Blowing Up the N-Body Problem IV

Realizing Syzygies
Parabolic Infinity

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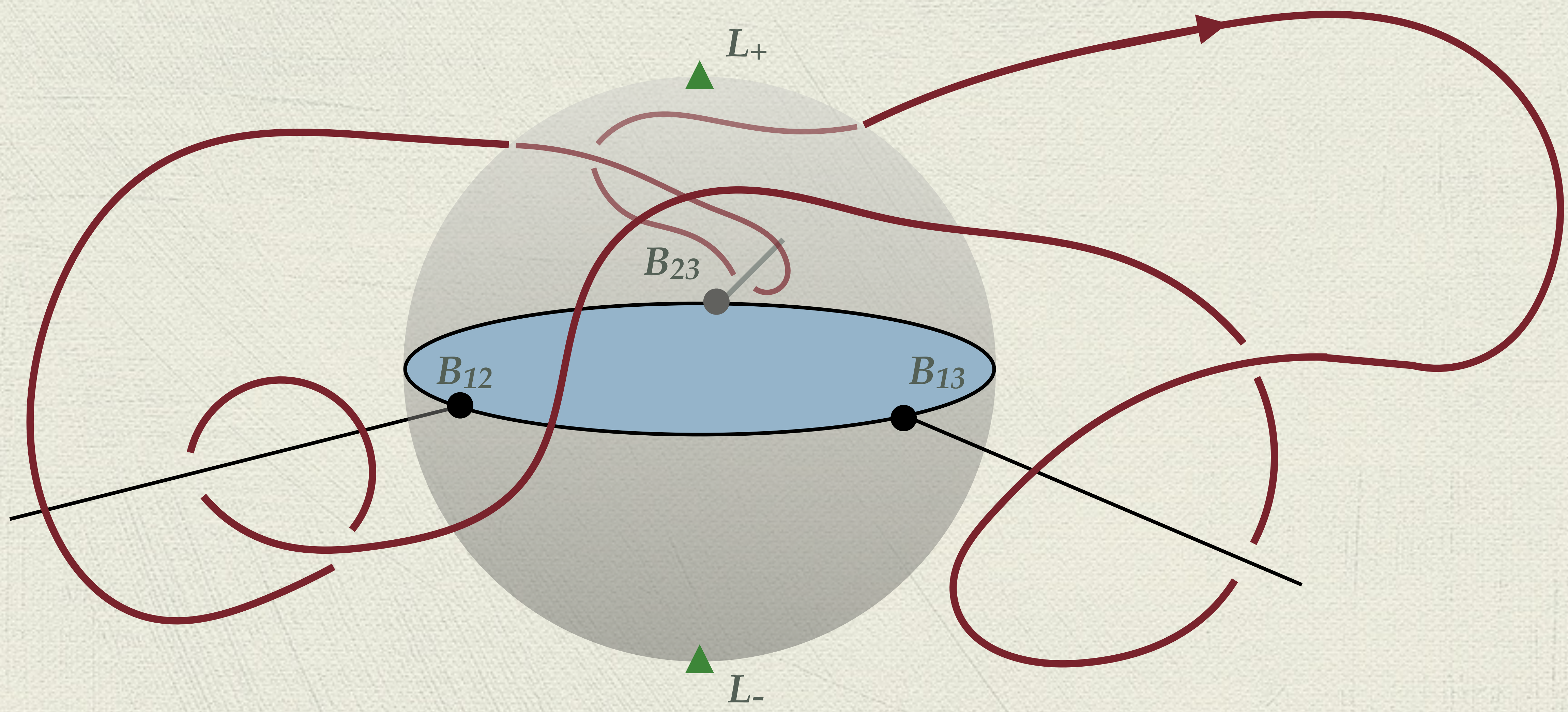
Last time: Chaos in the planar 3BP for small angular momentum

- Hyperbolic restpoints at r=0
- Stable and unstable manifolds of triple collision orbits
- Infinitely many solutions beginning and ending at equilateral triple collision caused by spiraling at the collinear restpoints
- Restpoint cycles and connection graph
- Chaotic invariant based on 4D windows

Next: Describe an application of this work to the problem of realizing syzygy sequences (joint with R.Montgomery)

Recall the Reduced Configuration Space

$Q = \mathbb{R}^+ \times \mathbf{S}^2$ can be visualized as the exterior of the unit sphere in $\mathbb{R}^3$. The unit sphere represents triple collision, that is, triangles of size $r = 0$. The singular set Σ consists of this sphere together with three binary collision rays. A solution of the reduced three-body problem sweeps out a curve in the complement $Q \setminus \Sigma$.



