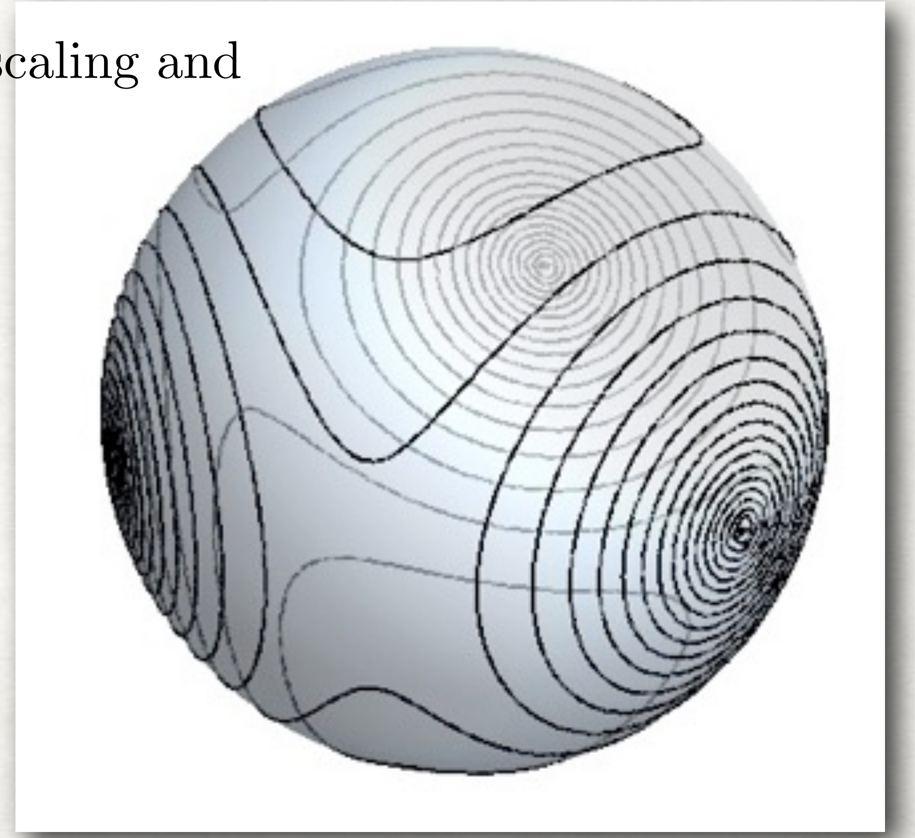
The quotient space

$$\mathcal{S} = \mathcal{E}/\mathbb{SO}(2) \simeq \mathbf{S}^2$$

is called the *shape sphere.*

If we view r as a radial coordinate then the reduced configuration space $\mathbb{R}^+ \times \mathbf{S}$ can be visualized as the exterior of the sphere with the sphere itself representing $r = 0$.

A point $p \in \mathcal{S}$ represents an equivalence class of triangles, up to scaling and rotation: a possible "shape" of a triangle.



Why is it a two-sphere ?

$s = (s_1, s_2, s_3) \in \mathbb{R}^6$

Center of Mass Subspace: $\dim X = 4$

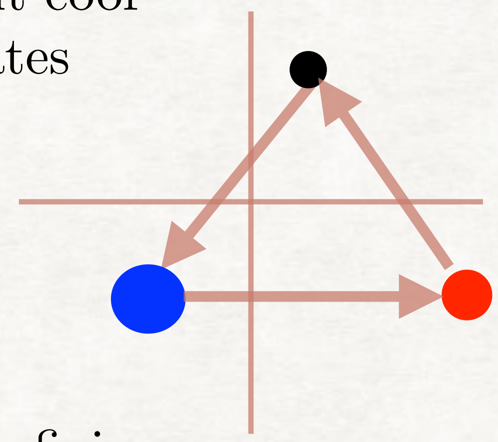
Size Normalization: $\|X\| = 1 \qquad \mathcal{E} \simeq \mathbf{S}^3$

Quotient Space: $\mathcal{S} = \mathcal{E}/\mathbb{SO}(2) \simeq \mathbf{S}^3/\mathbf{S}^1$

Hopf Map: $\mathbf{S}^3/\mathbf{S}^1 \simeq \mathbf{S}^2$

Explicit Formula for the Quotient Map

Here is an elegant way to carry out this reduction and obtain explicit coordinates for the shape sphere. First introduce mutual difference coordinates



$$X_1 = s_2 - s_3 \qquad X_2 = s_3 - s_1 \qquad X_3 = s_1 - s_2$$

which satisfy

$$(X_1, X_2, X_3) \in \mathcal{W} = \{X_1 + X_2 + X_3 = 0\}.$$

This reduces by translations in a symmetrical way and is an alternative to fixing the center of mass.

Next introduce a basis for $\mathcal{W}$. Using complex notation for $X_i \in \mathbb{R}^2 \simeq \mathbb{C}$ let

$$X_1 = z_1 - z_2 \qquad X_2 = \omega z_1 - \bar{\omega} z_2 \qquad X_3 = \bar{\omega} z_1 - \omega z_2 \qquad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Now every triangle is represented by $(z_1, z_2) \in \mathbb{C}^2$. To get shape space we need to quotient by scaling and rotations, or equivalently by relation

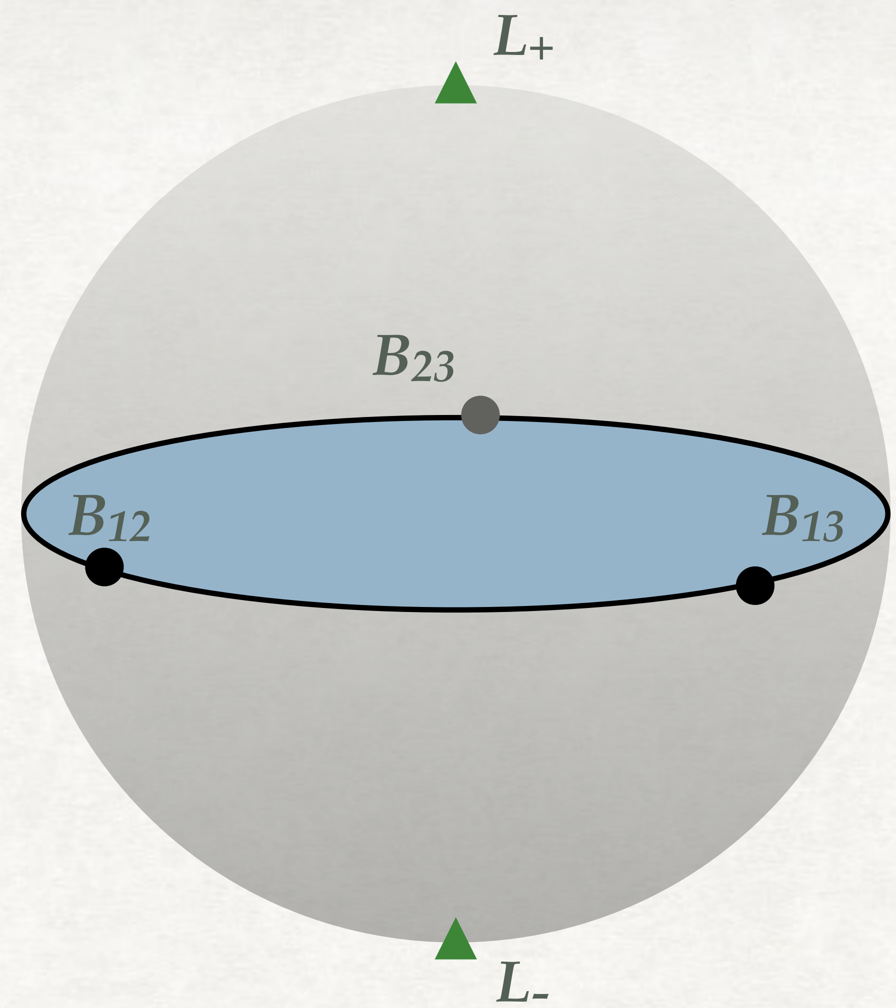
$$(z_1, z_2) \sim (kz_1, kz_2) \qquad k = |k|e^{i\theta} \in \mathbb{C} \setminus 0$$

This can be done using a Hopf map $(z_1, z_2) \mapsto (w_1, w_2, w_3) \in \mathbb{R}^3$:

$$w_1 = \frac{2\,\mathrm{re}\,\bar{z}_1 z_2}{|z_1|^2 + |z_2|^2} \qquad w_2 = \frac{2\,\mathrm{im}\,\bar{z}_1 z_2}{|z_1|^2 + |z_2|^2} \qquad w_3 = \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}$$

It is easy to check that $w_1^2 + w_2^2 + w_3^2 = 1$ so we have our two-sphere.

Geography of the Shape Sphere



- *Binary collision shapes,* $\boldsymbol{B_{ij}}$
- *Collinear shapes*
- *Equilateral shapes,* $\boldsymbol{L_{+,-}}$

Mutual distances are quite simple:

$$|s_2 - s_3|^2 = |z|^2(1 - w_1) \qquad |s_3 - s_1|^2 = |z|^2(1 + \frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2) \qquad |s_3 - s_1|^2 = |z|^2(1 + \frac{1}{2}w_1 + \frac{\sqrt{3}}{2}w_2)$$

Equilateral Triangles: $w_1 = w_2 = 0 \qquad w_3 = \pm 1$

Binary Collisions: $w_3 = 0$ and $(w_1, w_2) = (1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Collinear Shapes: $w_3 = 0$

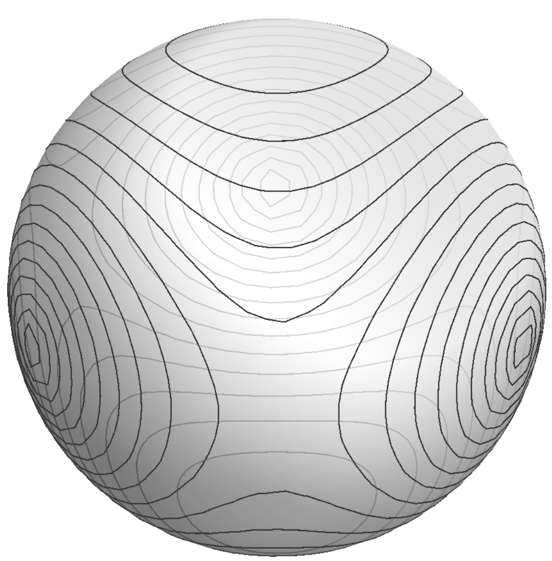
Shape Potential $U(w_1, w_2, w_3)$

Restricting $U(s)$ to $\|s\| = 1$ is equivalent to considering the scale invariant

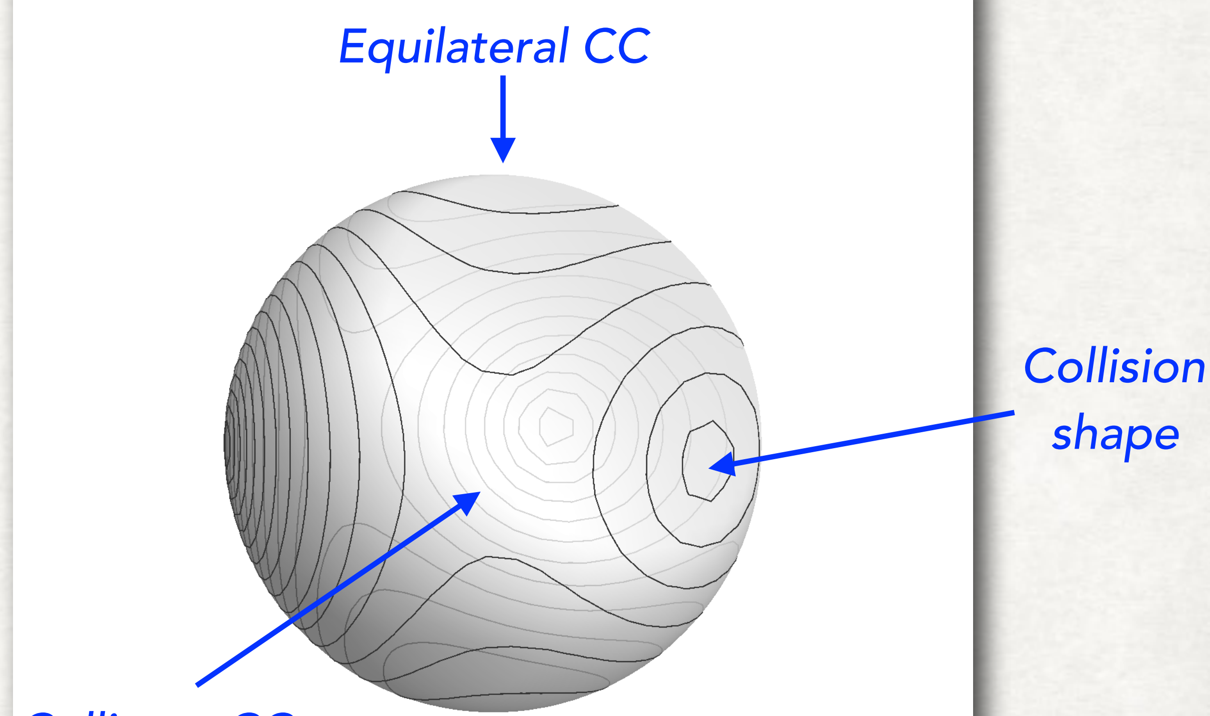
$$V(s) = \|s\| U(s) = \sqrt{\frac{m_1 m_2 \rho_3^2 + m_3 m_1 \rho_2^2 + m_2 m_3 \rho_1^2}{m_1 + m_2 + m_3}} \left(\frac{m_1 m_2}{\rho_3} + \frac{m_3 m_1}{\rho_2} + \frac{m_2 m_3}{\rho_1} \right)$$

where $\rho_1^2 = 1 - w_1 \qquad \rho_2^2 = 1 + \frac{1}{2} w_1 - \frac{\sqrt{3}}{2} w_2 \qquad \rho_3^2 = 1 + \frac{1}{2} w_1 + \frac{\sqrt{3}}{2} w_2$

Plotting the level curves on the sphere reveals the locations of the 5 critical points (the central configuration shapes).



