Blowing Up the N-Body Problem III

Chaos in the Planar 3BP

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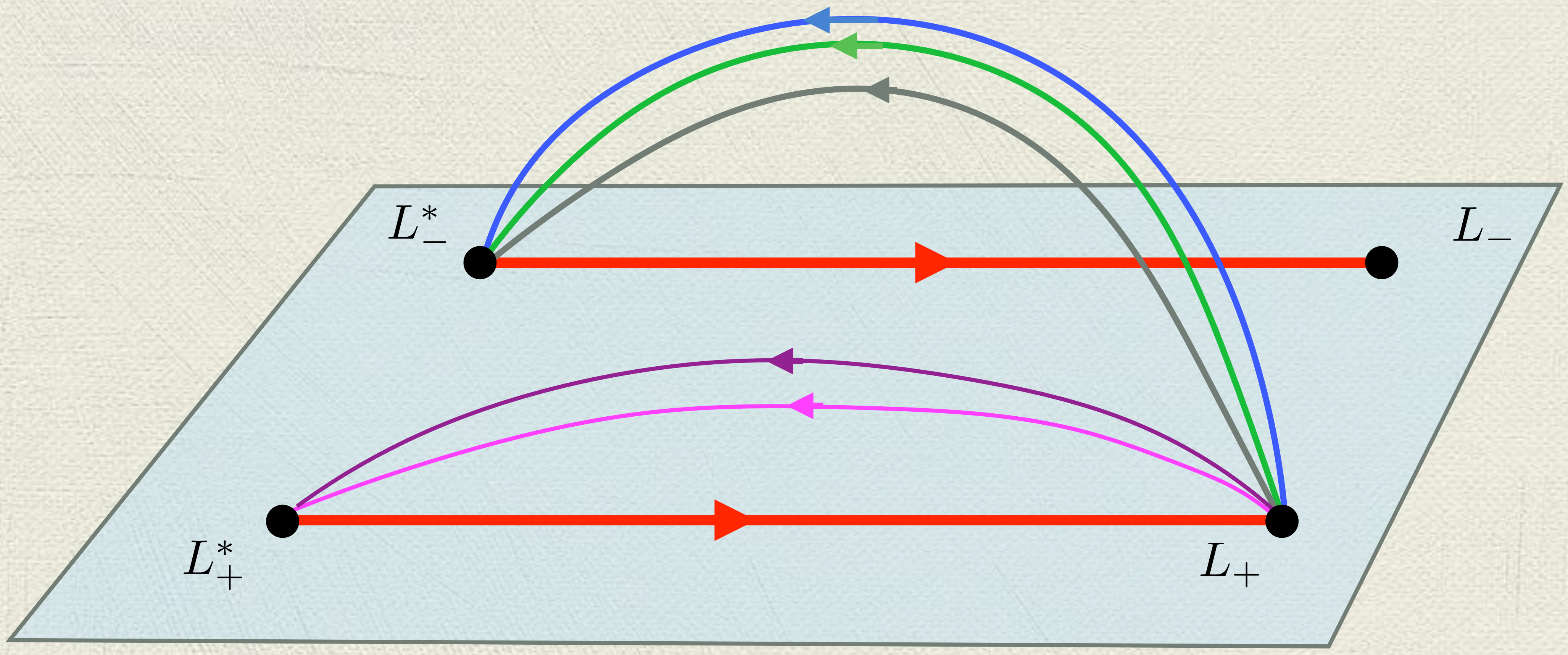
Last time: Zero angular momentum, planar 3BP.

- Hyperbolic restpoints at $r=0$
- Stable and unstable manifolds of triple collision orbits
- Infinitely many solutions beginning and ending at equilateral triple collision caused by spiraling at the collinear restpoints

Now we will see how by perturbing to small, nonzero angular momentum, we can produce a chaotic invariant set described by symbolic dynamics. The phase space has dimension 5, Poincaré sections have dimension 4. The symbolic dynamics is based on a network of 4D windows which are stretched across one another by the flow.

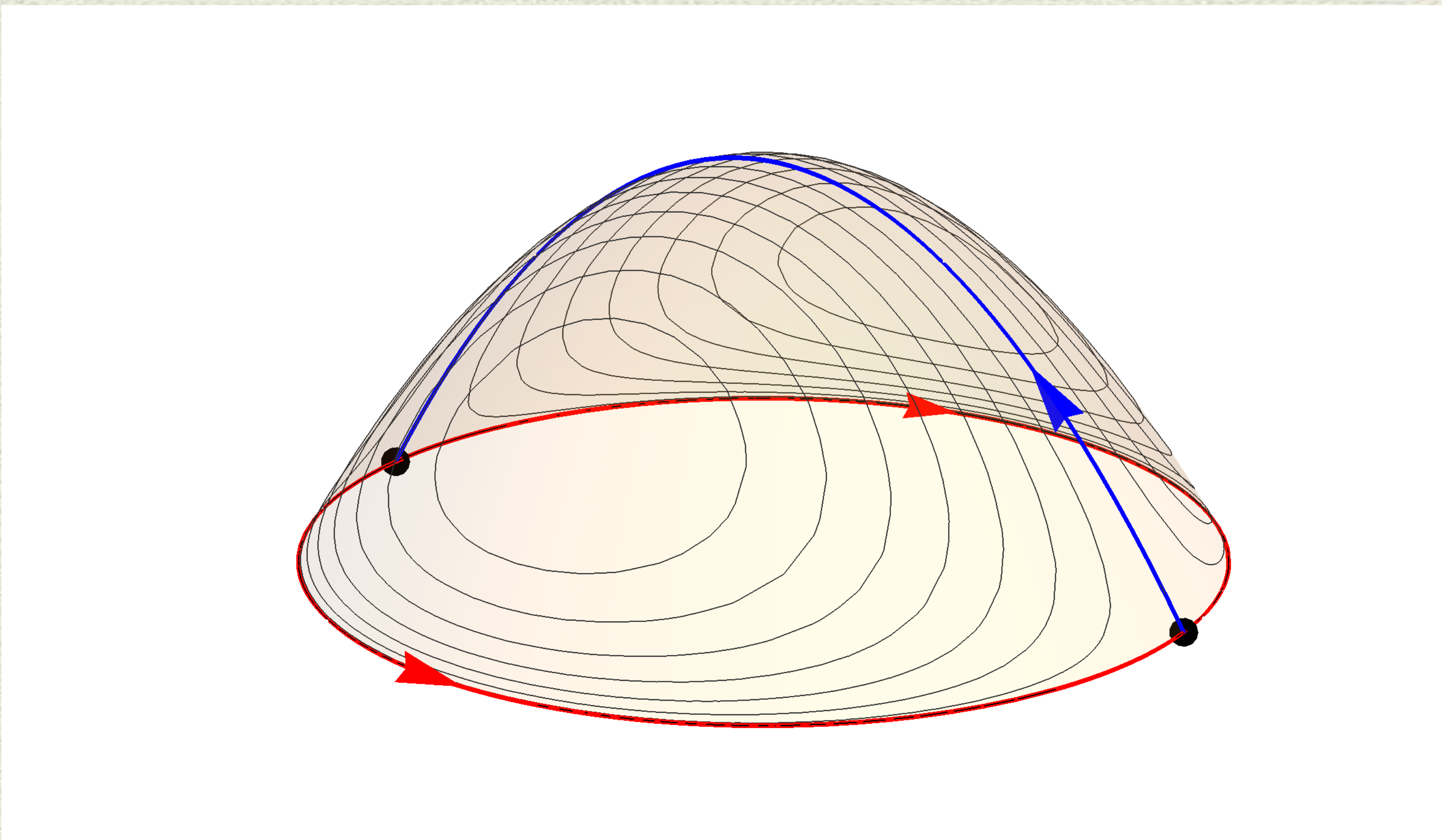
Zero Angular Momentum — Restpoint Cycles

We have restpoint connections $L_\pm \Longrightarrow L_\pm^*$ between the equilateral restpoints in the collision manifold. There are also connections in the other direction inside the collision manifold r=0. Together these cycles of restpoints make a framework for the recurrent phenomena when we perturb to nonzero angular momentum.



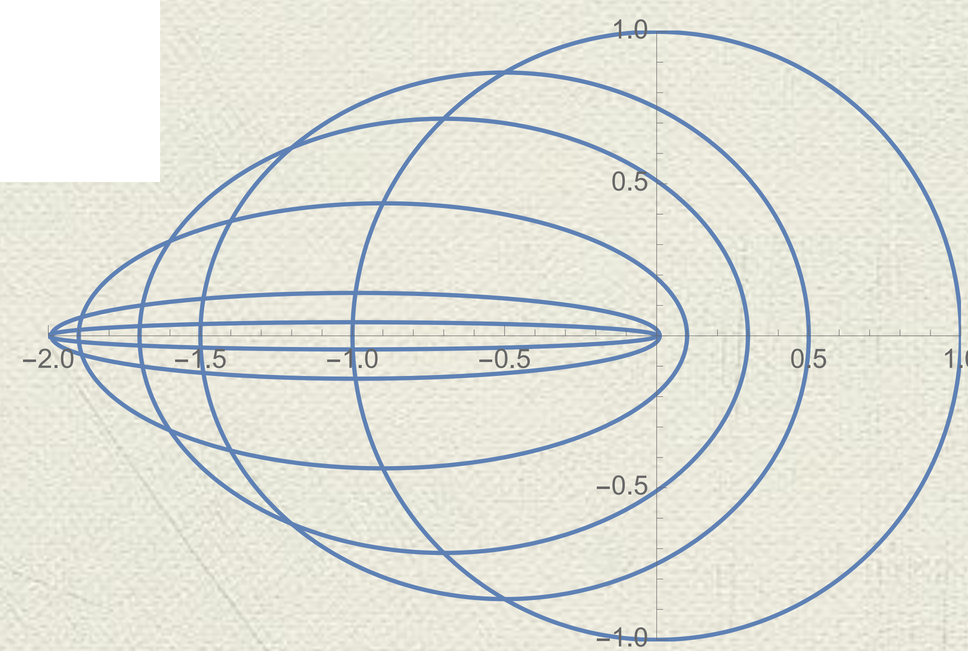
Restpoint Cycles in the 2BP for $\lambda = 0$

The negative energy 2BP already contains such a cycle of restpoints



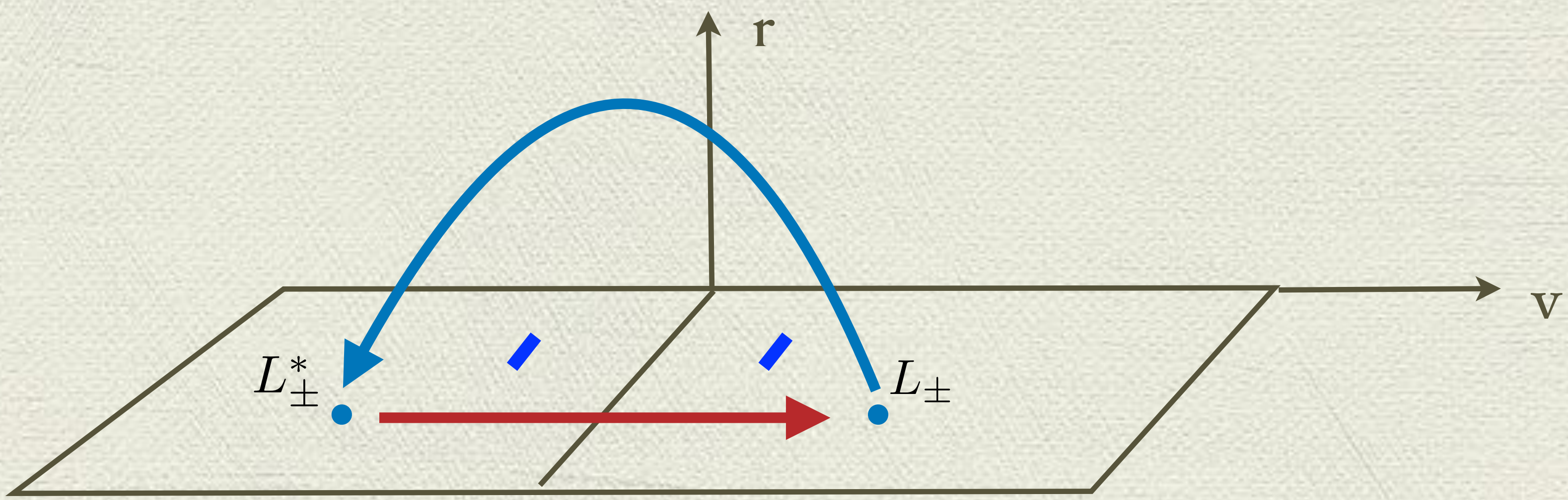
The flows on $\mathcal{M}(h, \lambda)$ converge to the restpoint cycles as $\lambda \to 0_+, \lambda \to 0_-$

The red orbits in r=0 represent the two bodies spinning around by 2π near collision.