Free homotopy classes

Periodic solution of the reduced, planar three-body problem

➞ Closed loop $\gamma(t)$ in $Q \setminus \Sigma$

➞ Free homotopy class $[\gamma] \in [\mathbf{S}^1, Q \setminus \Sigma]$

Theorem: *For equal or near equal masses, every free homotopy class is realized by a periodic solution of the reduced planar three-body problem*

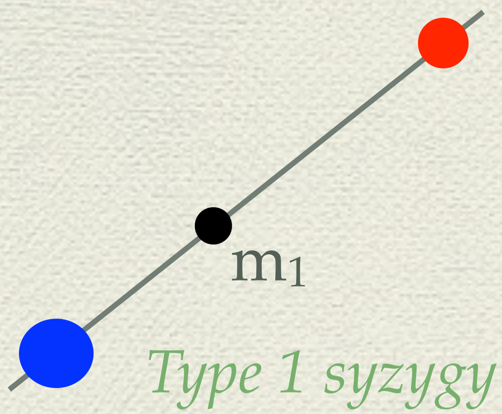
- The solutions may not be periodic for the *unreduced* problem
- Fixed negative energy and small, nonzero angular momentum
- Repeated close approaches to triple collision
- Reminiscent of a theorem about minimal geodesics in differential geometry, but the proof is not variational. Trying to minimize action just leads to collisions.