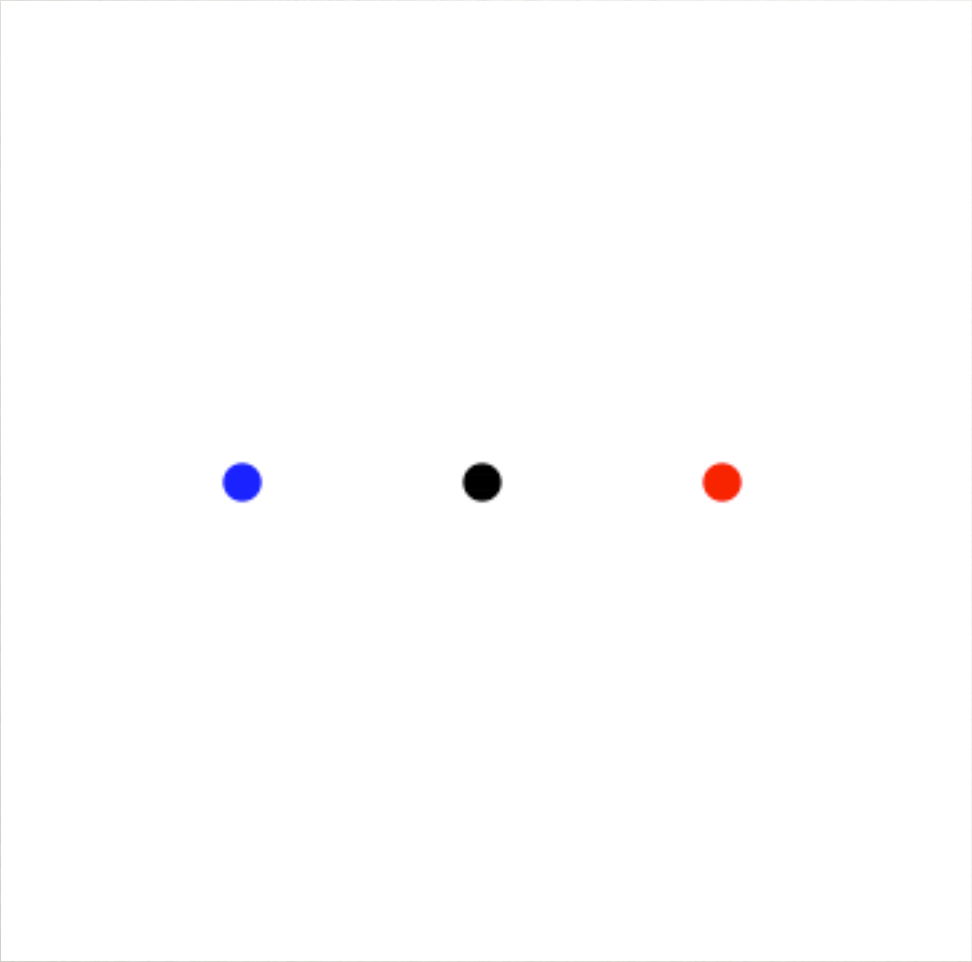
Equal masses

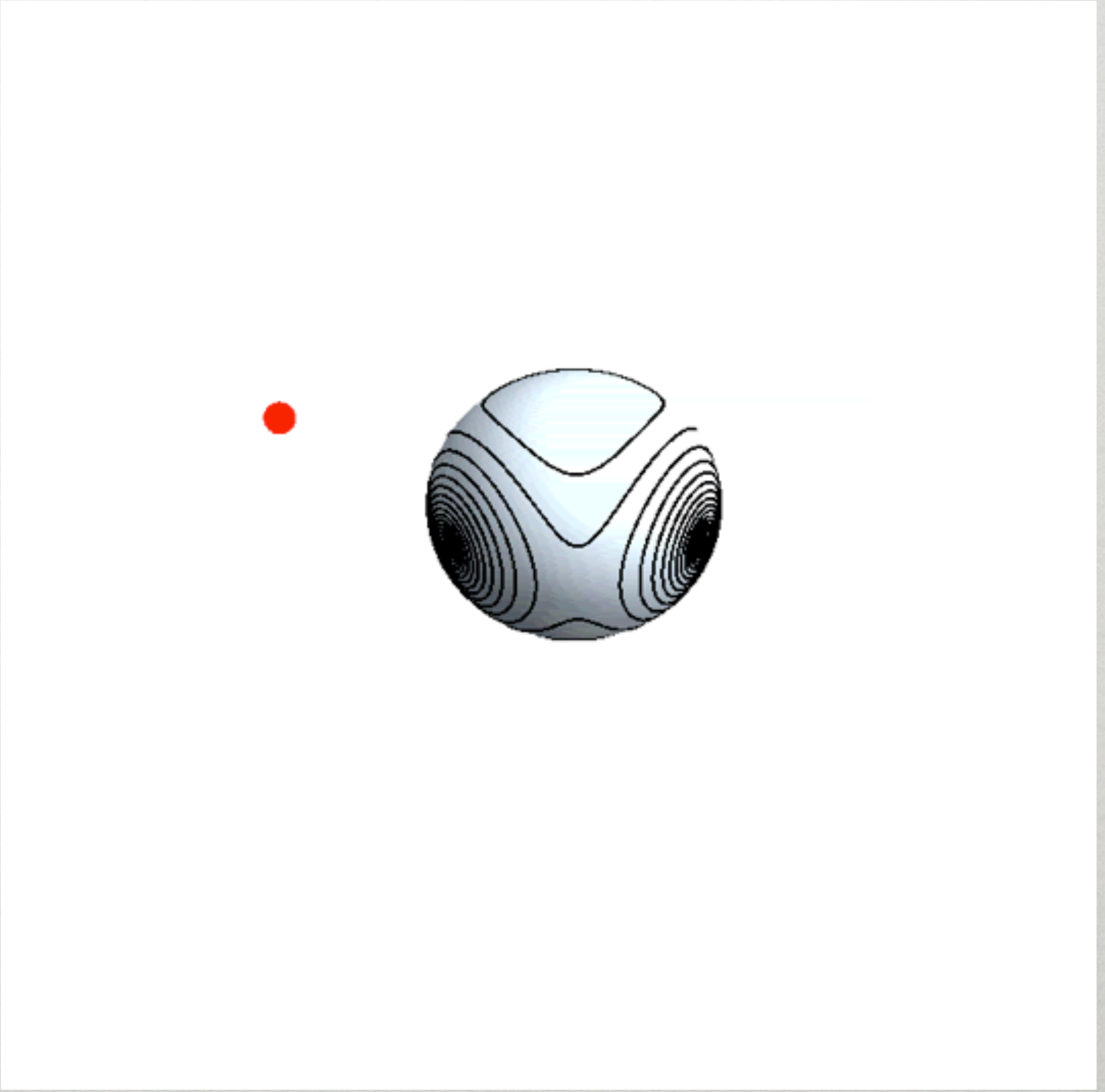


$m_1 = 4, m_2 = 2, m_3 = 1$

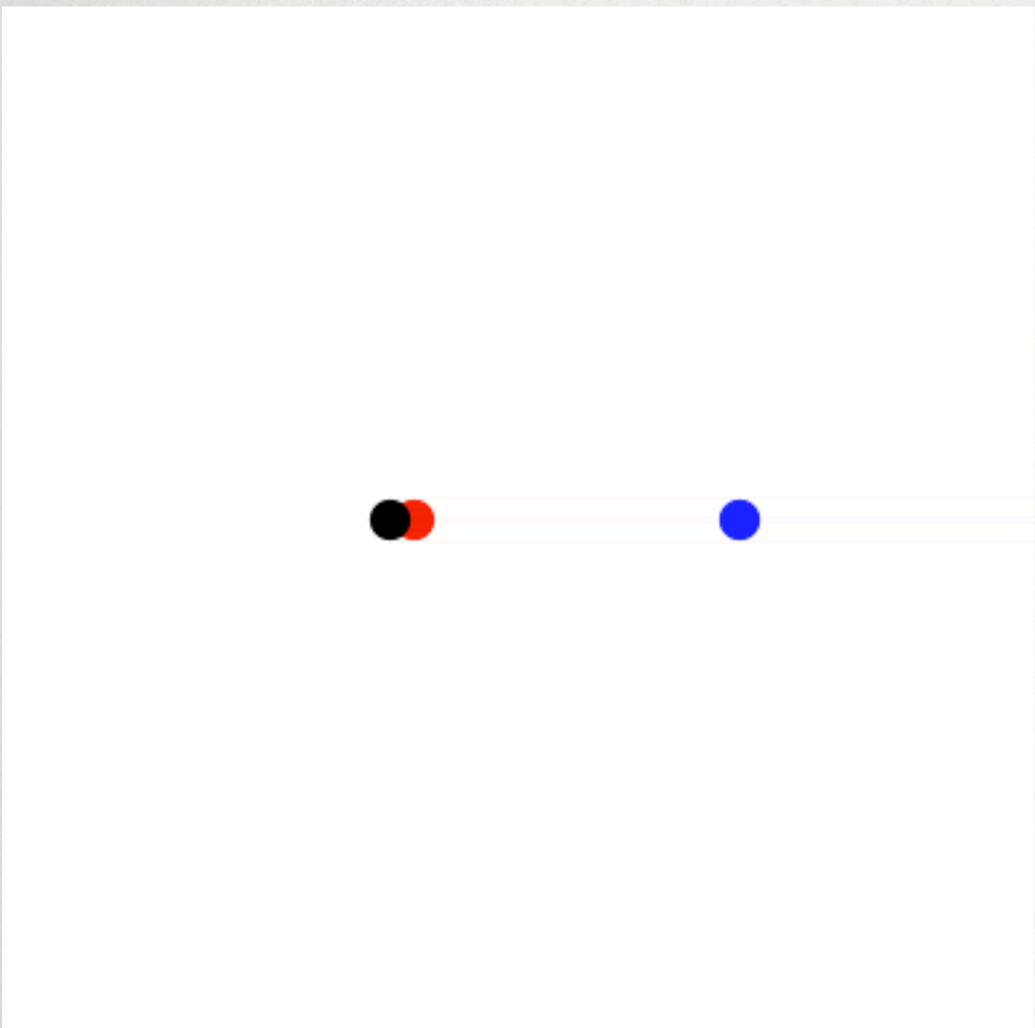
For fun —Some orbits plotted in reduced space