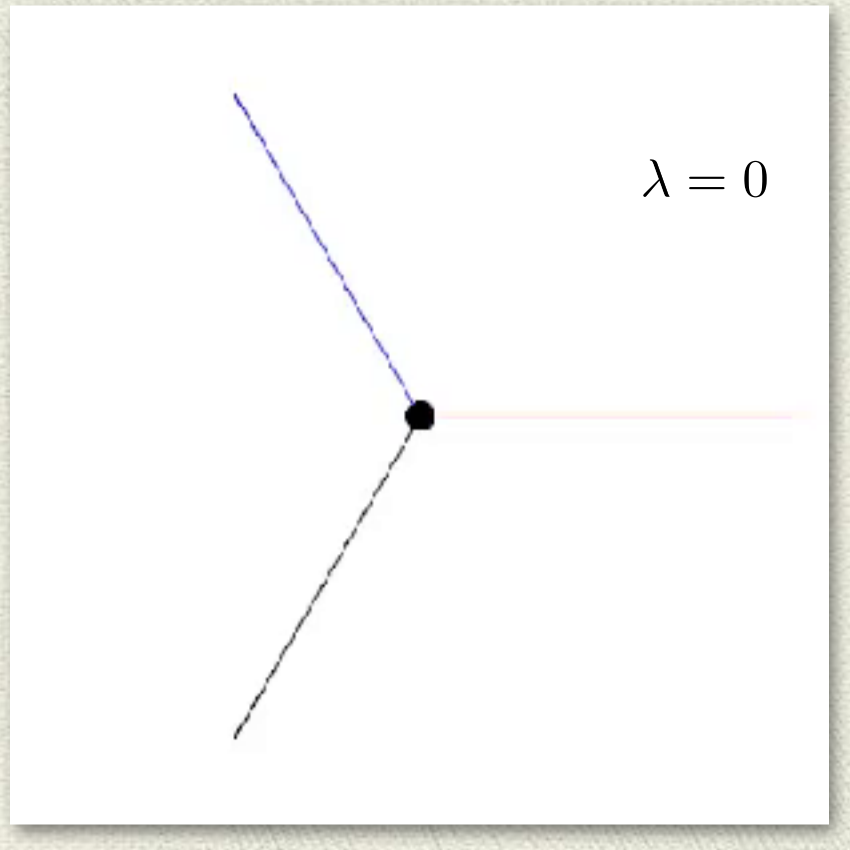


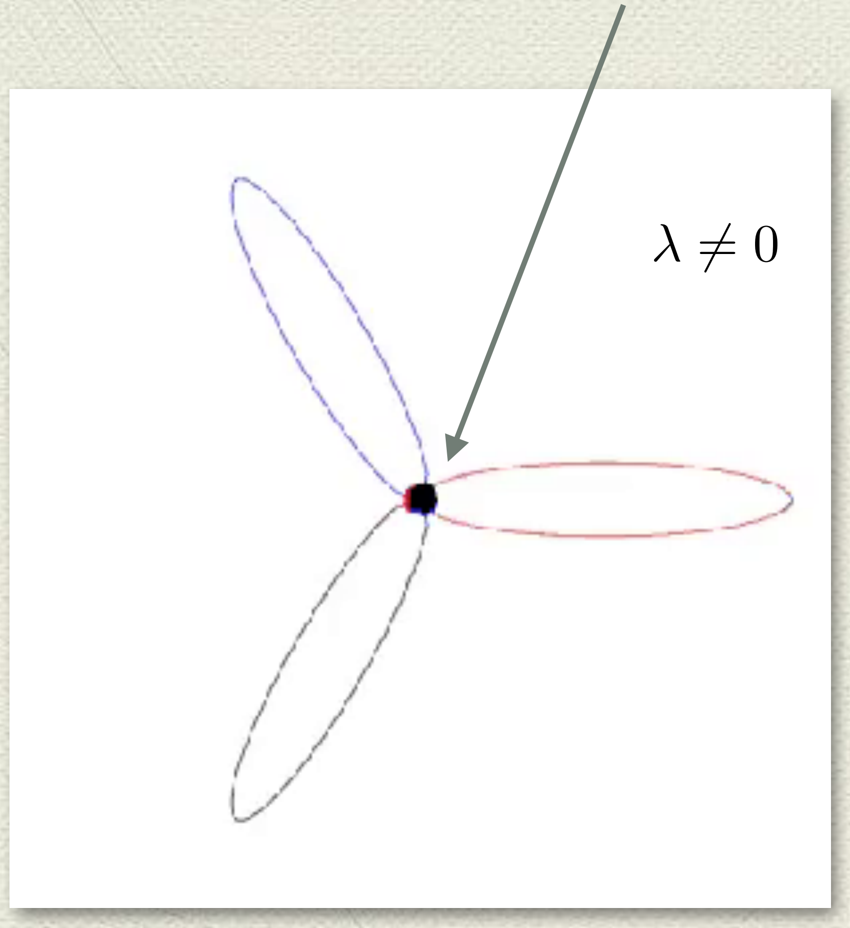
Planar 3BP — Homothetic Restpoint Cycle

Each of the five CCs has a similar restpoint cycle. Here is an equilateral one.



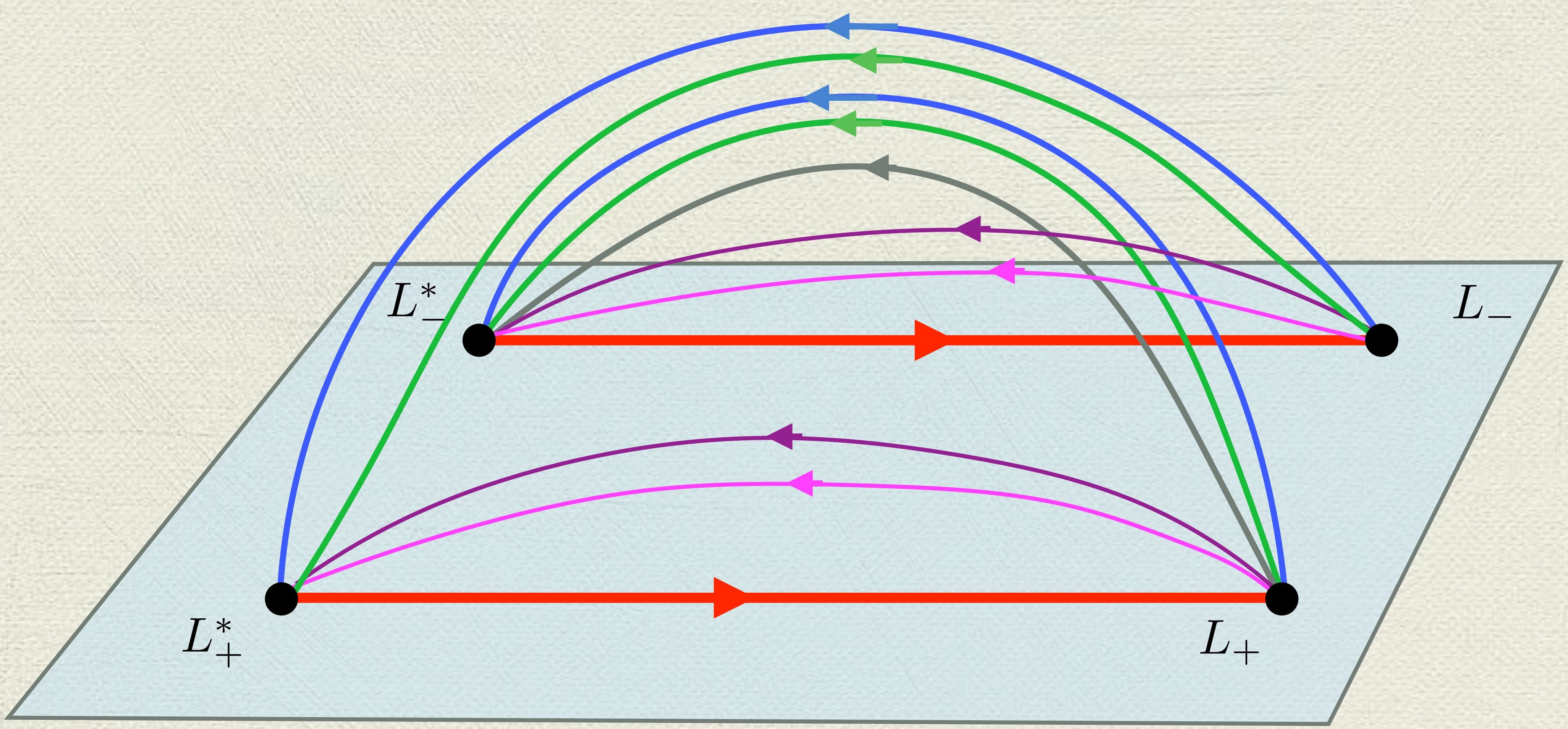
The red orbit in r=0 represent the equilateral triangle spinning around by 2π near collision.





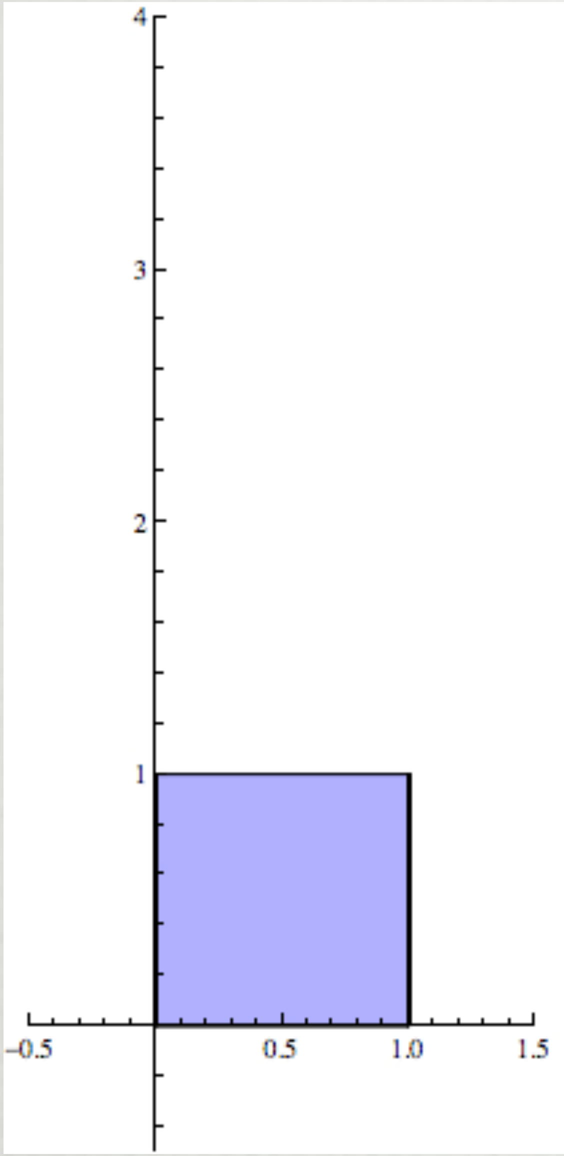
Zero Angular Momentum — Framework of connections

Of course for the 3BP we have many other restpoint connections $L_\pm \Longrightarrow L^*_\pm$ besides the homothetic ones.

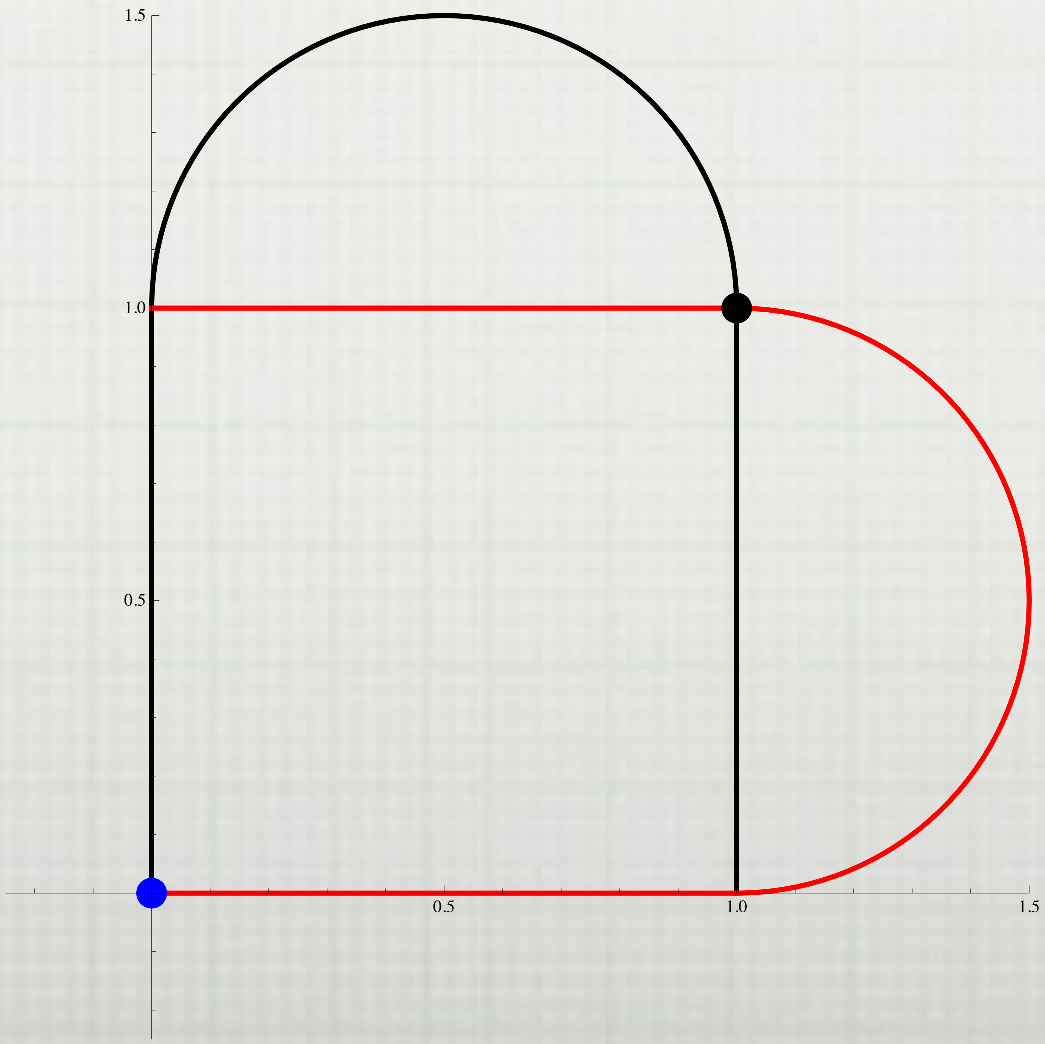


Goal: Show that for nonzero angular momentun, we can shadow any sequence of these connections to get a chaotic invariant set.