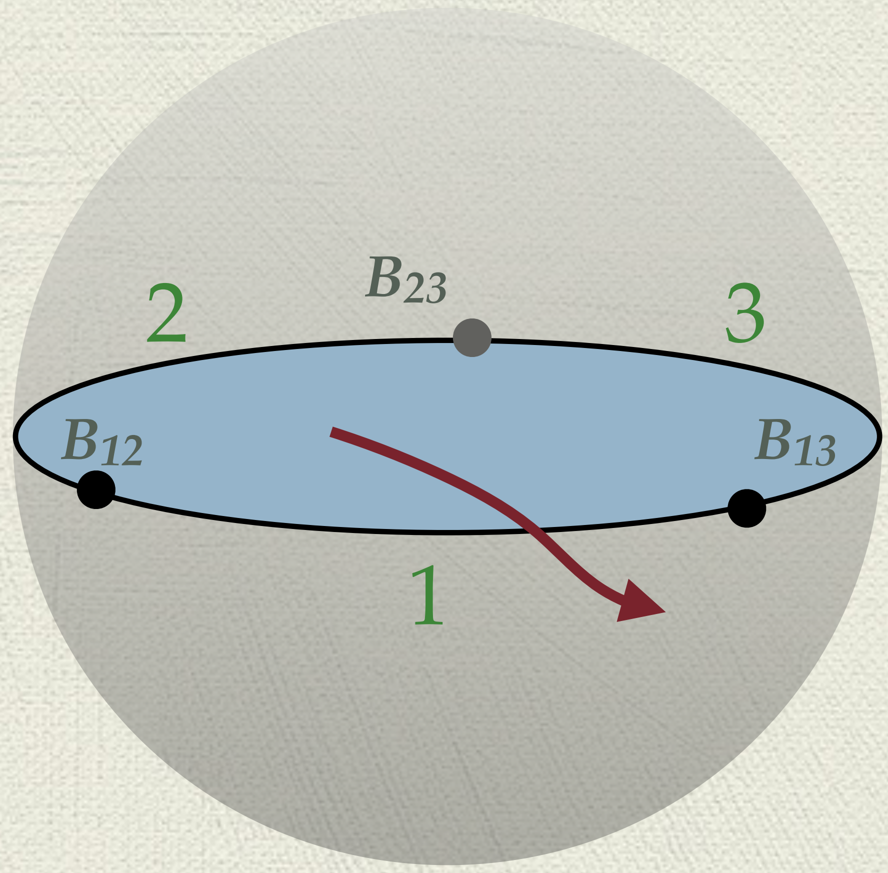
Syzygy Sequences

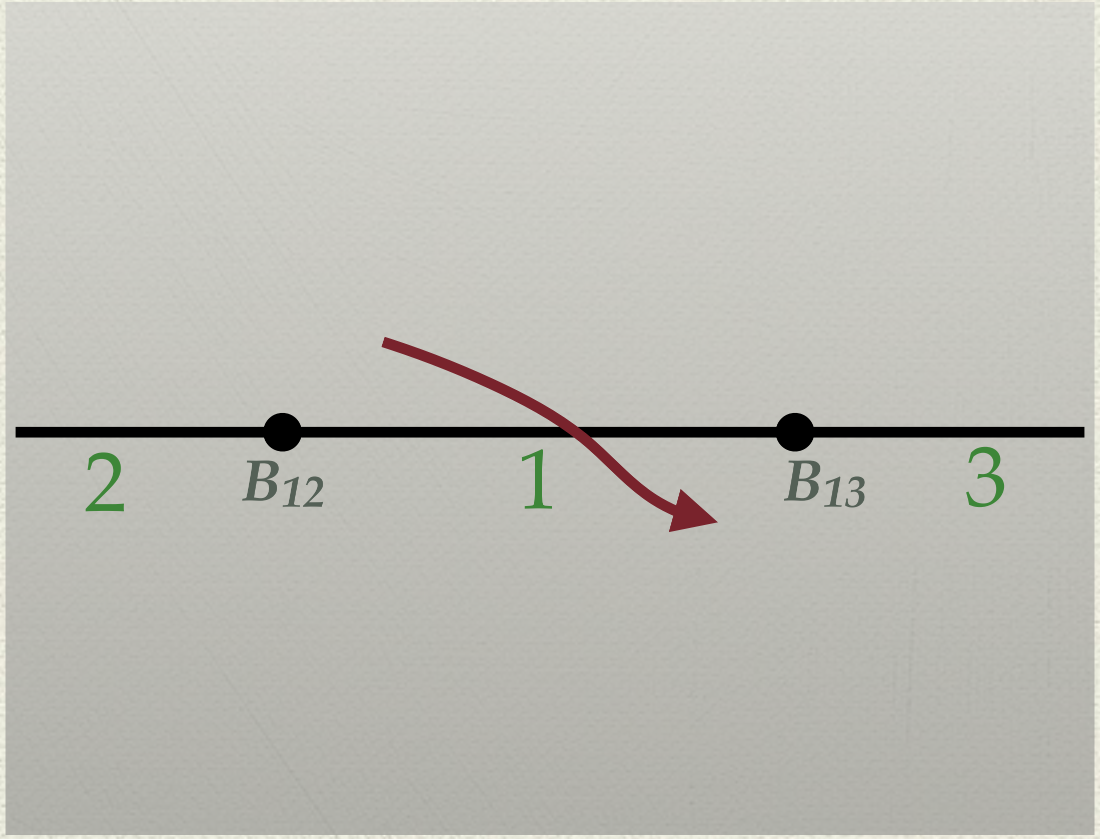
Up to homotopy equivalence

$$Q \setminus \Sigma \simeq \mathbf{S}^2 \setminus \{B_{12}, B_{13}, B_{23}\} \text{ (radial projection)}$$
$$\simeq \mathbb{R}^2 \setminus \{B_{12}, B_{13}\} \text{ (stereographic projection)}$$

We can code homotopy classes using *syzygy* sequences.