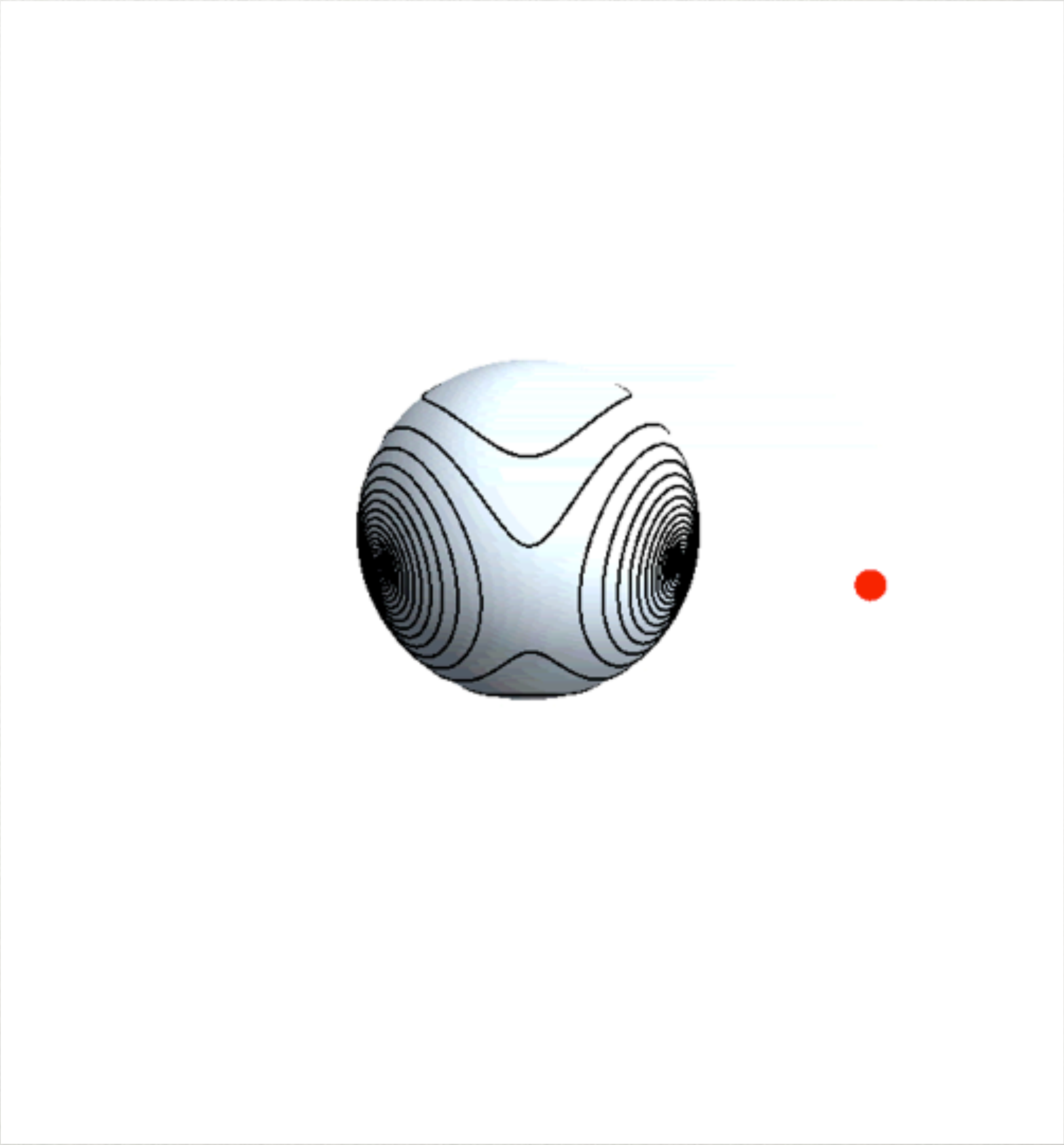
Figure eight orbit of Chenciner and Montgomery





Broucke-Henon Orbit

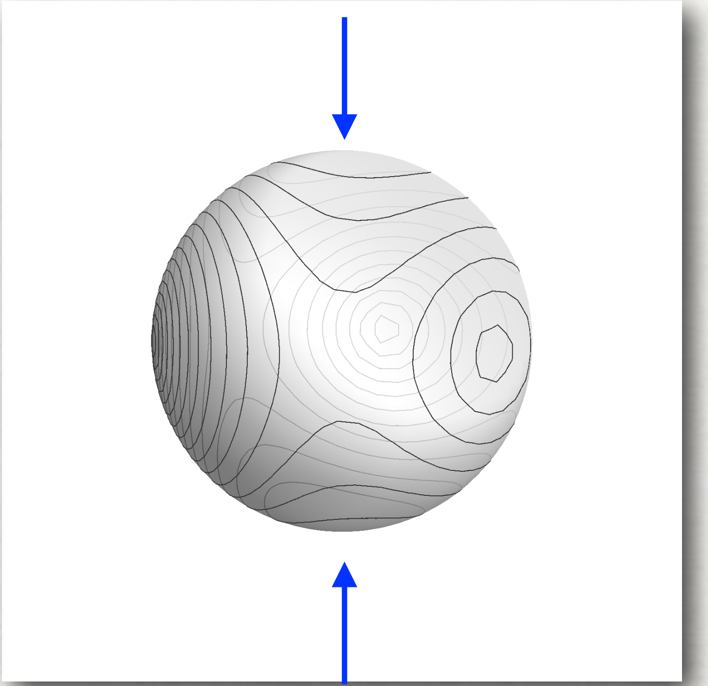




Equilateral Restpoints — Stable and Unstable Manifolds

The equilateral central configurations are the minima of U on $\mathcal{E}$. The Hessian eigenvalues $0, \alpha_1 > 0, \alpha_2 > 0$ are

$$\alpha_1, \alpha_2 = \frac{3U(s)}{2}\left(1 \pm \sqrt{k}\right)$$



where

$$k = \frac{(m_1 - m_2)^2 + (m_1 - m_3)^2 + (m_2 - m_3)^2}{2(m_1 + m_2 + m_2)^2}$$

$$U(s) = \frac{(m_1 m_2 + m_3 m_1 + m_1 m_2)^{\frac{3}{2}}}{(m_1 + m_2 + m_3)^{\frac{1}{2}}}$$

Triple collision restpoints: $p_\pm = (r, s, v, w) = (0, s, v, 0), v = \pm\sqrt{2U(s)}$ The eight eigenvalues are

$$\lambda = v, v, 0, -\frac{v}{2}, \frac{-v}{4}\left(1 \pm \sqrt{13 \pm 12\sqrt{k}}\right)$$

Recall that $\lambda = 0$ comes from the $\mathbb{SO}(2)$ symmetry.

The value of k always satisfies $0 \leq k < 1$ for any positive masses so the eigenvalues are always real. The signs are

Forward time triple collision:	$p_- : -, -, 0, +$	$-, -, +, +$	$\dim W^s(p_-) = 4$
Backward time triple collision:	$p_+ : +, +, 0, -$	$-, -, +, +$	$\dim W^u(p_+) = 4$

The other manifolds are inside r=0

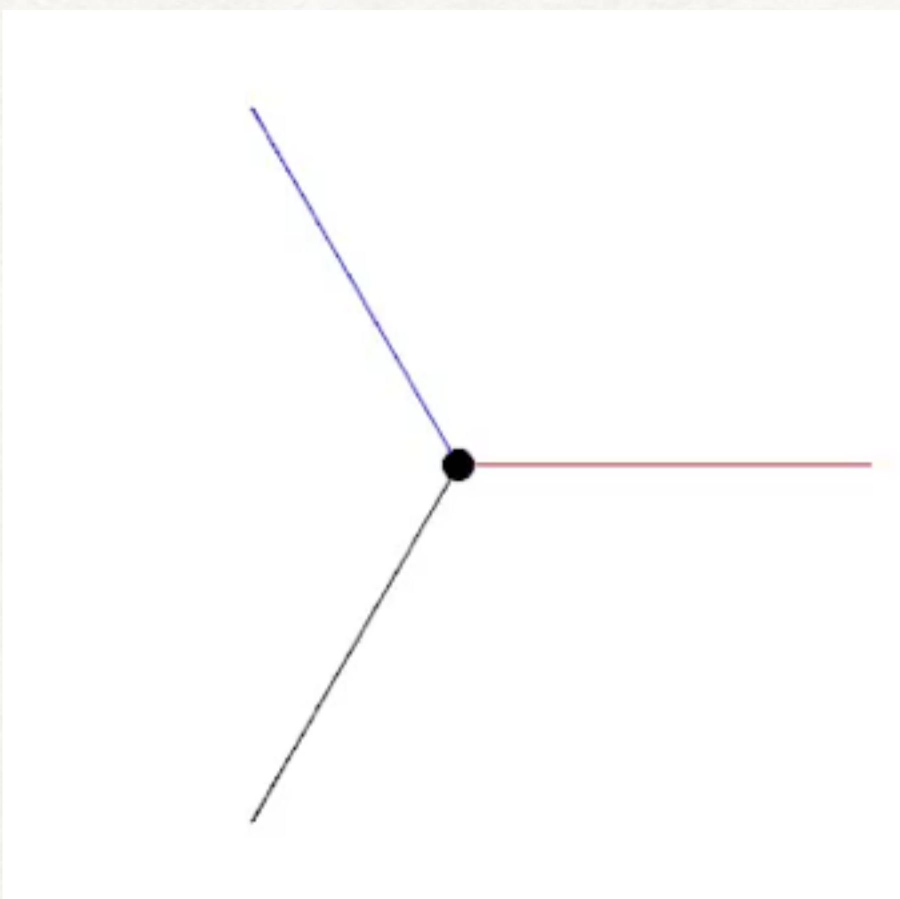
In the five-dimensional reduced space $\mathcal{M}(h,0)/\mathbb{SO}(2)$ we have

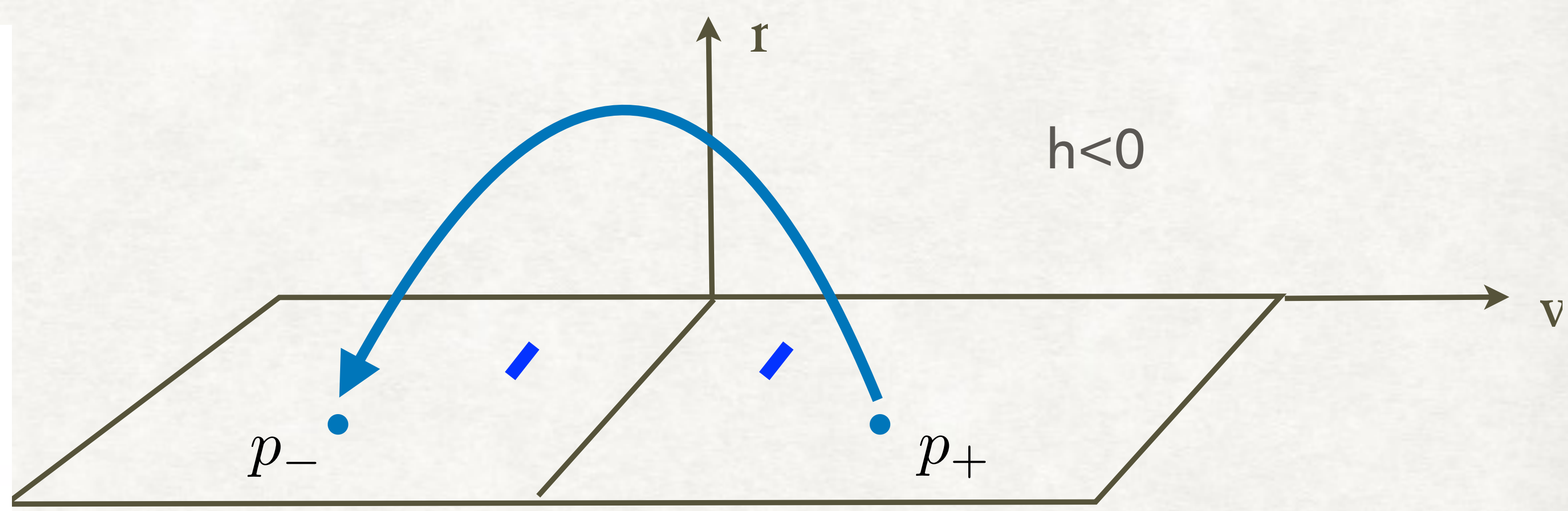
Triple collision restpoints: $p_\pm = (r,s,v,w) = (0,s,v,0), v = \pm\sqrt{2U(s)}$

Forward time triple collision: $p_- : -(,-,0,+)$ $\quad -,-,+,+ \quad \dim W^s(p_-) = 3$

Backward time triple collision: $p_+ : +,(+,0,-)$ $\quad -,-,+,+ \quad \dim W^u(p_+) = 3$

Note: it's possible to have transverse heteroclinic orbits $p_+ \Longrightarrow p_-$. In fact we already know about such an orbit, the homothetic collision orbit.





Devaney, Simo-Llibre proved transversality of this connecting orbit

Later: existence of many more transverse connections for certain masses

Equilateral Restpoints at Parabolic Infinity

Parabolic infinity restpoints: $q_\pm = (u,s,v,w) = (0,s,v,0), v = \pm\sqrt{2U(s)}$
The eight eigenvalues are the same except for the sign of the first one

$$\lambda = -v, v, 0, -\frac{v}{2}, \frac{-v}{4}\left(1 \pm \sqrt{13 \pm 12\sqrt{k}}\right)$$

$$\begin{array}{rlll} \text{Forward time parabolic:} & q_+ : -,+,0,- & -,-,+,+ & \dim W^s(q_+) = 4 \\ \text{Backward time parabolic:} & q_- : +,-,0,+ & -,-,+,+ & \dim W^u(q_-) = 4 \end{array}$$

The other manifolds are inside u=0

Generalization — Minimal Central Configurations

Return for a moment to the N-body problem in $\mathbb{R}^d$ with $\mathbb{SO}(d)$ symmetry with phase space $T(X-\Delta)$. Suppose s is a nondegenerate minimum of U on $\mathcal{E}$. Then the Hessian will have $\dim O(s)$ eigenvalues $\alpha = 0$ with all other eigenvalues $\alpha_i > 0$. The eigenvalues will be

$$\lambda = 0, \ldots, 0, -\frac{v}{2}, \ldots, -\frac{v}{2}, \frac{-v \pm \sqrt{v^2 + 16\alpha_i}}{4}$$

Since $\alpha_i > 0$, all eigenvalues are real and half of the eigenvalues involving the α_i are of each sign. Hence

$$\text{Forward time parabolic: } q_+ : \dim W^s(q_+) = \frac{1}{2}\dim T(X-\Delta)$$

$$\text{Backward time parabolic: } q_- : \dim W^u(q_-) = \frac{1}{2}\dim T(X-\Delta)$$