Digression on Symbolic Dynamics —The Smale Horseshoe

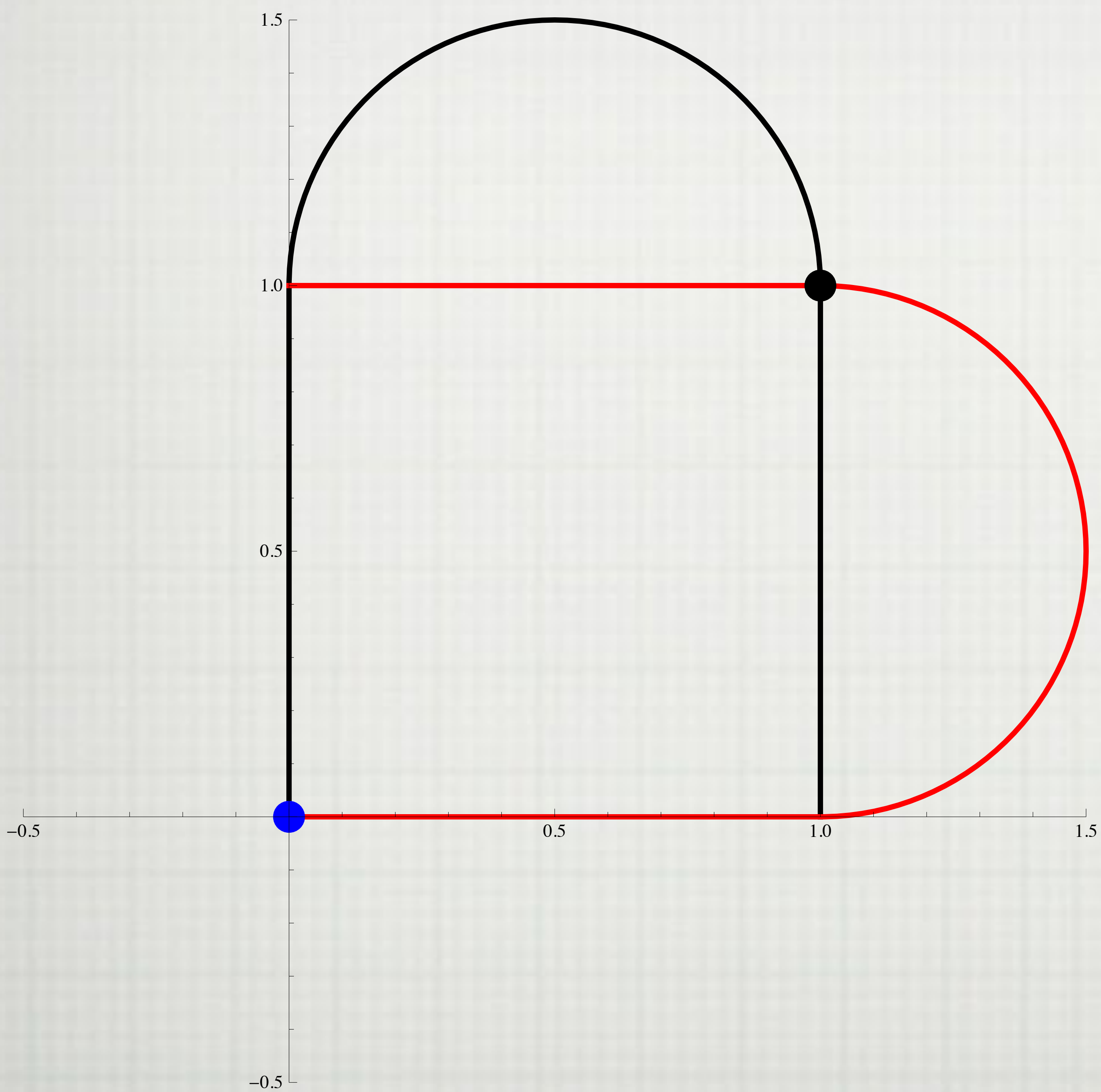


A simple example of shadowing and the use of symbolic dynamics to describe a chaotic invariant set is provided by Smale's horseshoe map



There is a hyperbolic fixed point at the origin with stable manifold (red) and unstable manifold (black) along the axes. Folding produces a transverse homoclinic point.

A Simple Trellise



Repeated folding and stretching of the invariant manifolds produces a beautiful trellise just as Poincaré imagined for the 3BP:

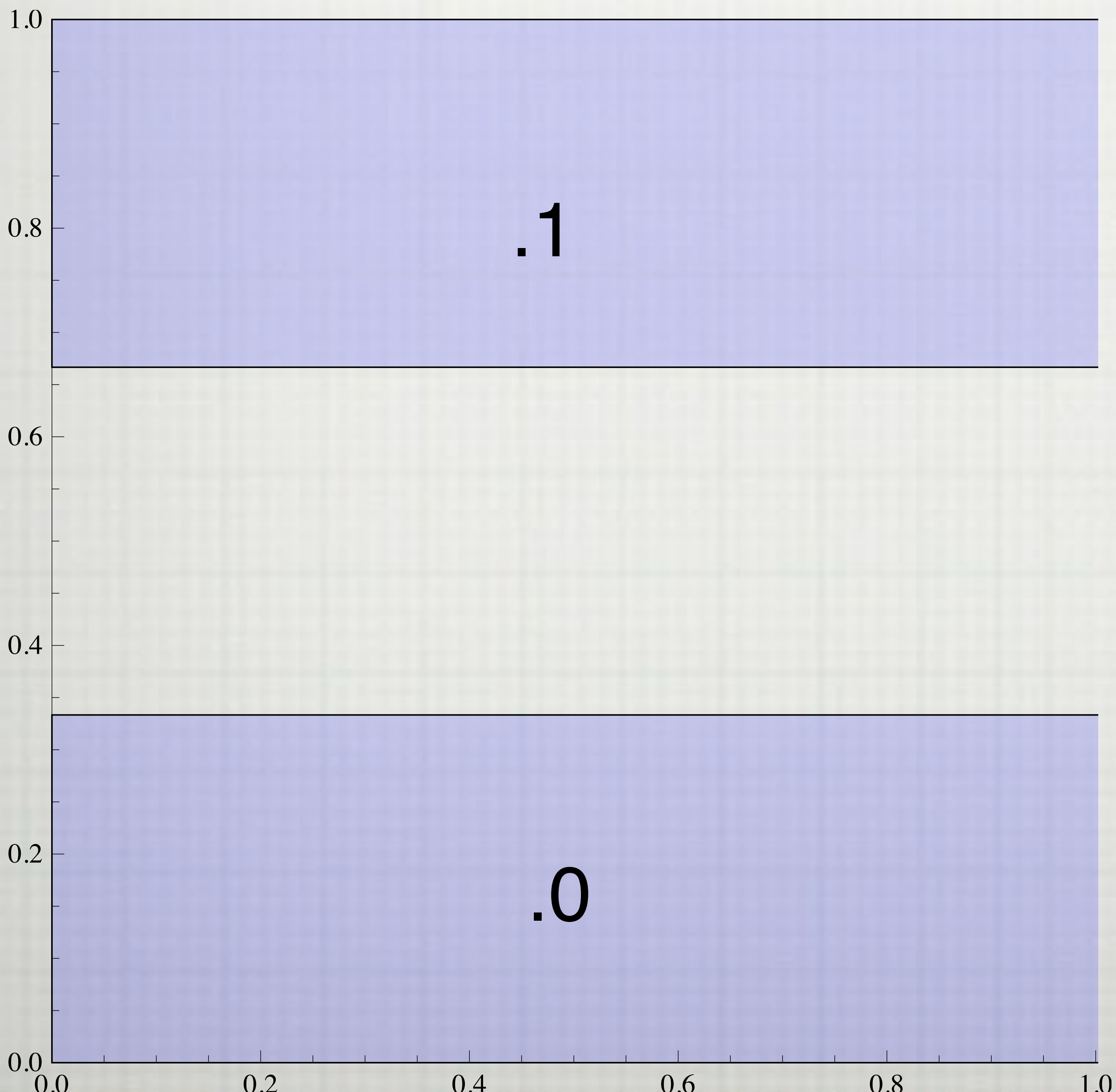
each of the two curves must not cross itself but it must fold on itself in a very complicated way to intersect all of the meshes of the fabric infinitely many times.

In particular we get infinitely many other intersections (bi-asymptotic orbits) nearby.

Symbolic Dynamics and Chaos

Poincaré may not have known the full story about the nearby dynamics. Nowadays we use symbolic sequences (itineraries) to code the trajectories.

$$\ldots \epsilon_{-2}\, \epsilon_{-1}\, .\epsilon_0\, \epsilon_1\, \epsilon_2 \ldots \qquad \epsilon_n = 0, 1$$



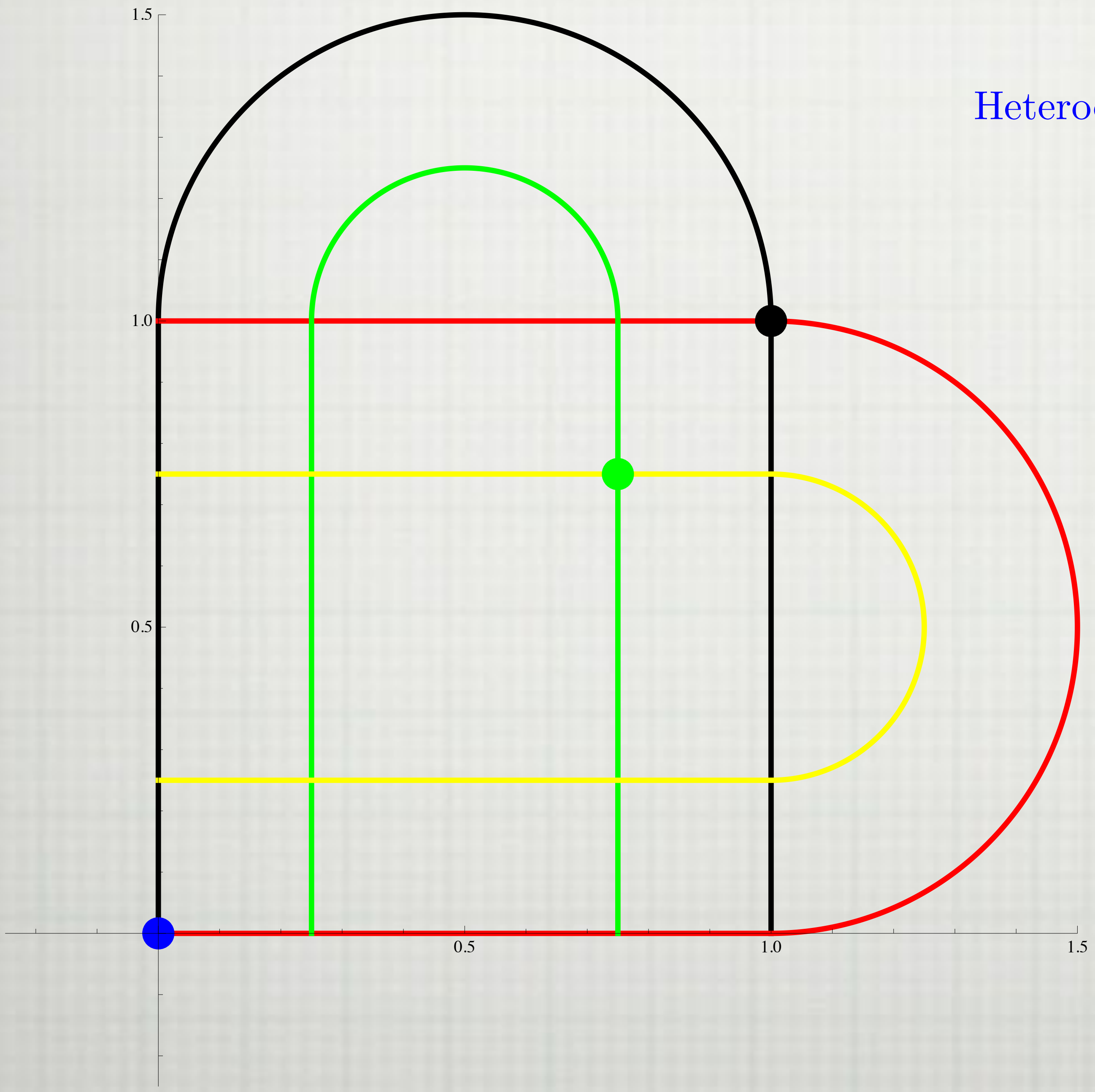
We have a one-to-one correspondence between orbits of the horseshoe map which remain in the unit square and bi-infinite sequences of 0′s and 1′s which describe how the orbit “hops” between box 0 and box 1.

This is an (uncountably) infinite set of orbits which includes all of the (countably infinite) intersection points of the stable and unstable curves but many other orbits as well.

All itineraries are realized by an orbit, even “random” ones produced, say, by a coin toss.