Reduced Syzygy Sequences

Free homotopy class $[\gamma] \in [\mathbf{S}^1, \mathbf{S}^2 \setminus \{B_{12}, B_{13}, B_{23}\}]$

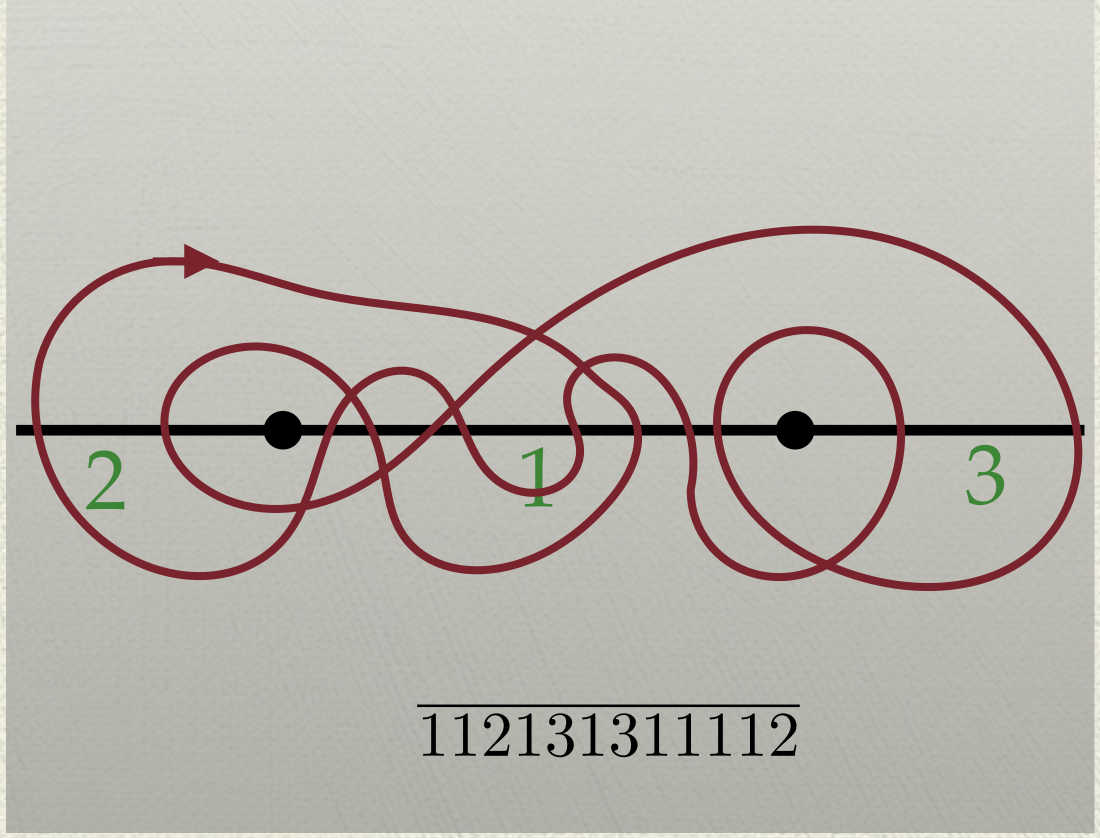
→ representative starting in N hemisphere and transverse to equator