Another Example: Recall 2BP where the dimensions are 2 and 4

Collinear Restpoints — Stable and Unstable Manifolds

The shapes of the collinear restpoints depend on the masses. Consider the collinear central configuration with m_2 between m_1 and m_3. It turns out that distance ratio $r = r_{23}/r_{12}$ for this configuration is the positive root to the fifth degree equation

$$(m_2 + m_3)r^5 + (2m_2 + 3m_3)r^4 + (m_2 + 3m_3)r^3 \\ - (3m_1 + m_2)r^2 - (3m_1 + 2m_2)r - (m_1 + m_2) = 0.$$

(This root is unique by Descartes' rule of signs.) Let

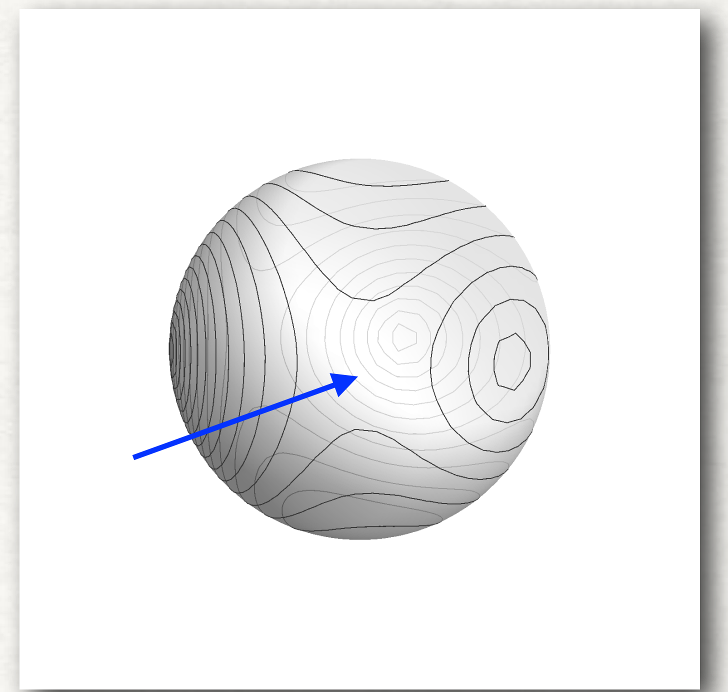
$$\nu = \frac{m_1(1 + 3r + 3r^2) + m_3(3r^3 + 3r^4 + r^5)}{(m_1 + m_3)r^2 + m_2(1 + r)^2(1 + r^2)}.$$

The eigenvalues of $D\tilde{\nabla}U(s)$ at this collinear CC are

$$\alpha_1, \alpha_2 = -U(s)\nu,\, U(s)(3 + 2\nu)$$

and the eight eigenvalues of the collision restpoints are

$$\lambda = v, v, 0, -\frac{v}{2}, \frac{-v}{4}\left(1 \pm \sqrt{1 - 8\nu}\right), \frac{-v}{4}\left(1 \pm \sqrt{25 + 16\nu}\right)$$



First computed by Chazy, Siegel

Collinear Triple Collision

$$\lambda = v, v, 0, -\frac{v}{2}, \frac{-v}{4}\left(1 \pm \sqrt{1-8\nu}\right), \frac{-v}{4}\left(1 \pm \sqrt{25+16\nu}\right)$$

The signs of the real parts of these eigenvalues are

Forward time triple collision $p_- : -, -, 0, + \qquad +, +, +, - \qquad \dim W^s(p_-) = 3$

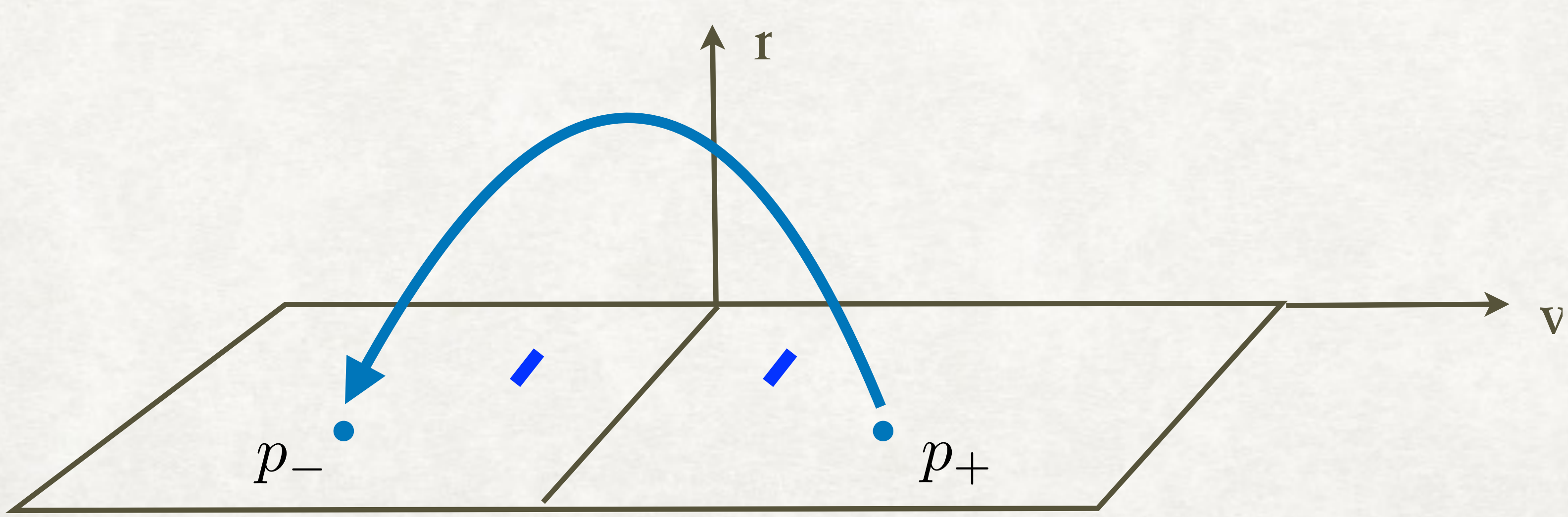
Backward time triple collision $p_+ : +, +, 0, - \qquad -, -, +, - \qquad \dim W^u(p_+) = 3$

or in $\mathcal{M}(h, 0)/\mathbb{SO}(2)$

Forward time triple collision $p_- : -\ (, -, 0, +) \qquad +, +, +, - \qquad \dim W^s(p_-) = 2$

Backward time triple collision $p_+ : +\ (, +, 0, -) \qquad -, -, +, - \qquad \dim W^u(p_+) = 2$

In spite of the low dimensions, there still exist homothetic connecting orbits $p_+ \Longrightarrow p_-$. The reason is that these stable and unstable manifolds are contained in invariant submanifolds of dimension 3 and the connection is a transverse intersection there.



Collinear 3BP (N=3, d=1) with all positions and velocities in R^1 is an invariant set of the planar problem (with zero angular momentum). It has codimension 2 or dimension 3 in the 5 dimensional spaces

$$\mathcal{M}(h, 0)/\mathbb{SO}(2)$$

Spiraling

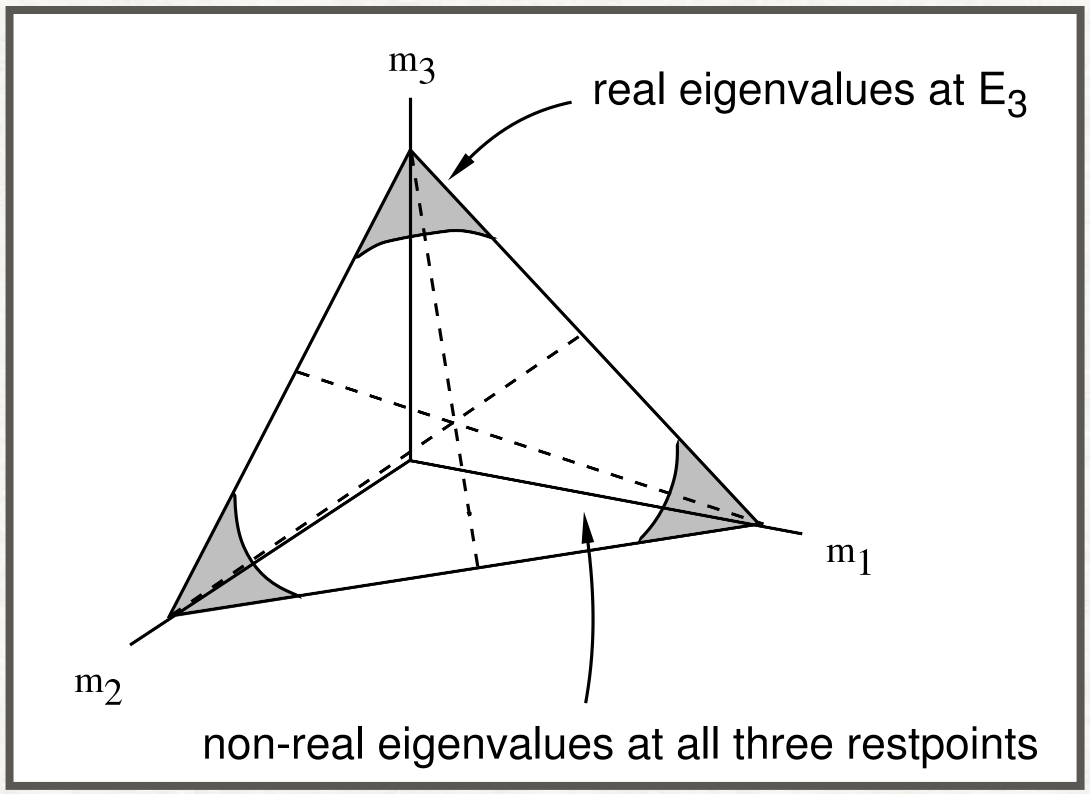
The eigenvalues associated to the CC E_2 with m_2 in the middle are

$$\lambda = v, v, 0, -\frac{v}{2}, \frac{-v}{4}\left(1 \pm \sqrt{1-8\nu}\right), \frac{-v}{4}\left(1 \pm \sqrt{25+16\nu}\right)$$

$$\nu = \frac{m_1(1+3r+3r^2)+m_3(3r^3+3r^4+r^5)}{(m_1+m_3)r^2+m_2(1+r)^2(1+r^2)}$$

$$(m_2+m_3)r^5+(2m_2+3m_3)r^4+(m_2+3m_3)r^3-(3m_1+m_2)r^2-(3m_1+2m_2)r-(m_1+m_2)=0.$$

For most choices of the masses, it turns (Robinson-Saari) out that $\nu > \frac{1}{8}$.

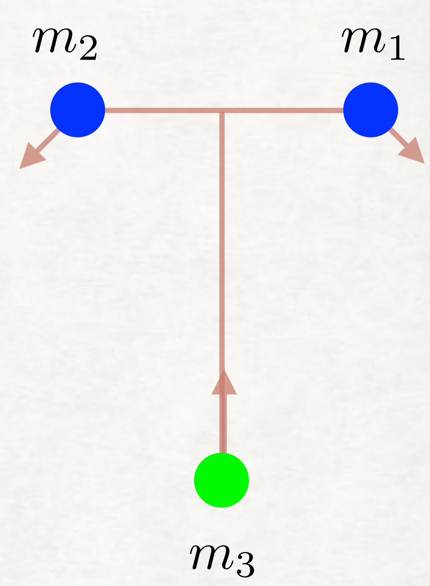


It turns out that this spiraling has many interesting dynamical consequences which we will explore.