Different Kinds of Itineraries

Fixed Point $(0,0)$	$\ldots 00000.00000\ldots$
Homoclinic Point $(1,1)$	$\ldots 00001.10000\ldots$
Other Homoclinic Points	$\ldots 00000\epsilon_m \ldots \epsilon_n 00000\ldots$
Fixed Point $(\frac{3}{4},\frac{3}{4})$	$\ldots 11111.11111\ldots$
Heteroclinic $(0,0)\to(\frac{3}{4},\frac{3}{4})$	$\ldots 00000\epsilon_m \ldots \epsilon_n 11111\ldots$

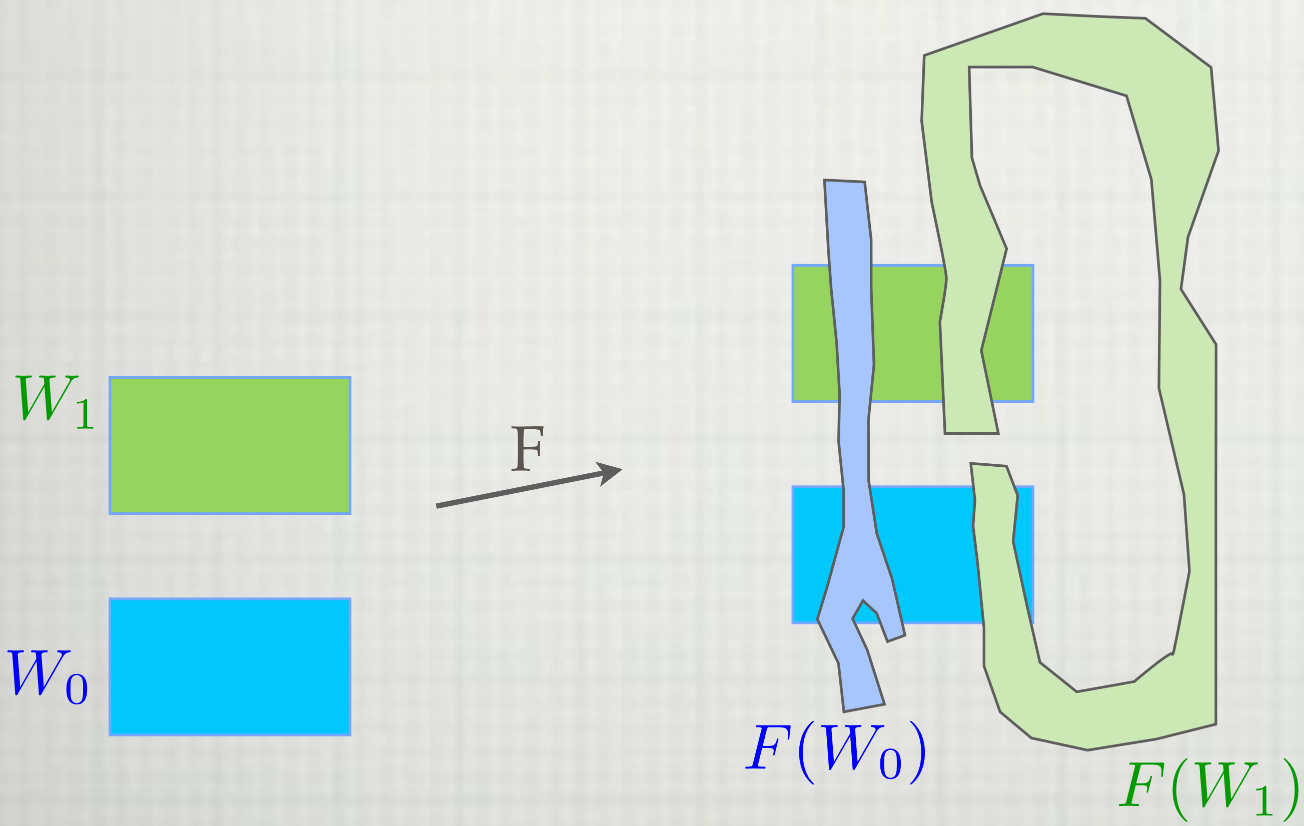


The fixed point (3/4, 3/4) has its own stable and unstable manifolds which form a separate trellise which interlace with the trellise of (0,0). For example the yellow and red curves must fold in such a way that they never cross.

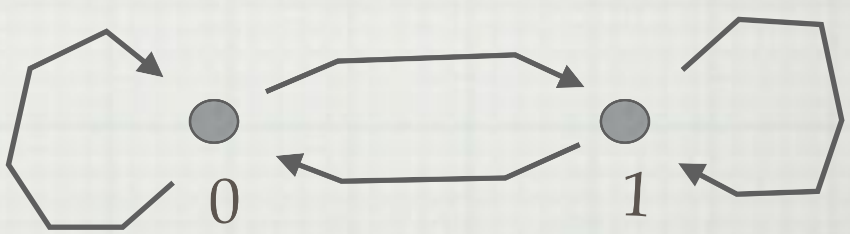
But the itineraries of all these point account for only countably many itineraries. The point is that all of the uncountably many itineraries are realized.

Windows and Connection Graphs

We can summarize the chaos in the horseshoe map by saying that there are two boxes or “windows” (Easton’s terminology) which are stretched across one another by the map. For the windows we will use in the 3BP, the windows will be 4-dimensional and stretching will not be as nice as in the horseshoe map. For example, we can set up windows when stable and unstable curves intersect non-transversely as long as they still “cross” (topological transversality).



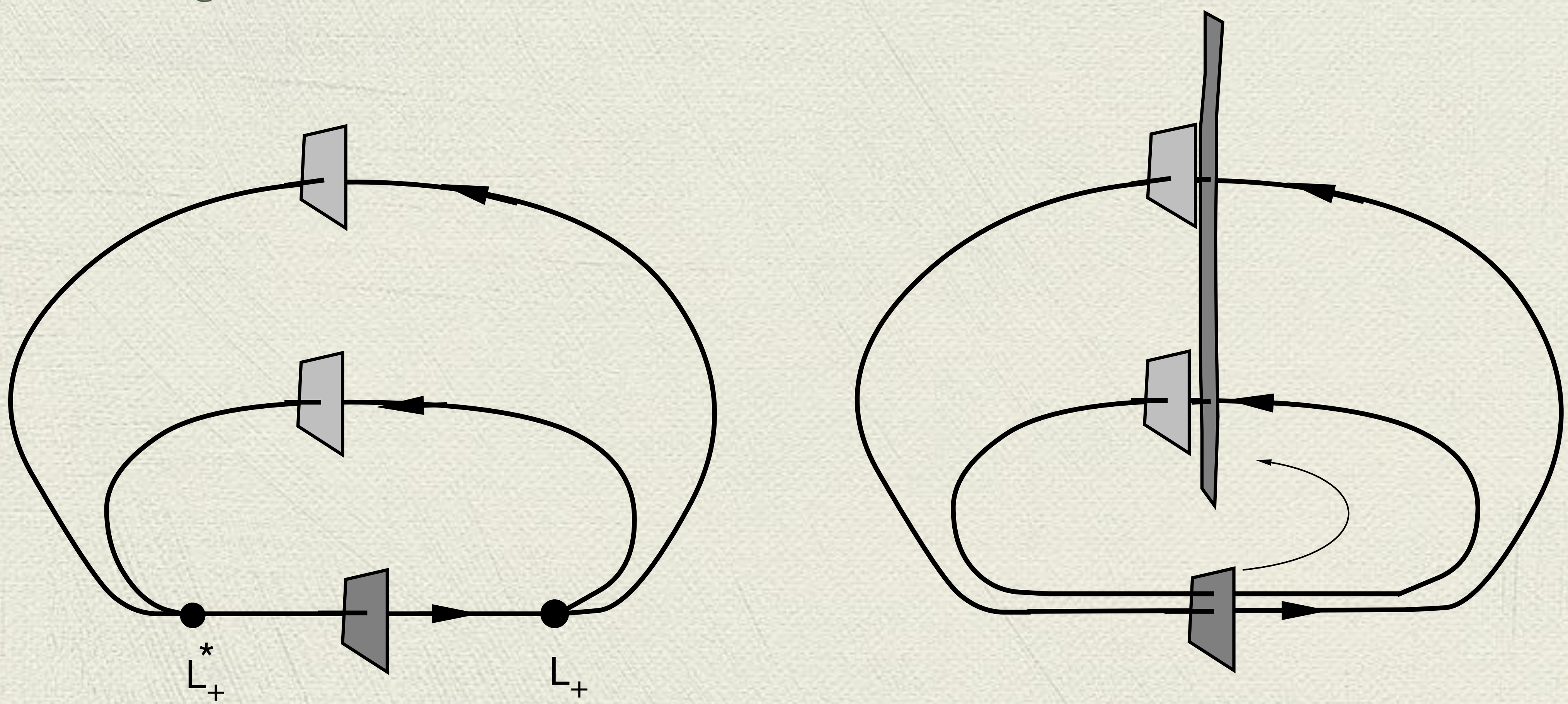
The situation is represented by a directed graph which describes which itineraries are realized by orbits:



Later will describe chaotic behavior in other situations using windows and connection graphs.

The Strategy for the Planar 3BP

Set up windows along the connecting orbits of our framework in the zero angular momentum problem. There will be many windows along the connections in $r > 0$ and one window along the connection in $r = 0$ (which represents the equilateral triangle spinning around by 2π. Then we will see that the corresponding windows in the nonzero angular momentum problem get stretched across one another as in the Smale horseshoe.

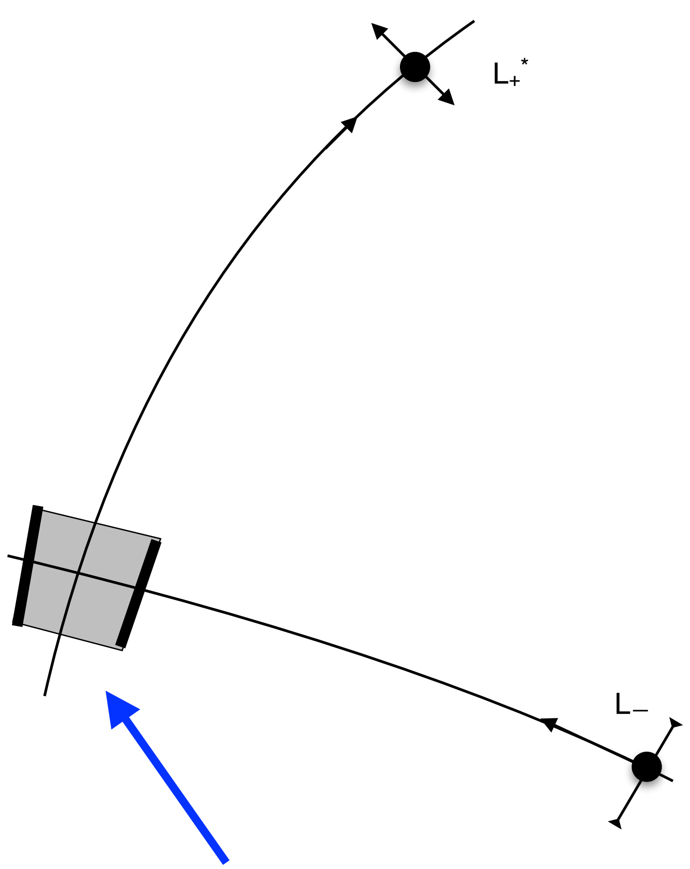


Zero angular momentum

Small nonzero angular momentum

Symbolic Dynamics with 4D "Windows"

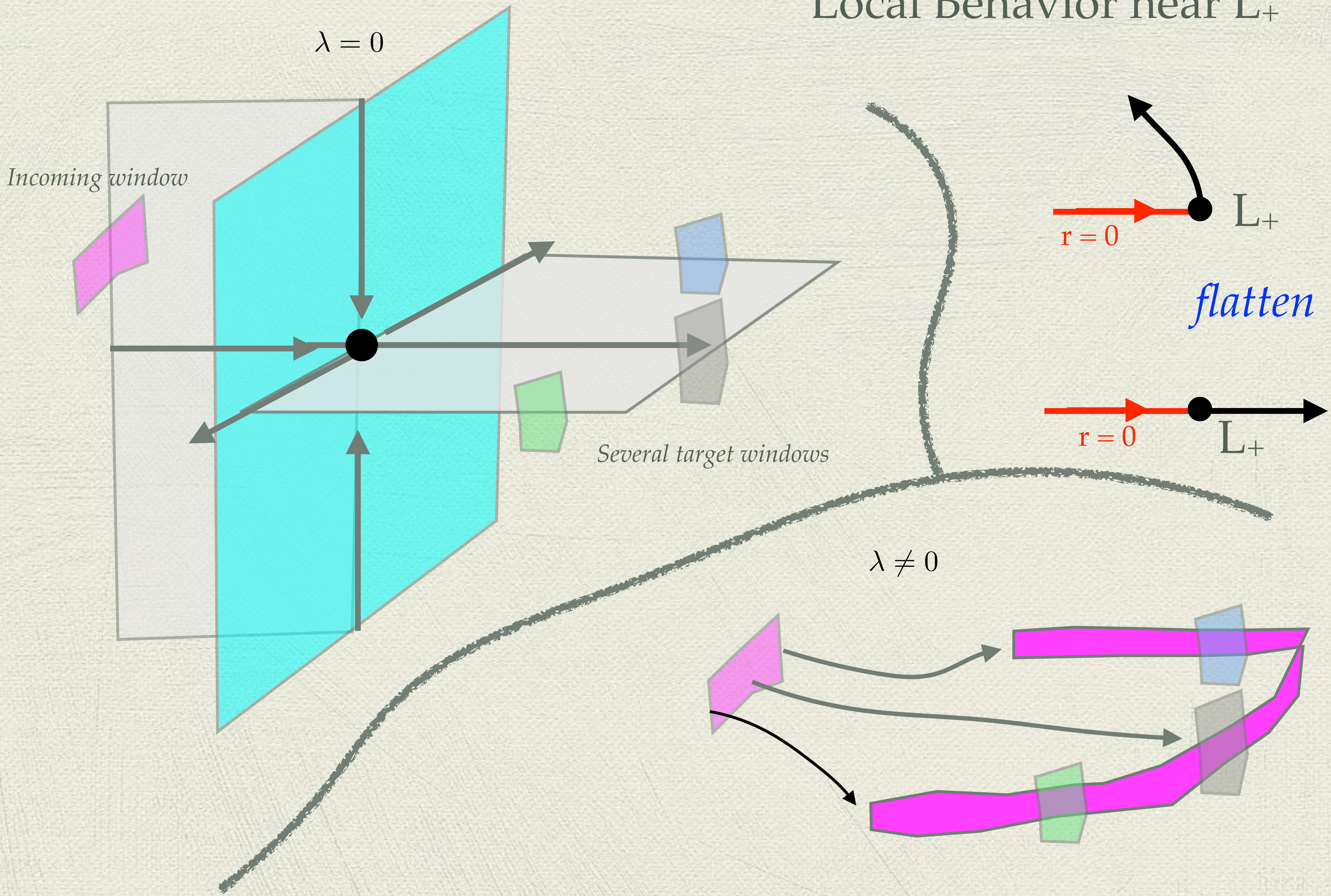
The energy manifolds $\mathcal{M}(h, \mu)$ are 5-dimensional and we can set up local Poincaré sections of dimension 4. We are going to set up a network of 4-dimensional boxes or *windows* which will be stretched across one another by the flow (as in the Smale horseshoe map). There will be one window near each of the restpoint connections in our framework. As for the horseshoe map, the stretching implies we can find orbits mapping through any given sequence of windows.