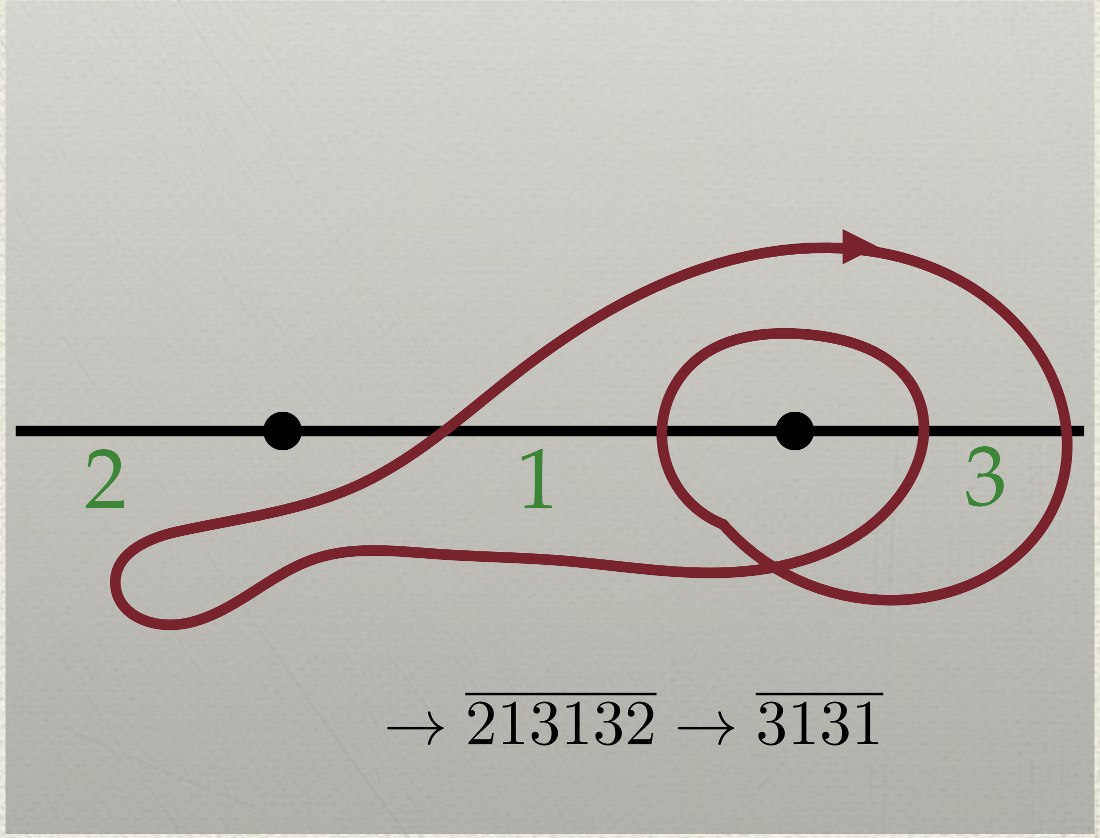
→ even, periodic syzygy sequence

Using homotopies we can

- shift cyclically by an even shift
- cancel “stutters” (repeated pairs)

Get a ***reduced syzygy sequence*** (no repeated pairs)





It suffices to show: ***Every even, periodic reduced syzygy sequence is realized by a periodic solution.***

In fact, we show: ***Every bi-infinite syzygy sequence with sufficiently long stutter blocks is realized and periodic ones are realized by a periodic orbit.***

$$\ldots \epsilon_{-1}^{n_{-1}} \epsilon_0^{n_0} \epsilon_1^{n_1} \ldots \qquad \epsilon_i \in \{1,2,3\} \qquad n_i \text{ suff.large}$$

Note: If n_i is odd then $\epsilon_i^{n_i}$ reduces to just ϵ_i

The proof uses the orbits in the chaotic invariant set from the last lecture together with some further observations about those orbits.

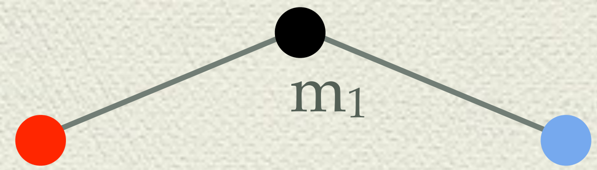
- for zero angular momentum and near equal masses there are orbits beginning and ending in triple collision realizing one syzygy block