- infinitely many transverse connections between equilateral restpoints at collision (later in this lecture)
- existence of a chaotic invariant set for the 3BP with small angular momentum (in the next lecture)

Spiraling in the Isosceles 3BP

Before considering the implications of spiraling in the planar 3BP, it is helpful to look at a lower dimensional subsystem: the isosceles problem



Suppose $m_1 = m_2$ and assume

$$s_1 = (x, y) \qquad s_2 = (-x, y) \qquad s_3 = (0, y_3)$$

Choosing the velocities symmetric in this way gives an invariant set of the zero angular momentum planar three-body problem.

$$s = (s_1, s_2, s_3) \in X \subset \mathbb{R}^3$$

$$\dim X = 2 \qquad \dim \mathcal{E} = 1 \qquad \mathcal{E} \simeq \mathbb{S}^1$$

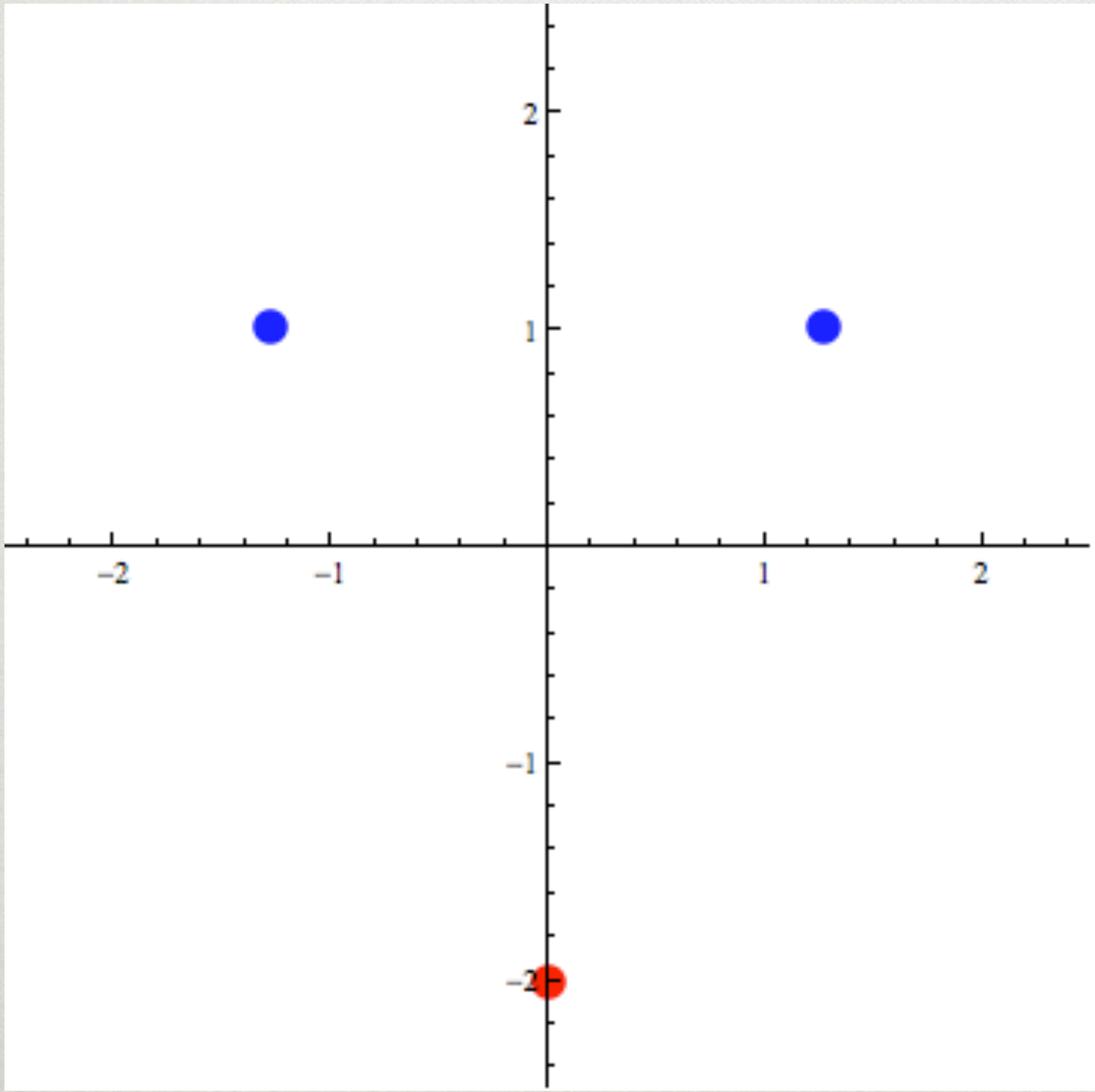
Phase Space: $(r, v, s, w) \in \mathbb{R}^+ \times \mathbb{R} \times T(\mathcal{E} \setminus \Delta) \qquad \dim = 4$

Integral Manifolds:

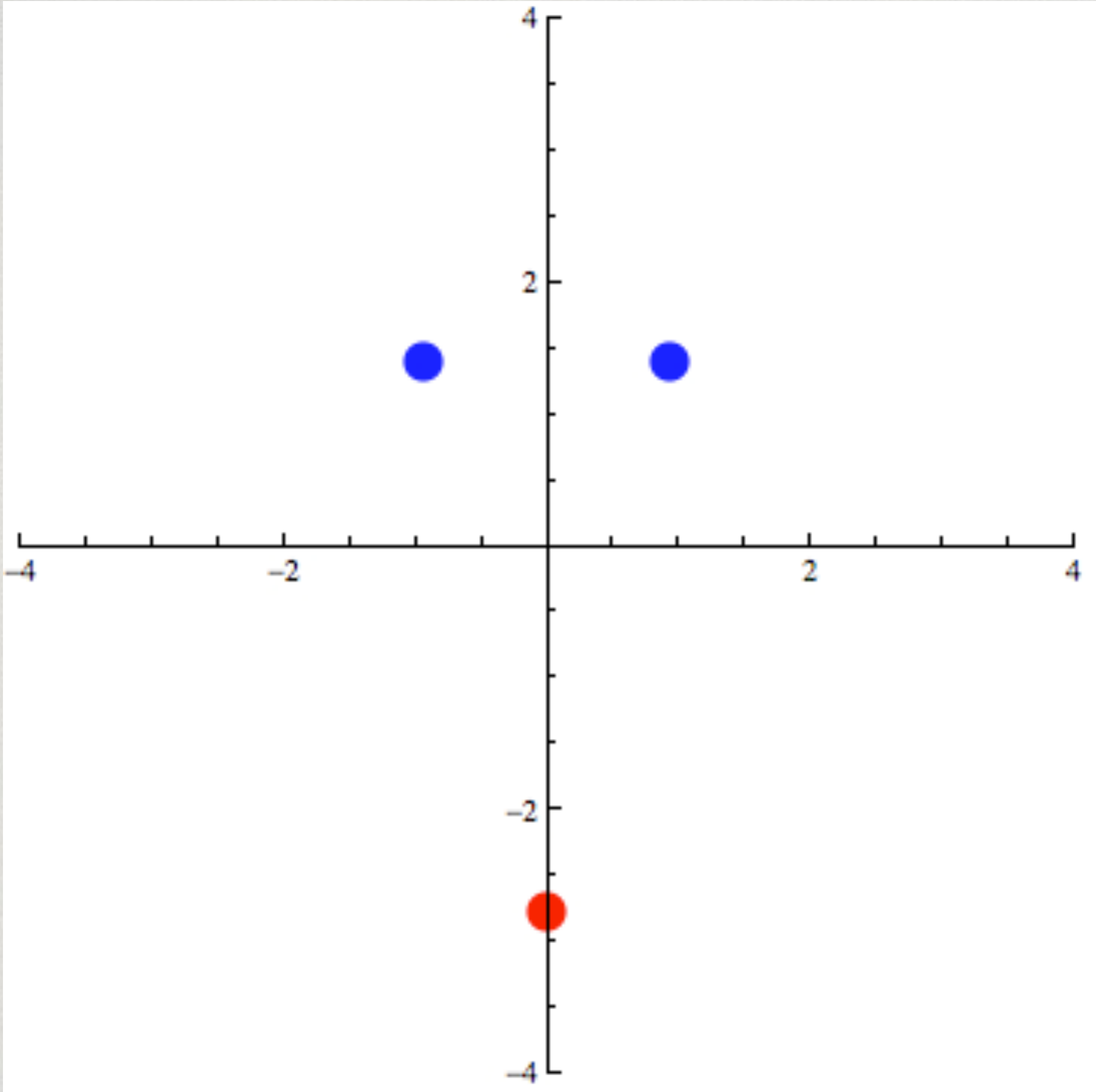
$$\mathcal{M}(h) = \tilde{\mathcal{M}}(h) \qquad \dim = 3$$

The lower dimension makes it possible to draw some nice pictures. Also, some of the features are useful for the discussion of the planar problem.

Some more fun: examples of orbits of the isosceles flow.

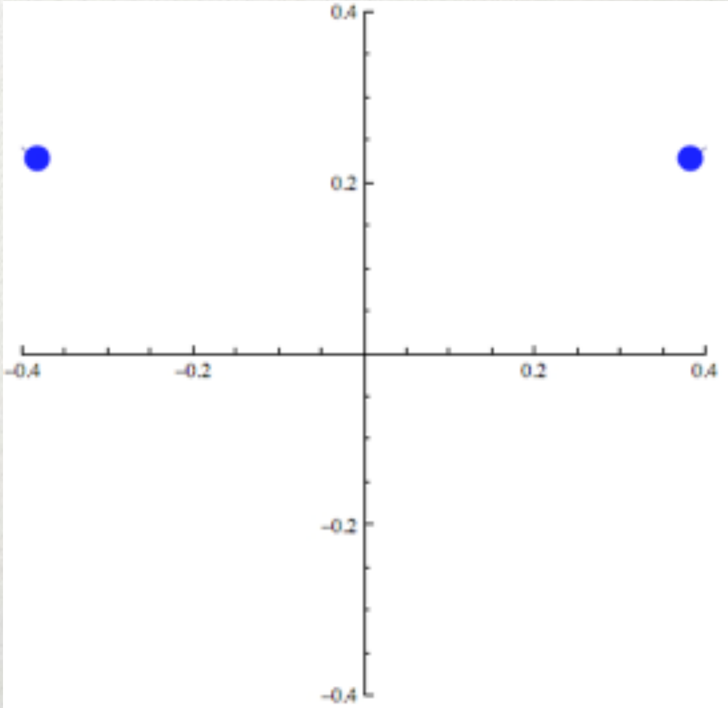


Simple periodic orbit of the isosceles 3BP. This orbit has binary collisions, which have been regularized.

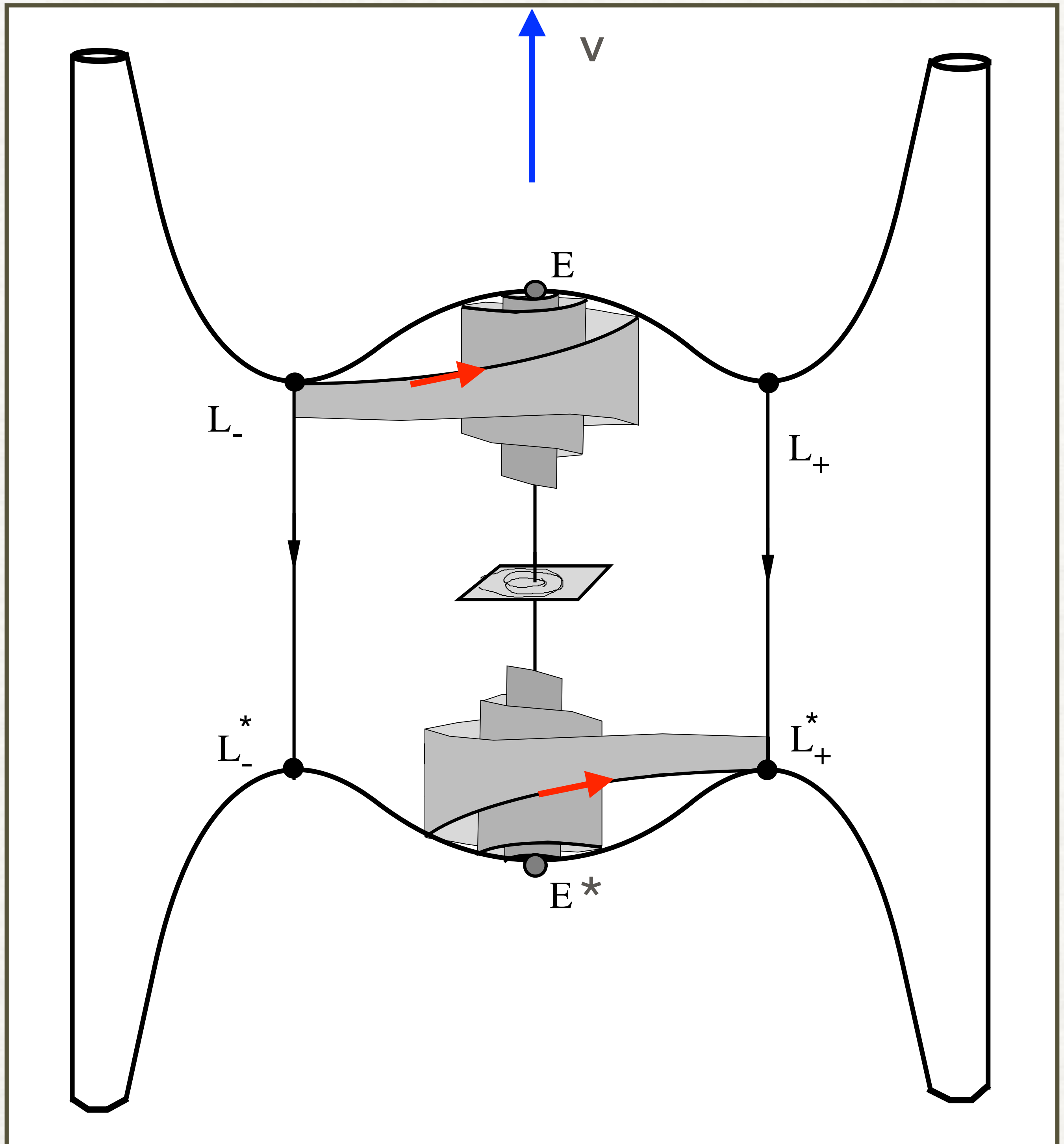


Isosceles periodic orbit with binary collisions and close approach to triple collision

Zoom showing behavior near triple collision



Collision Manifold for the Isosceles Problem



Studied by Devaney, Simo, R.M., ...

Features:

- 3 / 5 CC and 6 /10 restpoints at collision
- 3 homothetic connections
- dim = 3 (the inside of r=0 surface)
- Equilateral: $\dim W^s(p_-) = \dim W_u(p_+) = 2$
- Collinear: $\dim W^s(p_-) = \dim W_u(p_+) = 1$
- Gradient like on surface r=0
- Transverse restpoint connections collinear to equilateral in r=0 due to gradient-like flow $L_\pm \Longrightarrow E \quad E^* \Longrightarrow L_\pm^*$

These connections continue to exist and be transverse for almost all masses in the planar 3BP.