4D box aligned with $W^s(L_+^*)$ and $W^u(L_-)$

Viewed in a Poincaré section,
each restpoint connection is a transverse intersection of 2D stable and unstable manifolds.
Choose the 4D windows aligned with these.
Then the windows are stretched in a favorable way near the restpoints.
Two directions are hyperbolically stretched and two are contracted.

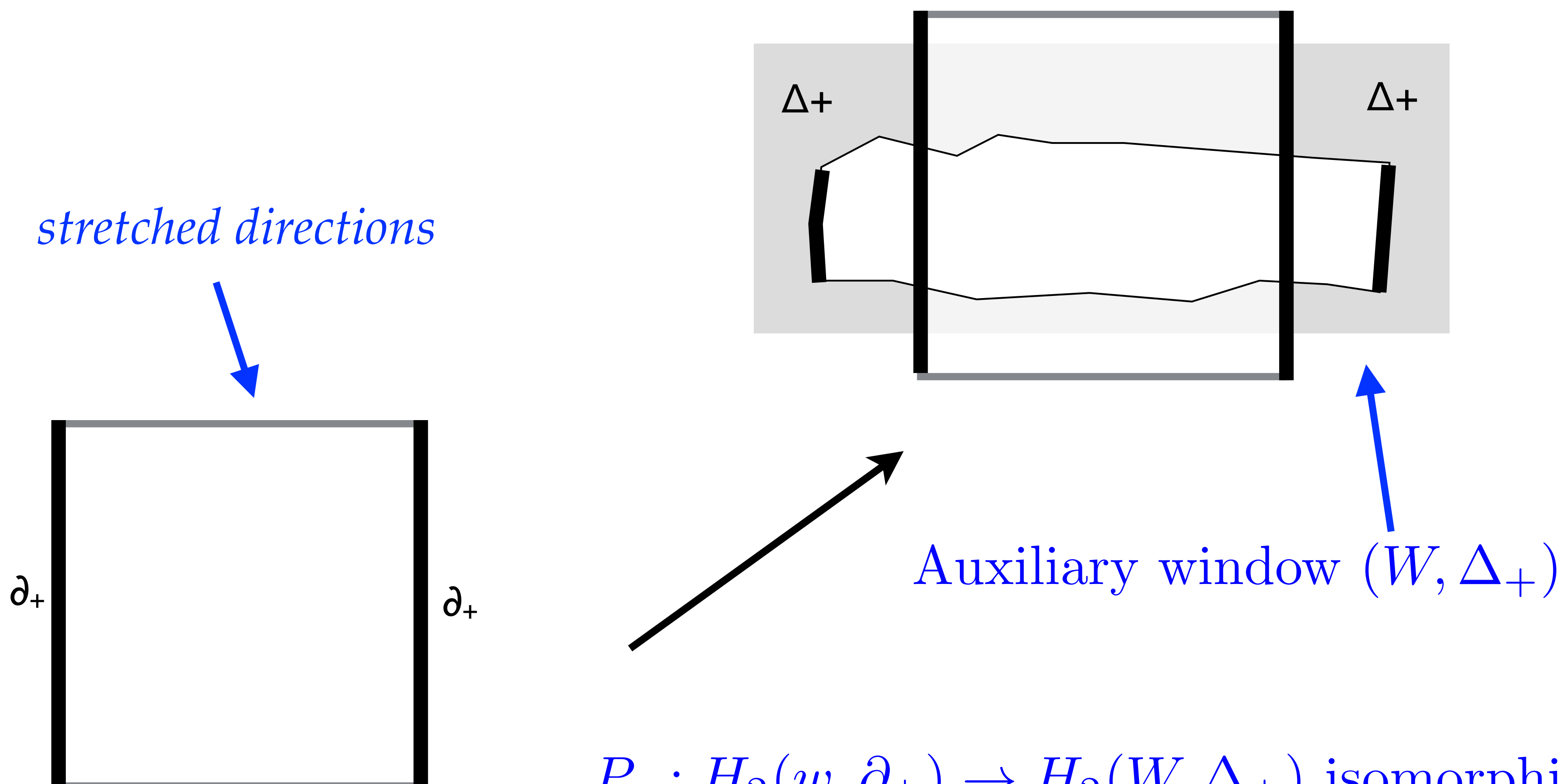
Local Behavior near L_+



Topological Stretching or Correct Alignment

A topological definition of what it means for one window to be stretched across another is easier to verify than the usual one and still implies the existence of orbits mapping from one box to the next (Easton developed a version of this idea). Our version is that the Poincaré map should induce an isomorphism of certain relative homology groups (and similarly for its inverse map, not shown).

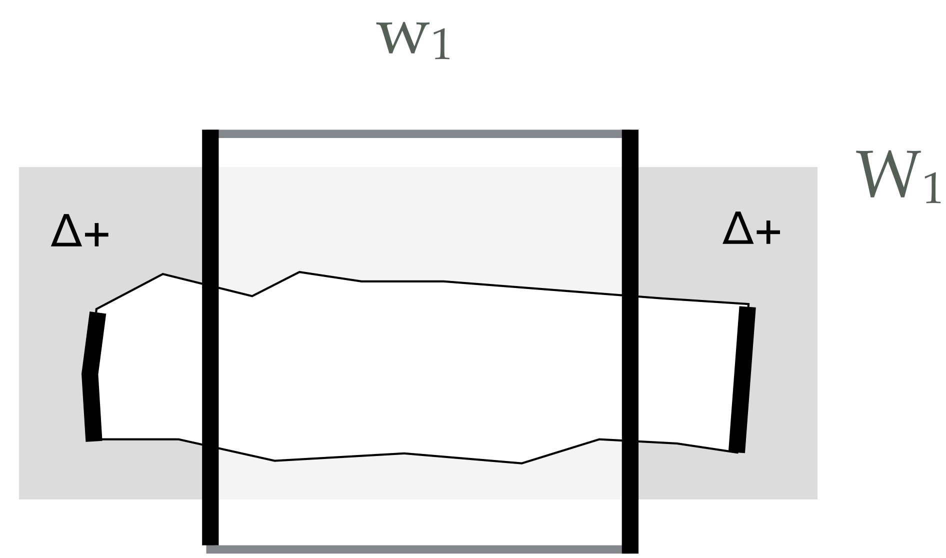
$$w \simeq D^2 \times D^2$$
$$\partial_+ \simeq S^1 \times D^2$$
$$H_2(w, \partial_+) \simeq \mathbb{Z}$$

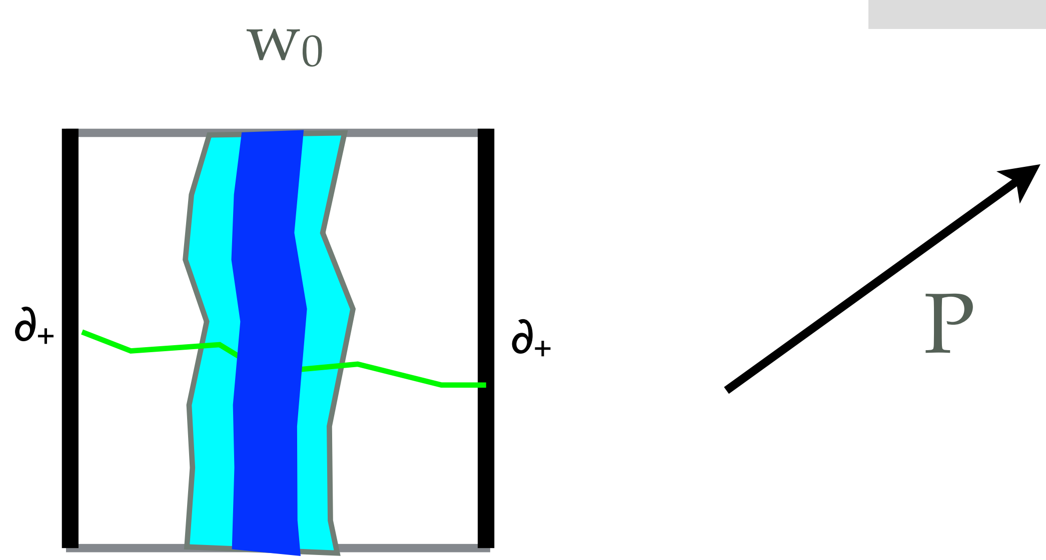


$P_* : H_2(w, \partial_+) \to H_2(W, \Delta_+)$ isomorphism

This kind of stretching is sufficient to show: Given a sequence of window, each stretched across the next one in this way, there will be a nonempty set of initial conditions which maps through the windows in the given order.

The proof uses some algebraic topology

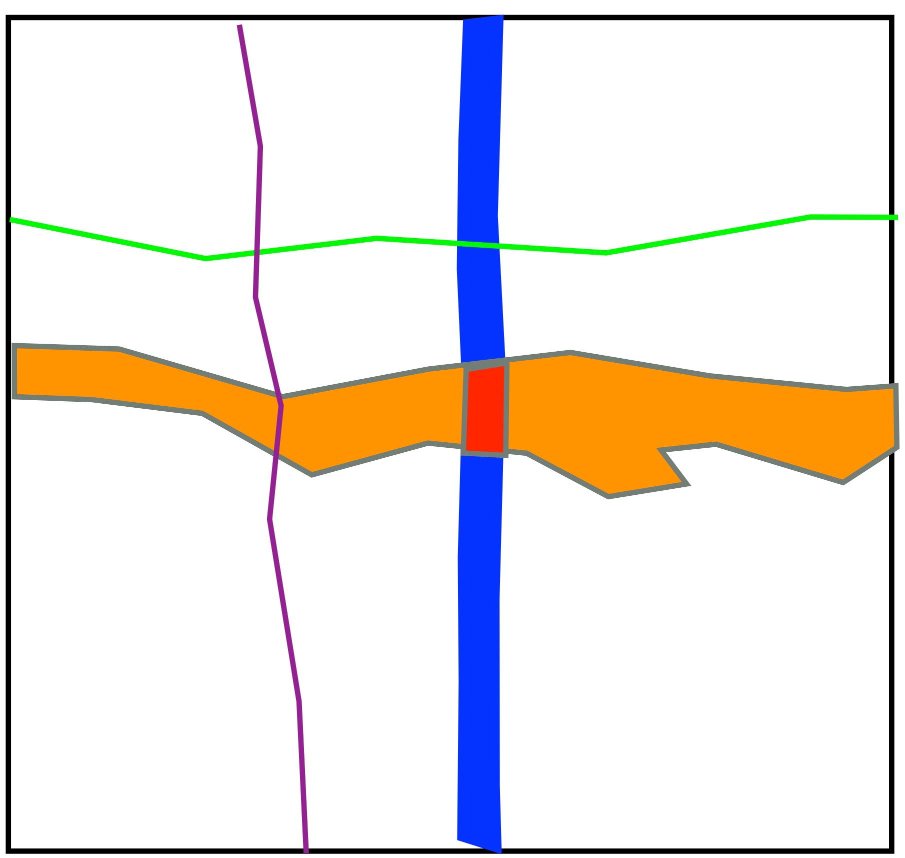




The auxiliary window pair (W_1,Δ_+) should retract onto the window pair (w_1,d_+)giving an isomorphism of homology groups. Apply this retraction then the Poincare map.

If w_0 is stretched across w_1 in this way then there the set of points in w_0 which map into w_1 is a nonempty compact set (light blue) with the property that it intersects the support of any generator of the homology $H_2(w_0,\delta_+)$ (green). If w_1 is then stretched across w_2 the set in w_0 which first maps to w_1 and then to w_2 gives a subset (dark blue) which still has this property and so on.

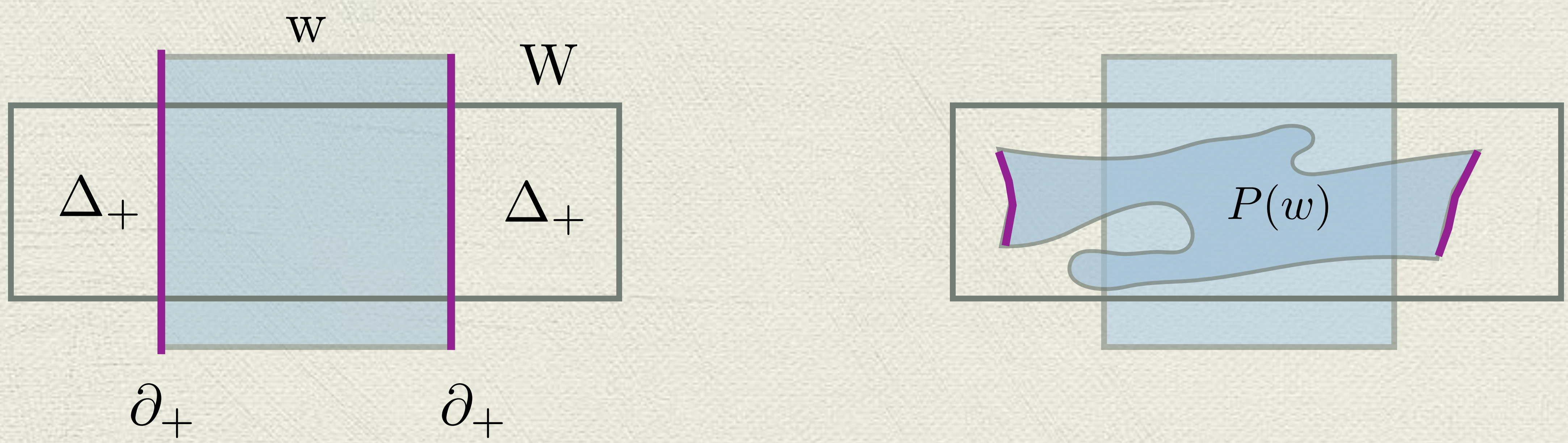
Thus specifying any finite sequence of windows in forward time gives such a subset of w_0. Similarly specifying a finite sequence of windows in backward time gives a similar set (orange) with the property that it intersects the support of any generator of $H_2(w_0,\delta_-)$ (purple). Then an algebraic topological argument shows that any two such sets must have a nonempty compact intersection (red) which realizes both the forward and the backward sequences. Finally one gets bi-infinite sequences as an intersection of these sets for longer and longer finite sequences.