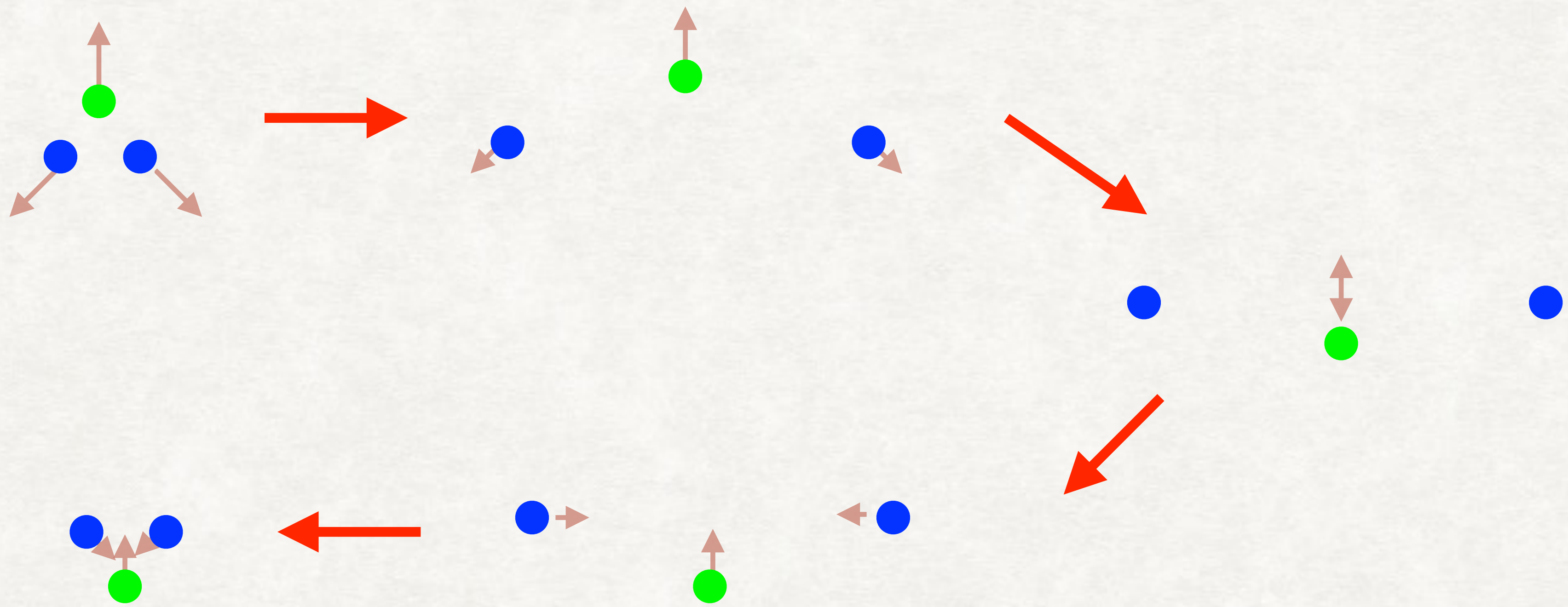
Change of notation for restpoints: p for v<0 and p for v>0*

Poincaré Section along Collinear Homothetic Orbit

The equilateral triple collision surfaces $W^s(L_+^*), W^s(L_-^*), W^u(L_+), W^u(L_+)$ form scrolls around the collinear homothetic orbit. In a Poincar'e section along this orbit, they form spiraling curves. This forces the existence of infinitely many heteroclinic connections

$$L_+ \Longrightarrow L_+^* \qquad L_+ \Longrightarrow L_-^* \qquad L_- \Longrightarrow L_+^* \qquad L_- \Longrightarrow L_-^*$$

The corresponding solutions emerge from an equilateral triple collision, oscillate around the collinear configuration and end at an equilateral triple collision. For example, a connection $L_+ \Longrightarrow L_-^*$ looks like



Note: These orbits do not involve binary collisions

What is a spiral (2D) ?

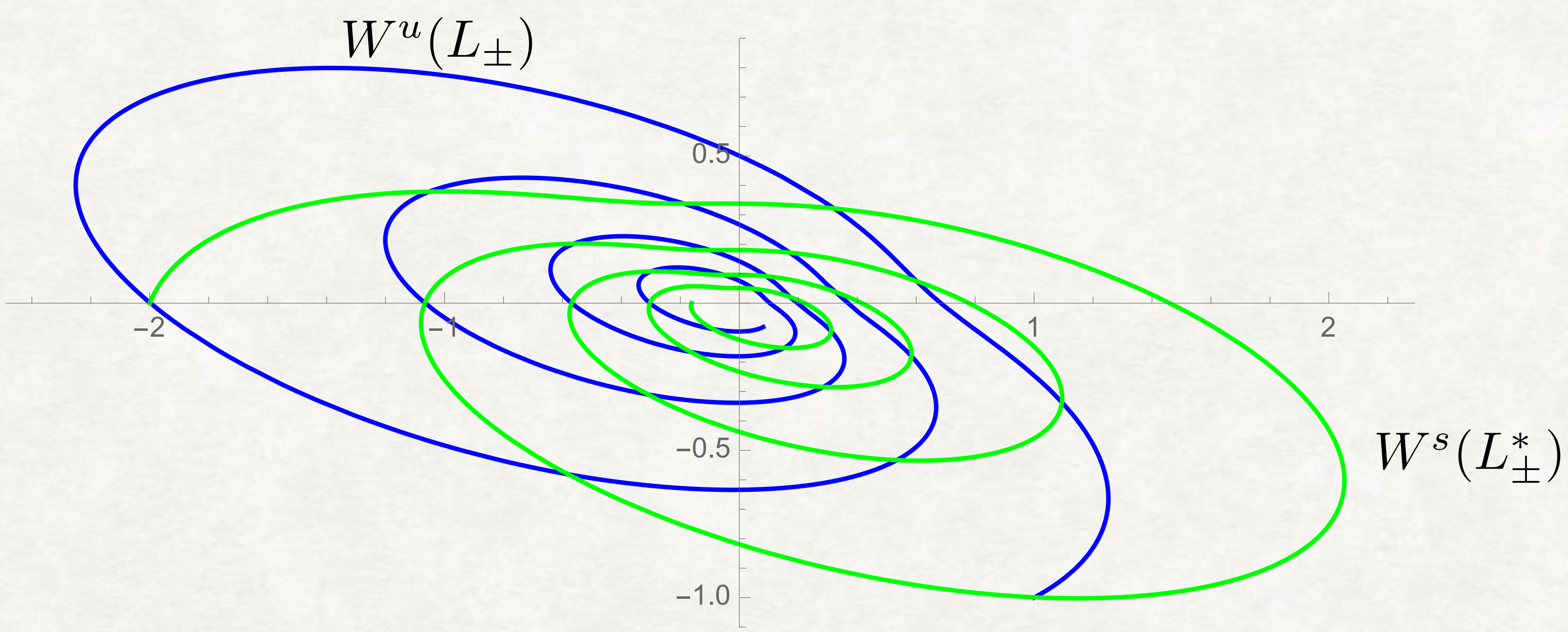
To make a rigorous existence proof one needs to be precise about what kind of spiraling occurs and how this produces crossings. Here is a definition for curves spiraling around a point.

Definition A curve in the plane is a spiral around the origin if there is a neighborhood $\mathcal{U}$ of the origin such that the part of the curve in $\mathcal{U}$ can be parametrized in polar coordinates as $r(\theta)$ with

$$r(\theta) \to 0 \qquad \frac{dr}{d\theta} \to 0$$

as $\theta \to \infty$ (counterclockwise) or as $\theta \to -\infty$ (clockwise).

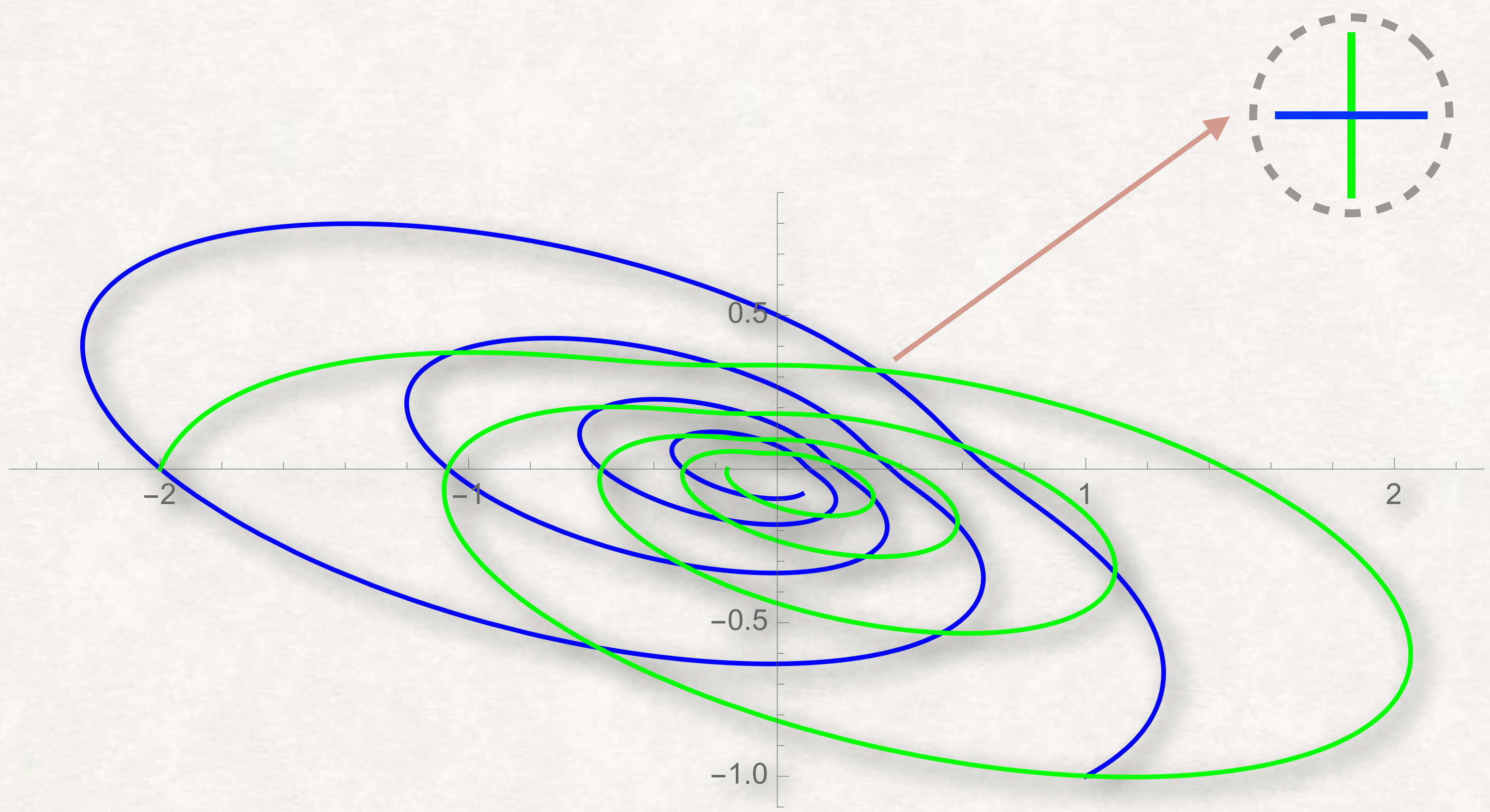
This can be shown to be invariant under diffeomorphisms (important for spirals in flows).



Topological Transversality (2D)

Definition An intersection point p of two plane curves is *topologically transverse* is there is a local homeomorphism taking p to the origin and the curves to the coordinate axes.

For example, the intersection of the x-axis and the graph of x^3 is topologically transverse but the intersection with the graph of x^2 is not. The spirals from the isosceles 3BP are real analytic and it can be shown that two oppositely oriented spirals must have infinitely many topologically transverse intersections.



Spiralling Surfaces in 4D ?

To generalize the result about existence of infinitely many heteroclinic orbits to the planar 3BP we need to understand spirals in 4D.

$$W^s(L_+^*), W^s(L_-^*), W^u(L_+), W^u(L_-)$$

have dimension 3. Their intersections with a 4D Poincaré section are 2D surface. We want to say that these 2D surface are spiraling and that this produces infinitely many topologically transverse intersections.

To have a spiral one needs something to spiral around:

- Codimension-two submanifold Y to spiral around
- Polar coordinates in a tubular neighborhood of Y

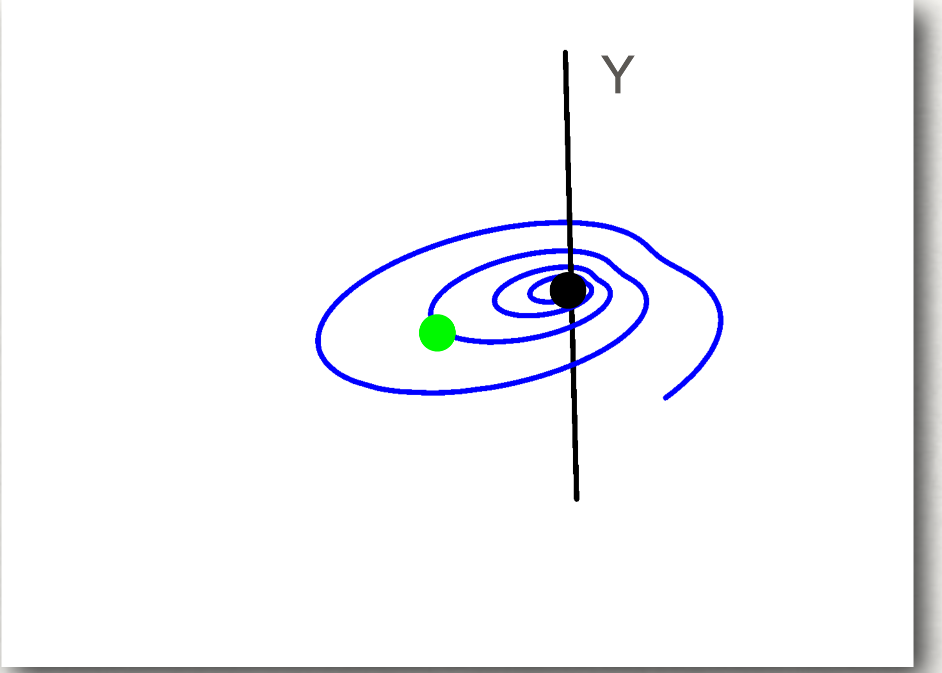
Then one can define:

- Spiraling surface intersects slices of constant angle in curves converging smoothly to a limit curve C (= Core) inside Y

Example of spiralling curve in R^3

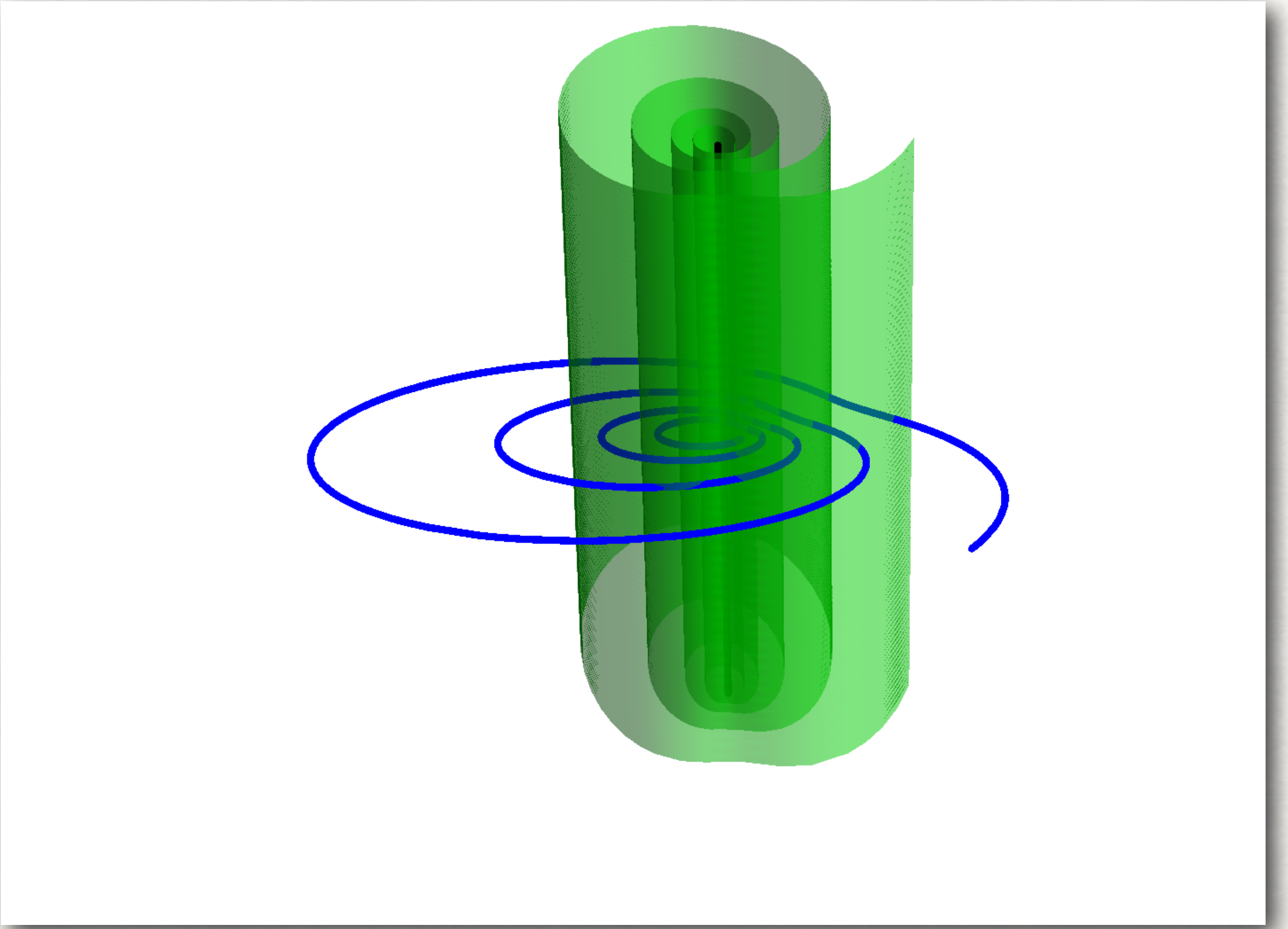
Slice

Core

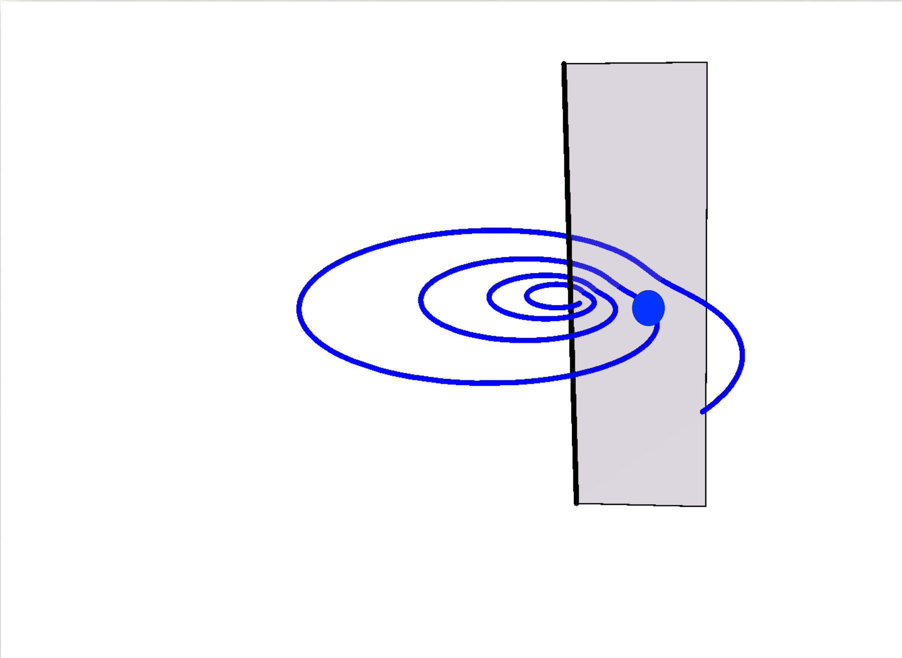


Intersecting Spirals — 3D

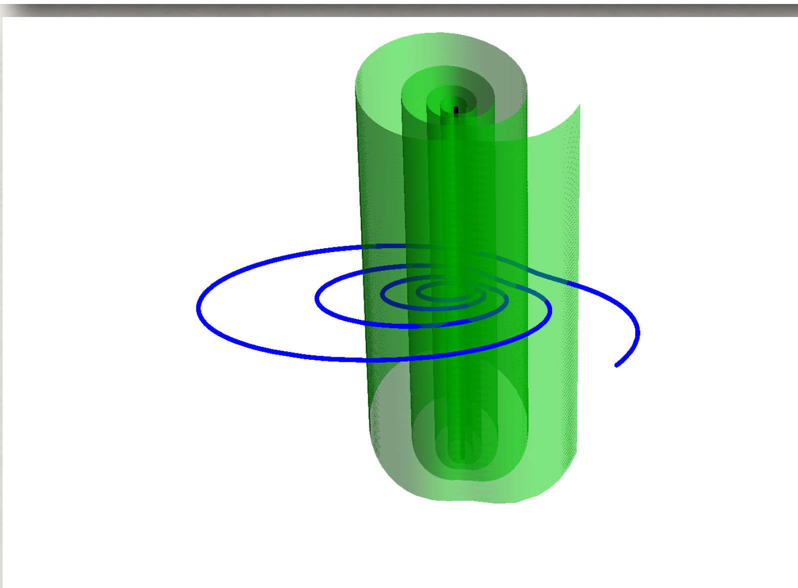
Before discussing spiraling surfaces in 4D, consider the analogous problem in 3D. Suppose we have a spiraling curve and an oppositely oriented spiraling surface, both spiraling around the same curve Y in R^3 (codimension two). Intuitively, there should be infinitely many crossings (topologically transverse intersections). One can understand this better by taking slices …

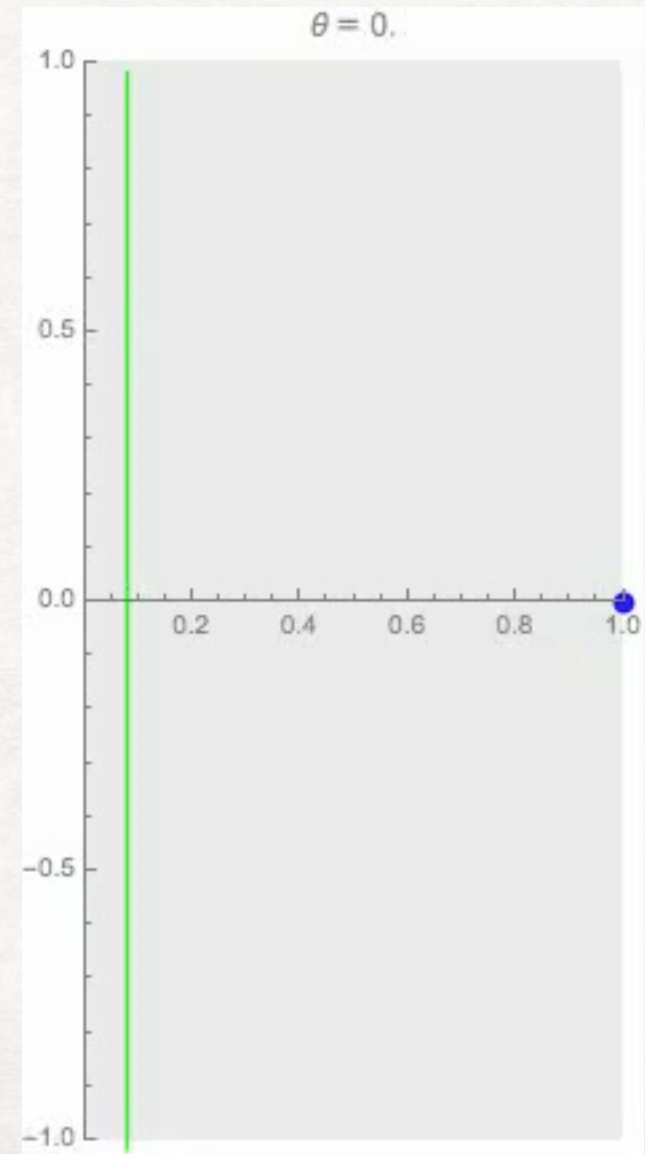


Spirals in 3D — the movie



Intersect the spiral with a plane θ = constant. Using the parametrizations, the spiraling curve will be seen as a point and the spiraling surface will be a curve. Actually many points and curves, but choose one.