If the blue set intersects every "horizontal generator" and the orange set intersect every "vertical generator" then the orange set and the blue set must meet.

Periodic Orbits

For a periodic sequence of windows, we get a composition of Poincaré maps taking a window into its own auxiliary window in a topologically nontrivial way. This is enough to guarantee at least one fixed point. Here is a 2D analogue. Suppose a rectangle maps into its own auxiliary rectangle as shown. Then one can show there is at least one fixed point inside.



$P : (w, \partial_+) \to (W, \Delta_+)$ induces an isomorphism $H_1(w, \partial_+) \simeq H_1(W, \Delta_+)$
or equivalently $H_0(\partial_+) \simeq H_0(\Delta_+)$

Symbolic Dynamics for the Planar 3BP

Choose masses such that the collinear equilibrium point E_j with the j-th mass in the middle exhibits spiraling. Then we have a network of restpoint connections for zero angular momentum as described above. Construct a connection graph as shown:

The symbols represent windows near some of the connecting orbits with $r > 0$.

$L_{+,-}$: along equilateral homothetic orbits 

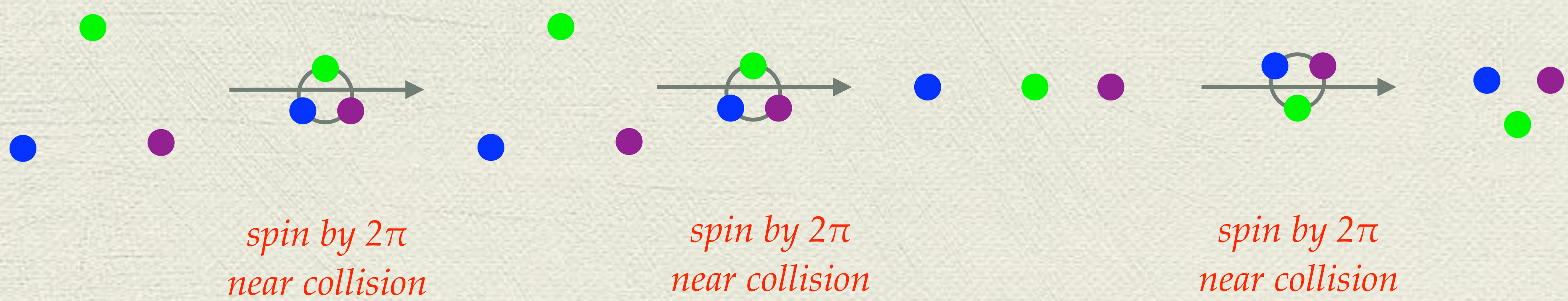
E_j : several windows near collinear homothetic orbit 

The arrows represent Poincaré maps stretching these windows across one another

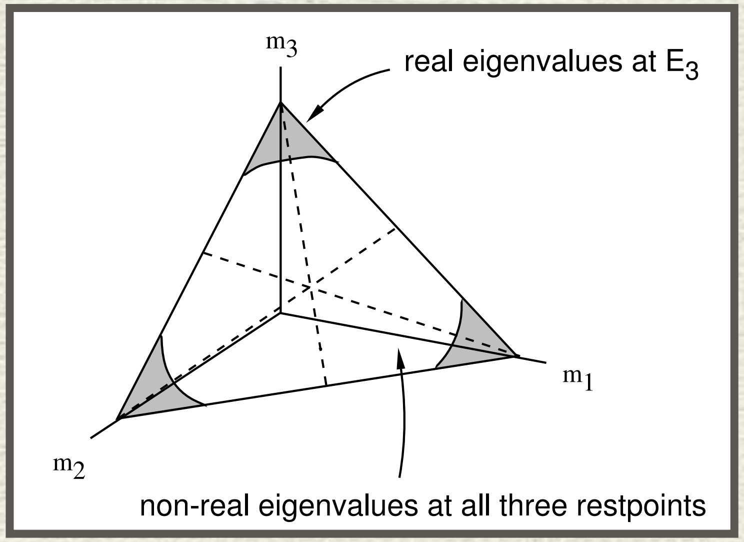
The main result is that all of the paths in the graph are realized by solutions of the 3BP. In other words, one can specify any sequence of symbols and find a orbit passing through the corresponding windows in the given order.

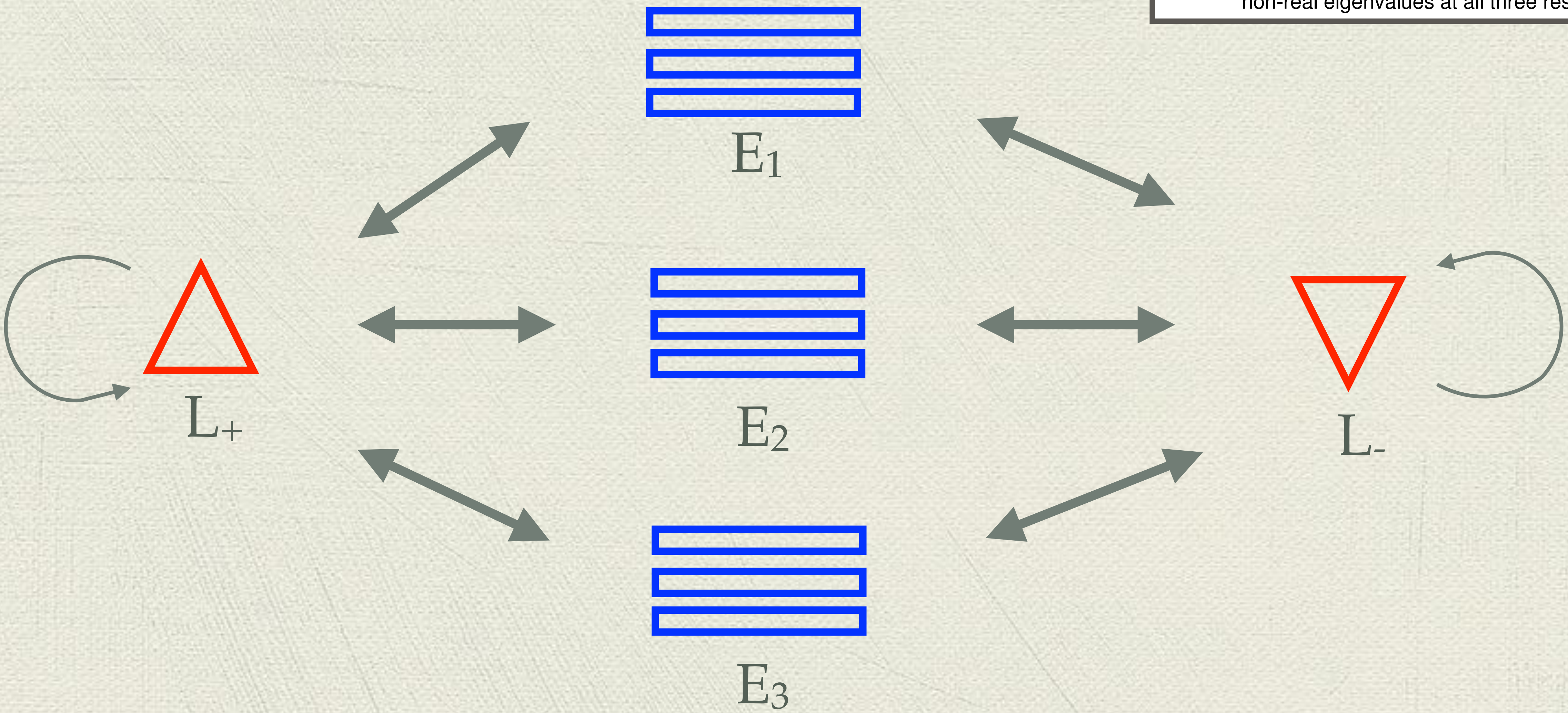
Theorem: For almost all masses with spiraling at E_j, for every negative energy $h<0$ and for nonzero λ sufficiently small, every bi-infinite path in the graph is realized by a nonempty compact set of orbits in $M(h,\lambda)/SO(2)$. Moreover, every periodic path is realized by at least one (relative) periodic orbit.

Example: sequence …. L_+ L_+ E_j L_- …..

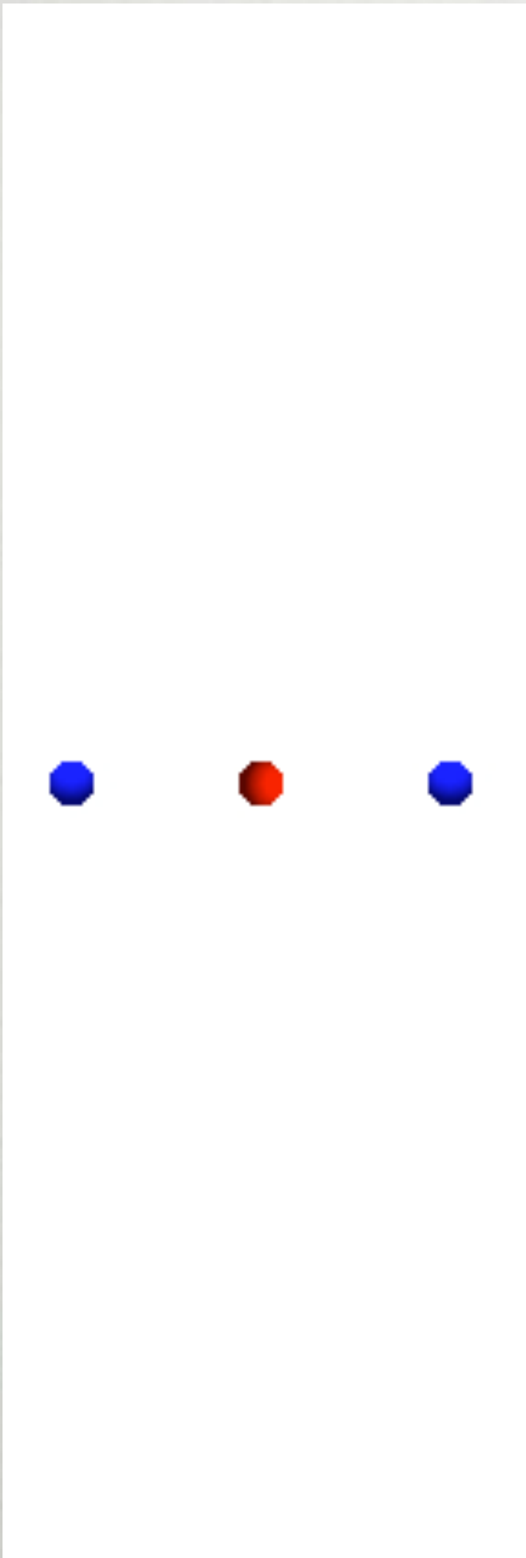


For most masses we have spiraling at all three collinear restpoints, in which case a more complicated symbol graph is realized. One can emerge from near collision with any of the three collinear CC shapes !





A simpler case: Chaos in Sitnikov's Problem

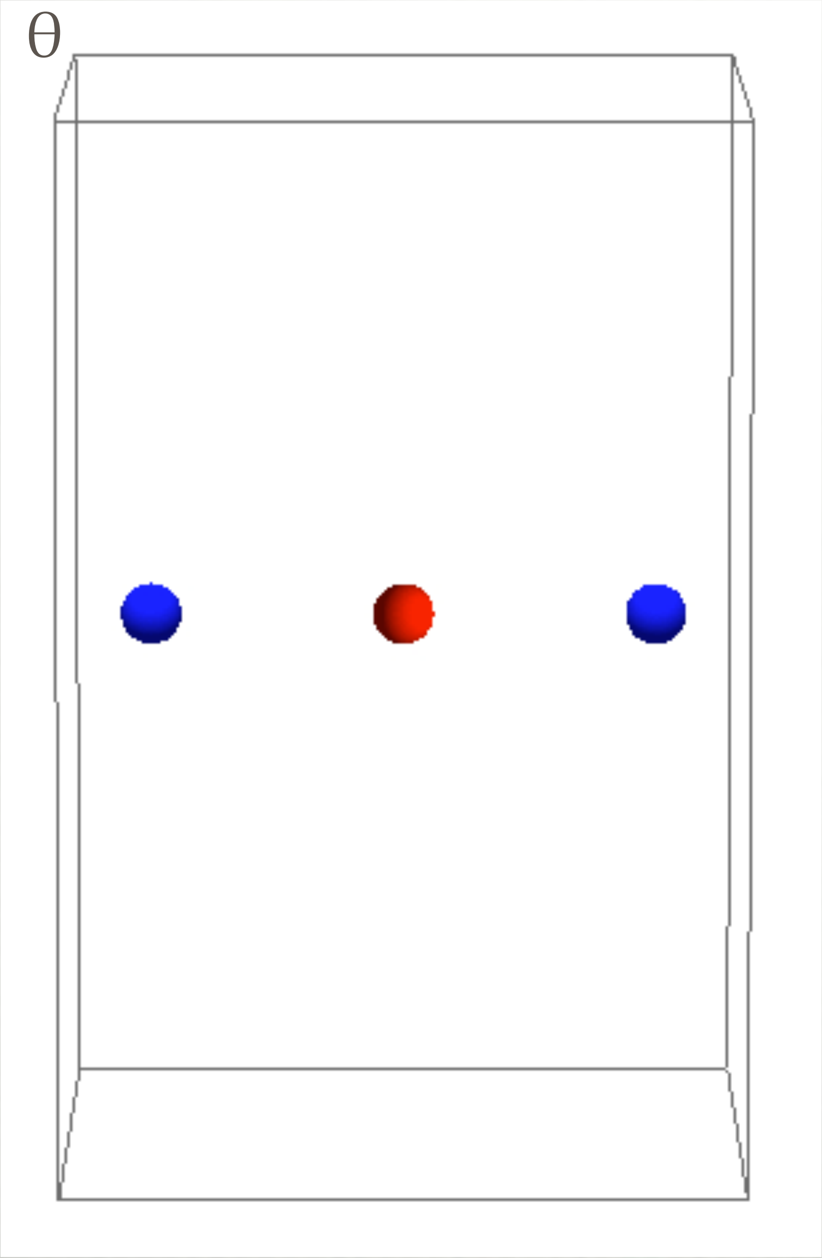


We will now focus on an even simpler three-body system where the homoclinic tangles are easier to see. In the 3D isosceles three-body problem we have two equal masses, say $m_1 = m_2 = 1$, moving symmetrically around the z-axis which a third body of mass m_3 moves up and down on the axis. The shape of is always an isosceles triangle.

The special case $m_3 = 0$ is the *Sitnikov* problem. Then m_1, m_2 move on symmetrical elliptical orbits in the (x, y)-plane. It is a dynamical system with $1\frac{1}{2}$ degrees of freedom. The state of the third body is determined by one position z and velocity $\dot{z}$ but there is a time-periodic forcing. We have a three-dimensional flow on $\mathbf{R}^2 \times \mathbf{S}^1 = \{(z, \dot{z}, t \bmod 2\pi)\}$.

Here is a typical orbit. We will be especially interested in orbits where z tends to infinity and then the third mass flies off the screen. So it is convenient of replace z by a bounded variable.

Sitnikov problem in θ coordinates



Setting $z = \frac{1}{2}\tan\theta$, Newton's laws give the following system of three ODE's:

$$\begin{aligned}\dot\theta &= 2\cos^2\theta\, v\\ \dot v &= -\frac{4\sin\theta\cos^2\theta}{(\cos^2\theta(1+\epsilon\cos u(t))^2+\sin^2\theta)^{\frac{3}{2}}}\\ \dot t &= 1\end{aligned}$$

where ϵ is the eccentricity and $u(t)$ is the eccentric anomaly of the two-body motion of the primaries. $u(t)$ satisfies Kepler's equation

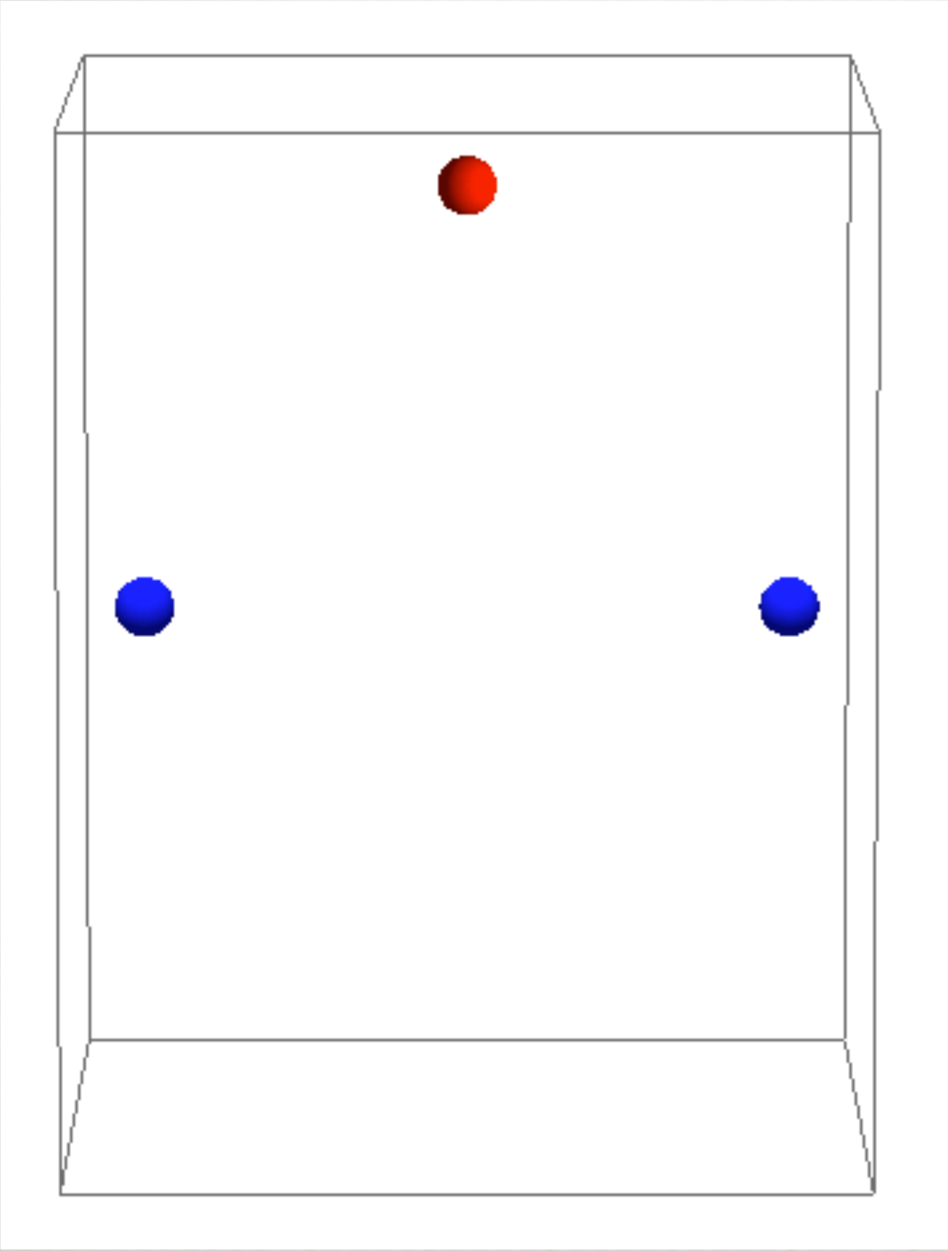
$$t = u(t) + \epsilon \sin u(t).$$

θ remains in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Orbits with $z \to \pm\infty$ now converge to the top or bottom of the box with $\theta \to \pm\frac{\pi}{2}$.

Near Triple Collision $\varepsilon = 0.96$

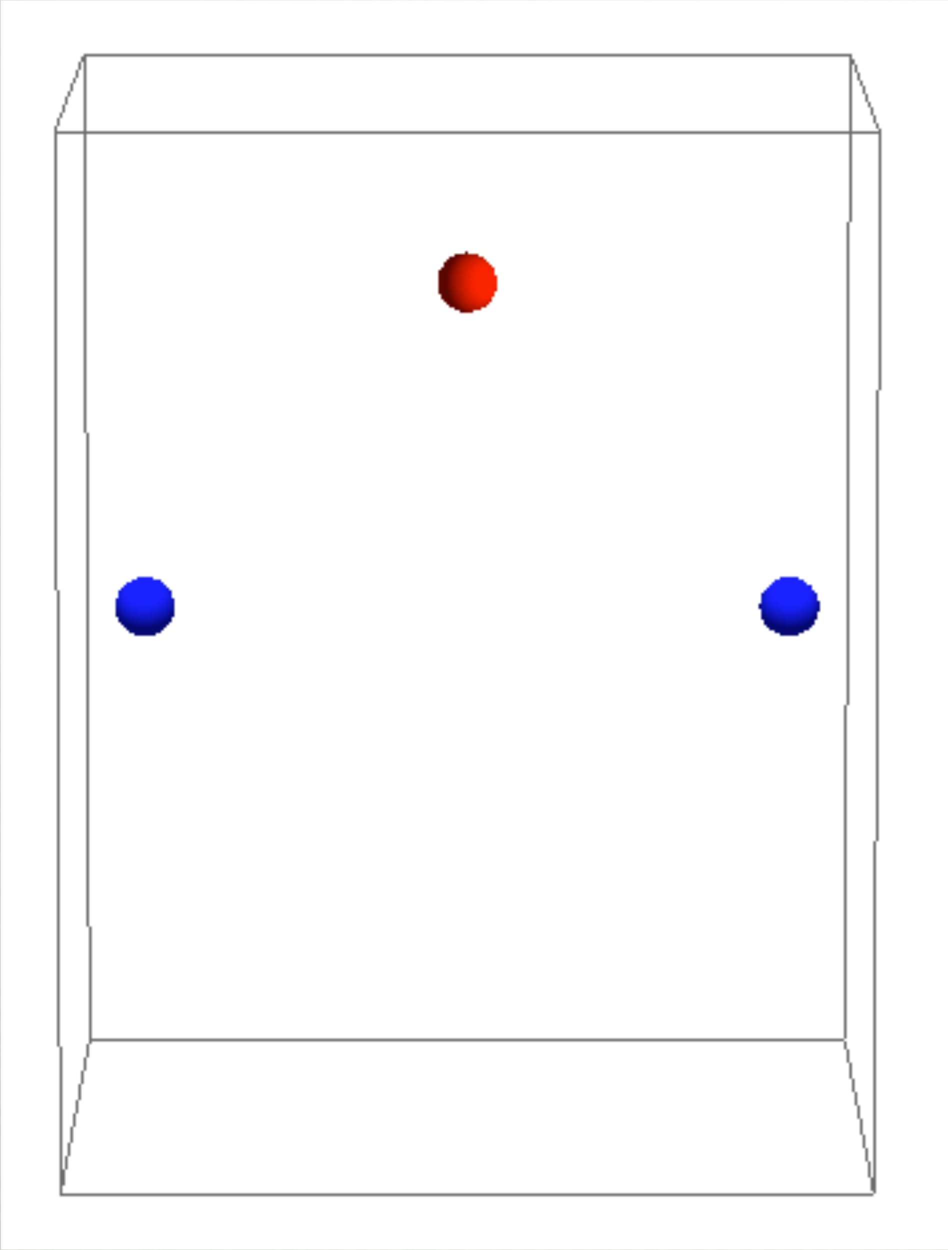
We can get near-triple collision orbits in Sitnikov's problem by choosing a high eccentricity for the orbit of the primary masses and then timing the third body to pass through the origin when the elliptical bodies are close.



This orbit is a highly unstable, hyperbolic periodic orbit with a close approach to triple collision (there is a corresponding hyperbolic fixed point of the Poincaré map).

It is close to the Lagrange equilateral collision solution in the planar problem.

More near-collision orbits



One can show that there are many more such near-triple-collision periodic orbits. They differ in their detailed behavior while approaching triple collision.

Here is another orbit for ε=0.96 which "wobbles" above and below the plane of the primaries near triple collision.

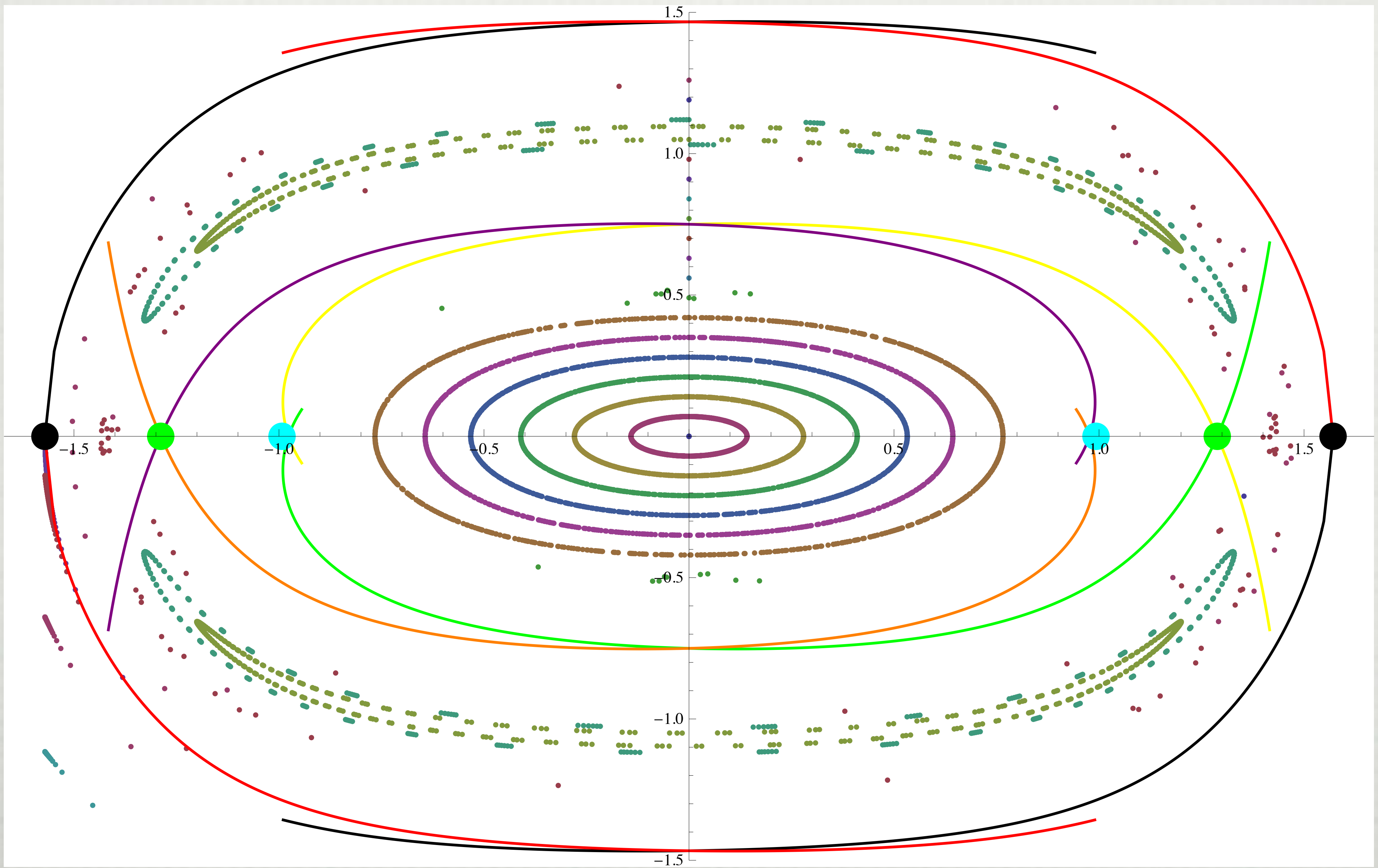
An infinite sequence of such hyperbolic fixed points of the Poincaré map is created as the eccentricity ε --> 1 (though for any fixed ε < 1, there will only be finitely many).