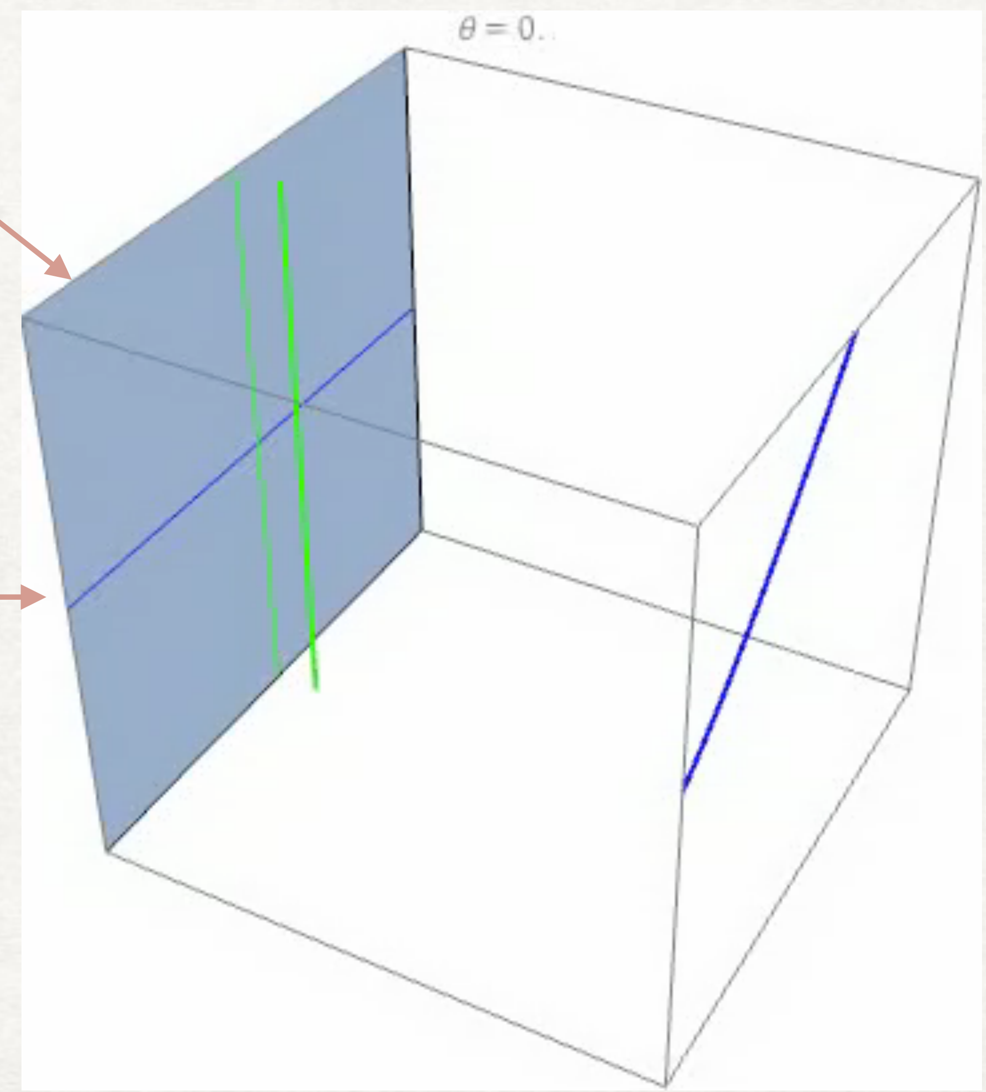
As θ varies, the point moves "in" and the curve moves "out" producing a topologically transverse intersection.

Spirals in 4D — the movie

Now consider two spiraling surfaces in R4, spiraling around a common surface Y (codimension two). Using the parametrizations, their intersections with a plane θ = constant can be repesented by two curves. If the spirals are oppositely oriented, then as θ varies, one curve moves "in" and the other "out". Assume, in addition that the Cores of the spirals (the curves in Y to which they converge) are transverse. Then the two moving slice curves must cross to produce a topologically transverse intersection of the surfaces.

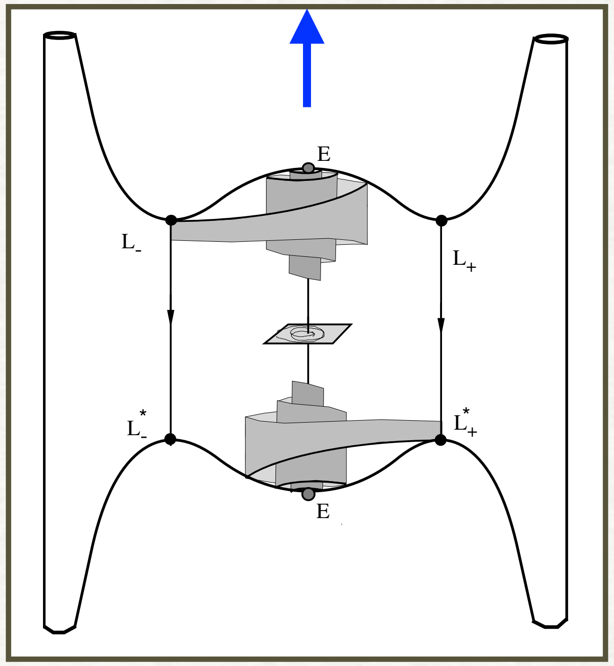
Codimension two manifold Y

Core curve of the Blue surface



The Spirals in the 3BP

Isosceles problem: In the 2D Poincaré section, Y is the collinear point, and the spirals are the stable and unstable manifolds of equilateral triple collision.



$$W^s(L^*_\pm) \qquad W^u(L_\pm)$$

Planar problem: Dimension are higher

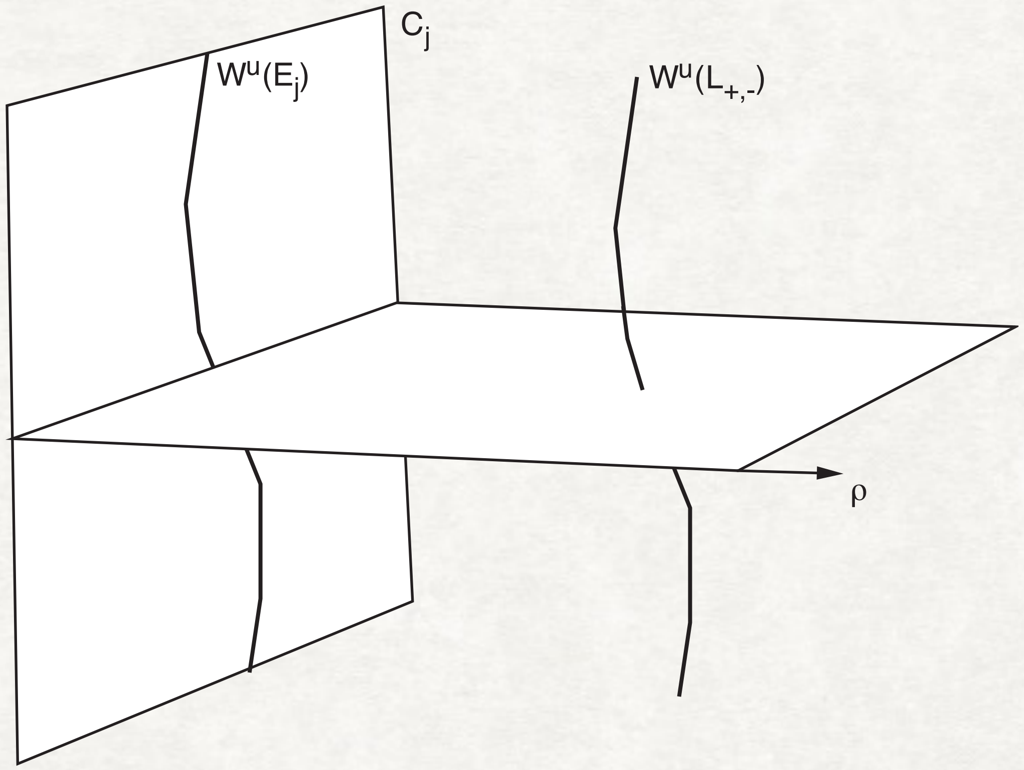
- Poincaré section is 4D
- Y = Collinear 3BP submanifold is 2D
- $W^u(L_\pm)$ are spiraling surfaces around

Y with core $W^u(E)$

- $W^s(L^*_\pm)$ are spiraling surfaces around

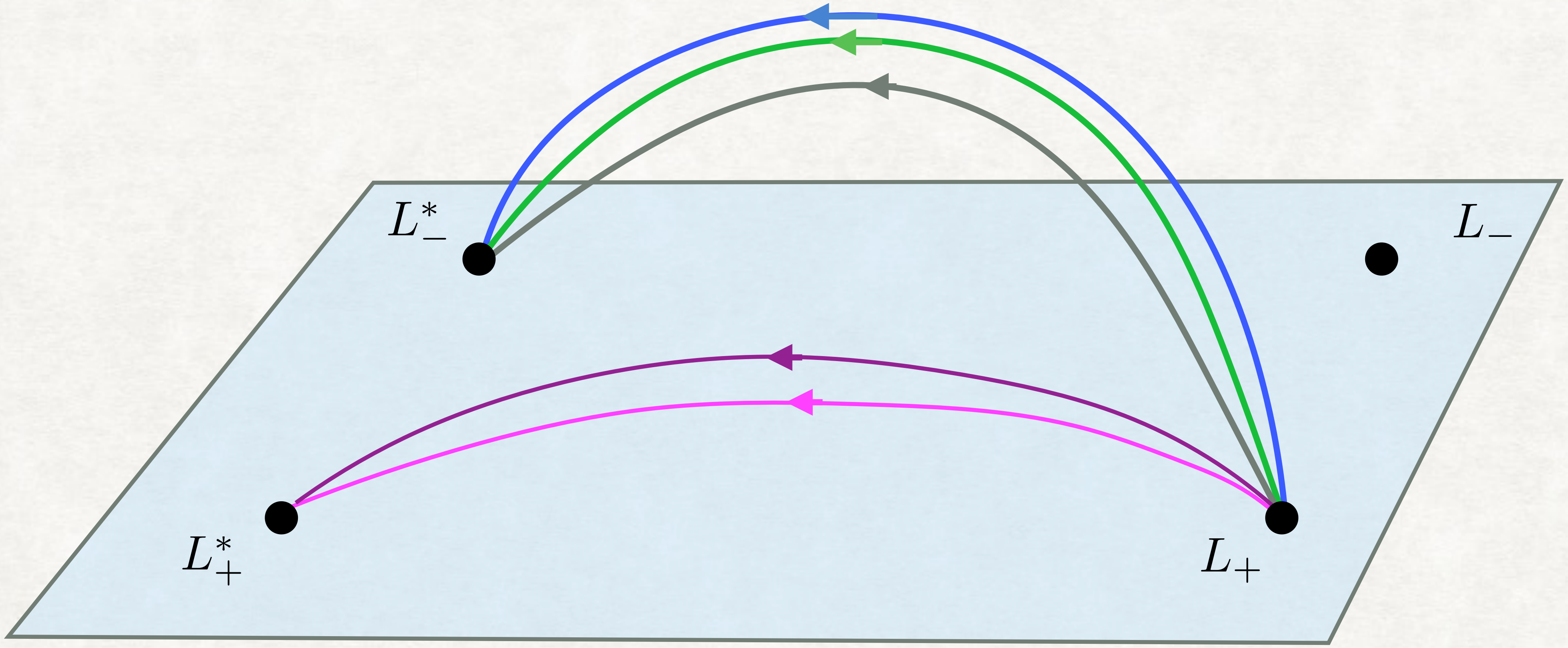
Y with core $W^s(E^*)$

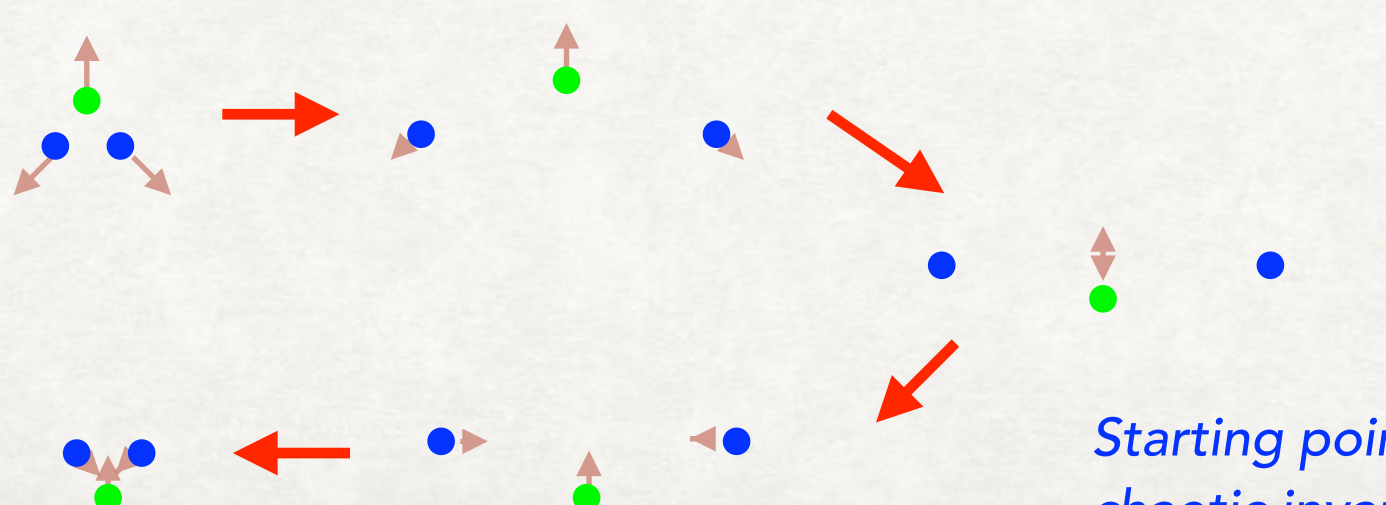
- the core curves are transverse (Devaney)



3D slice **θ** *= constant in the Poincaré section*

Theorem: Consider the zero angular momentum planar 3BP. For almost all masses such that spiraling occurs near the collinear restpoints, there exist infinitely many topologically transverse connections from each of $L_\pm$ to each of $L^*_\pm$. These can be found arbitrarily close to the collinear homothetic orbit.





Starting point for constructing a chaotic invariant set (next lecture)