Also, the reflections of these orbit are distinct solutions.

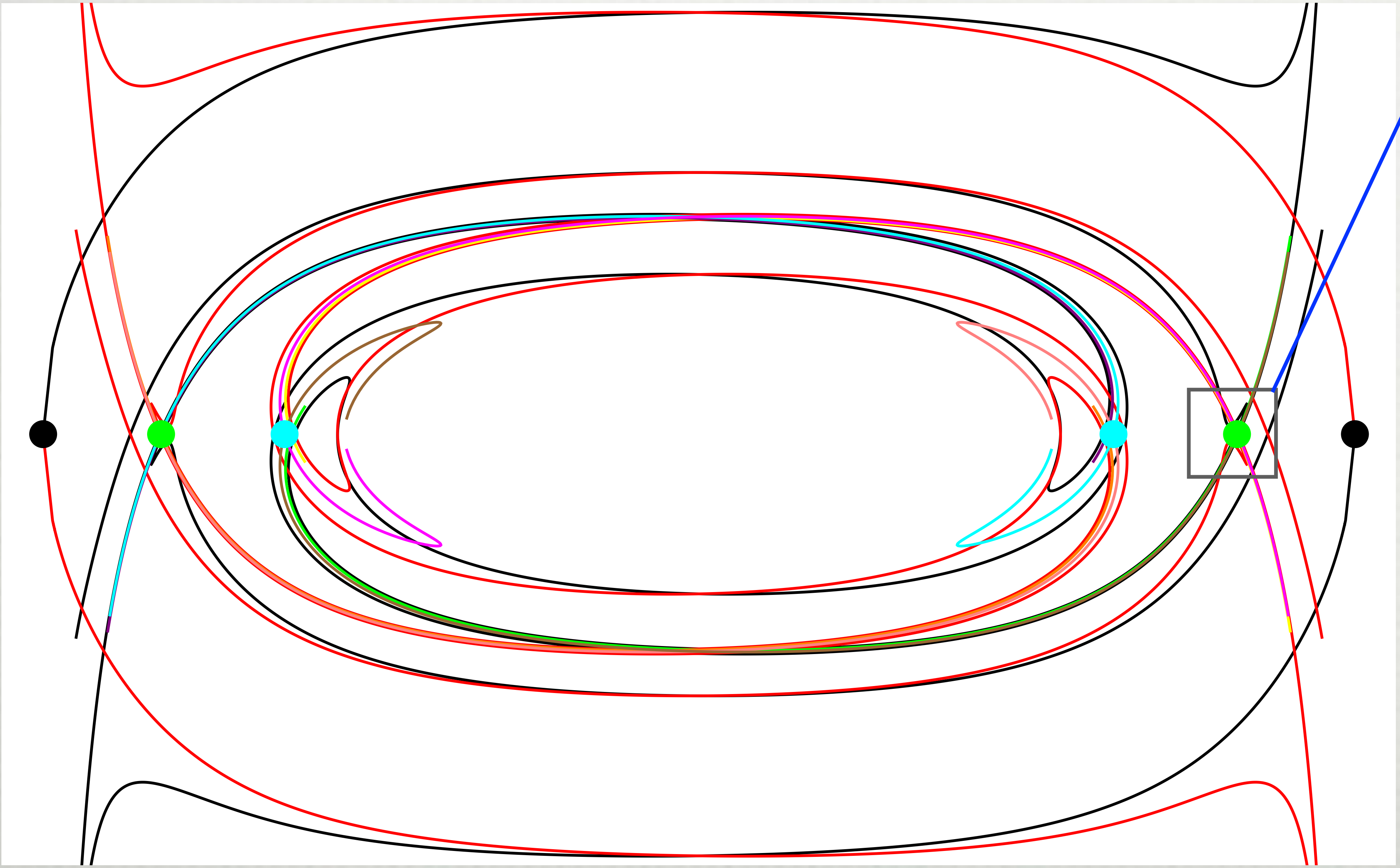
2D Poincaré map for $\varepsilon = 0.96$

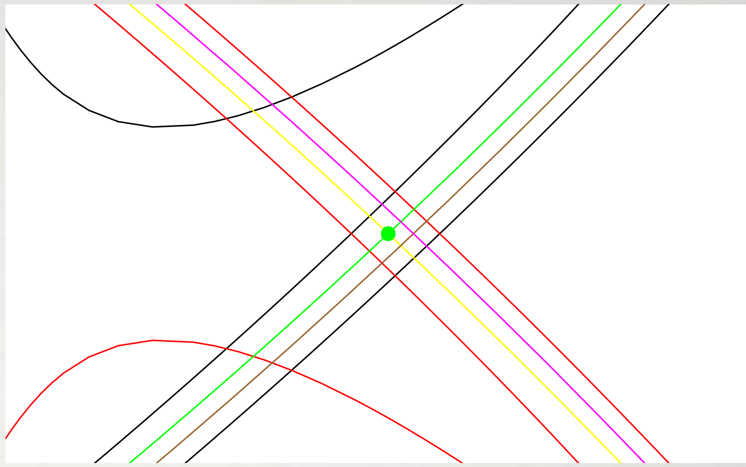
We have (at least) six hyperbolic fixed points all with stable and unstable curves.

● (black)	∞_+	∞_-	Parabolic Infinity
● (green)	P_+	P_-	Near triple collision
● (cyan)	Q_+	Q_-	Near triple collision with wobbles



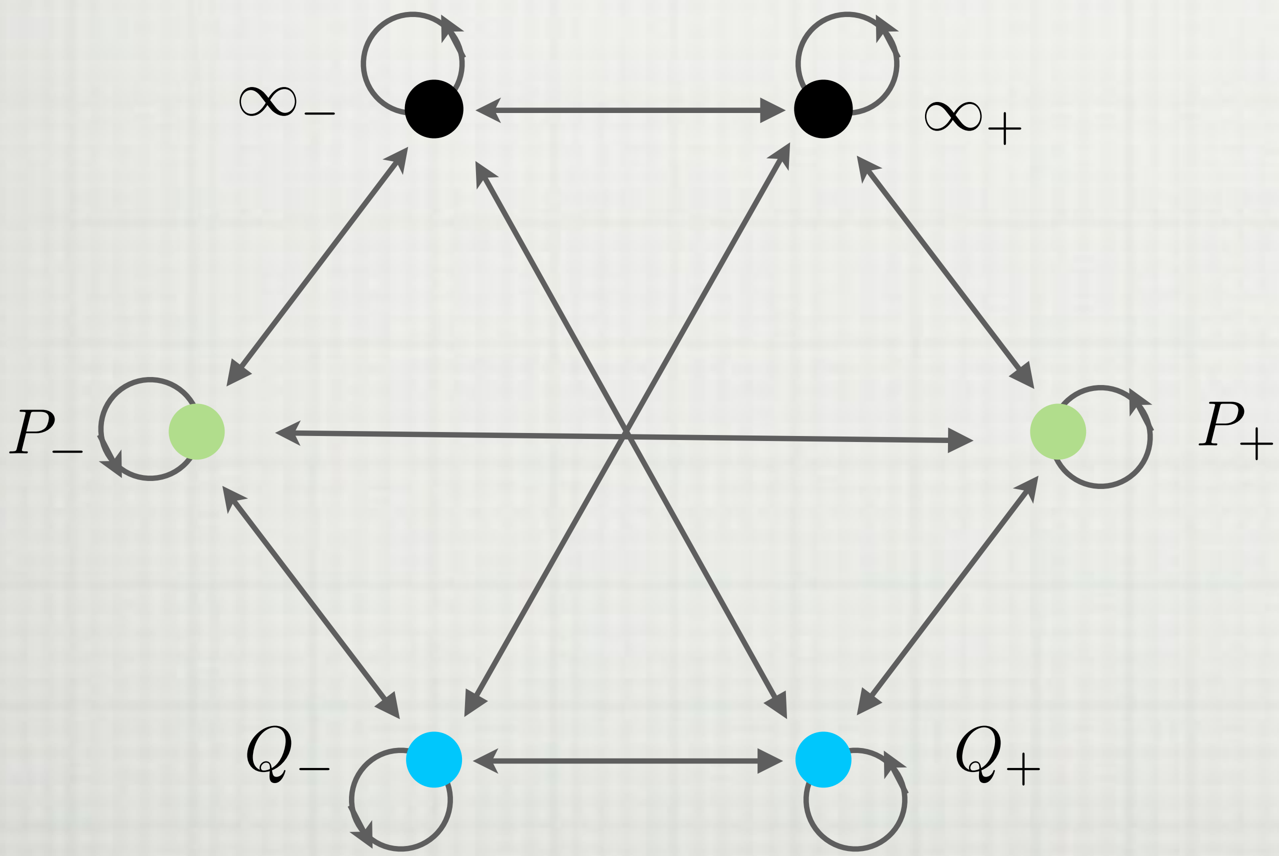
There is a rich network of transverse homoclinic and heteroclinic connections between the six fixed points





Symbolic Dynamics

As a result, there will be orbits realizing every path in the graph below. In other words one can choose any sequence of the six symbols and find an orbit which exhibits that sequence of behaviors.

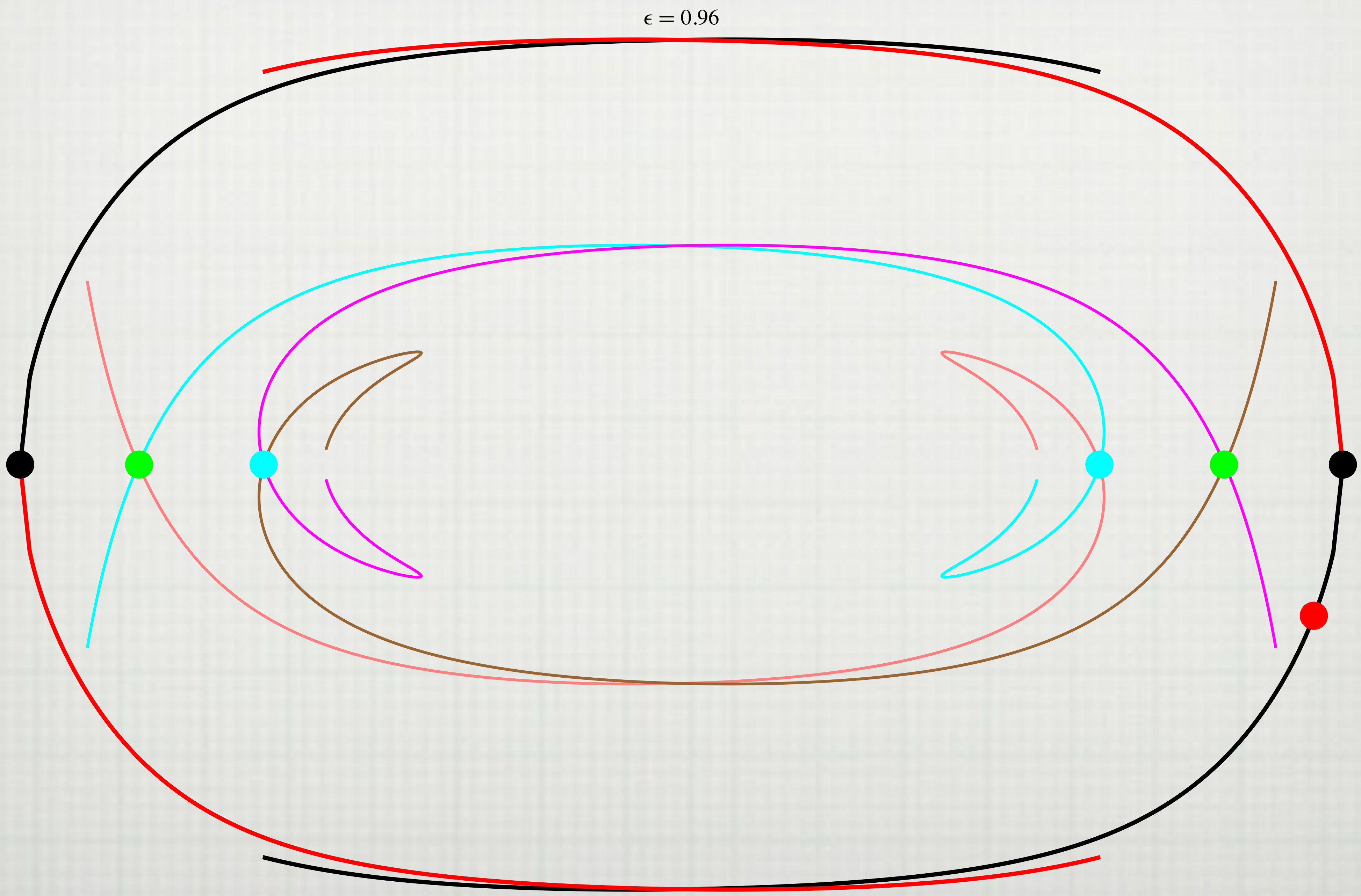


Each vertex represents a window near one of the six fixed points.

To realize a sequence means that the orbit passes through the corresponding sequence of windows for the Poincaré map.

This is rather delicate in practice since the behavior is sensitive changes in initial conditions.

Realizing the sequence $\ldots, \infty_+, Q_-, P_+, \infty_+, \ldots.$



Realizing the sequence $..., \infty_+, Q_-, P_+, \infty_+, ...$