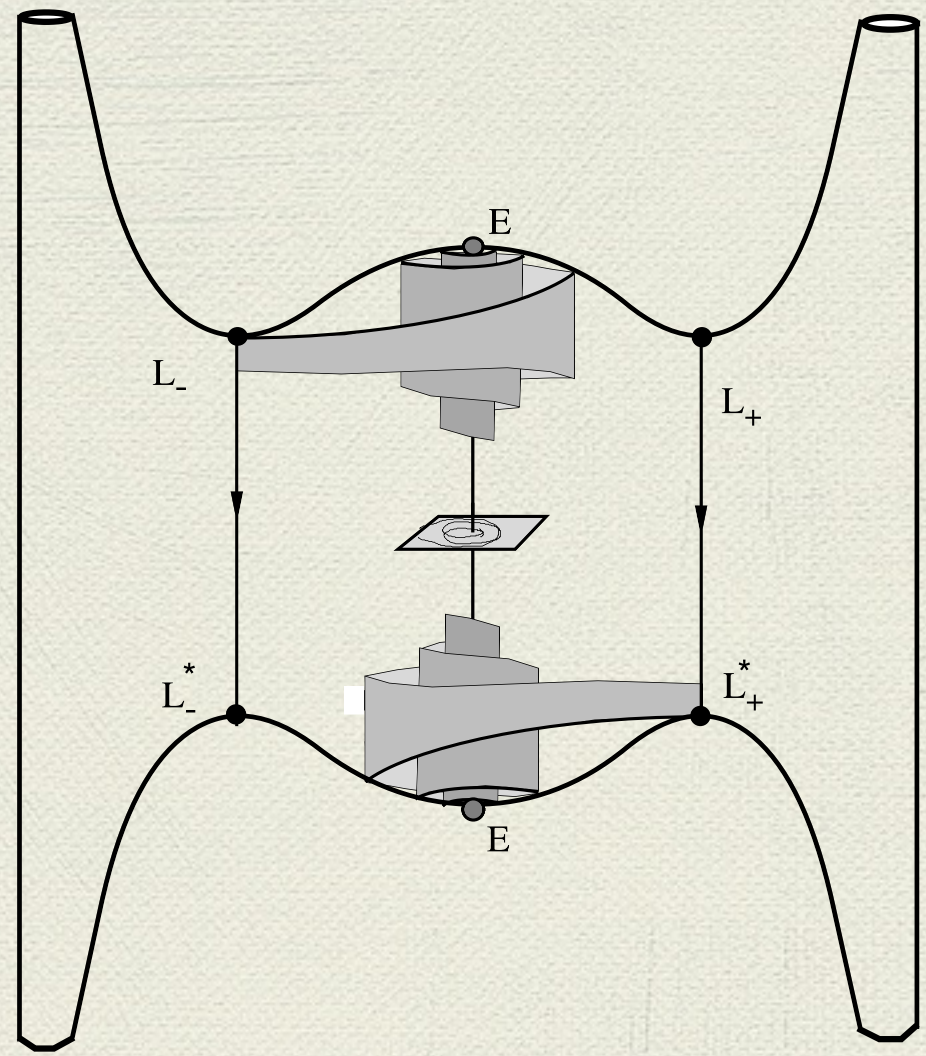
$$\epsilon^n \qquad \epsilon \in \{1,2,3\}$$

- perturbing to small nonzero angular momentum, triple collision is avoided and we can use symbolic dynamics to realize arbitrary concatenations of such blocks

Zero angular momentum — isosceles triple collision orbits

When the masses are all equal and angular momentum $\mu = 0$, there are three invariant isosceles subsystems of the planar three-body problem which have only two-degrees of freedom.

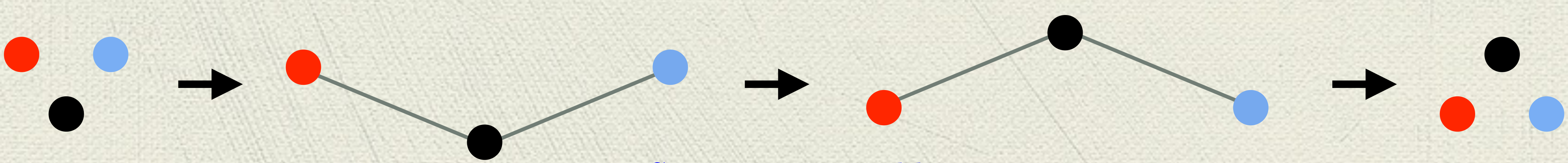




Using McGehee blow-up:

- r=0 invariant triple collision set
- triple collision orbits form stable and unstable manifolds of restpoints
- Lagrange (equilateral) restpoints yield spiraling surfaces with many intersections

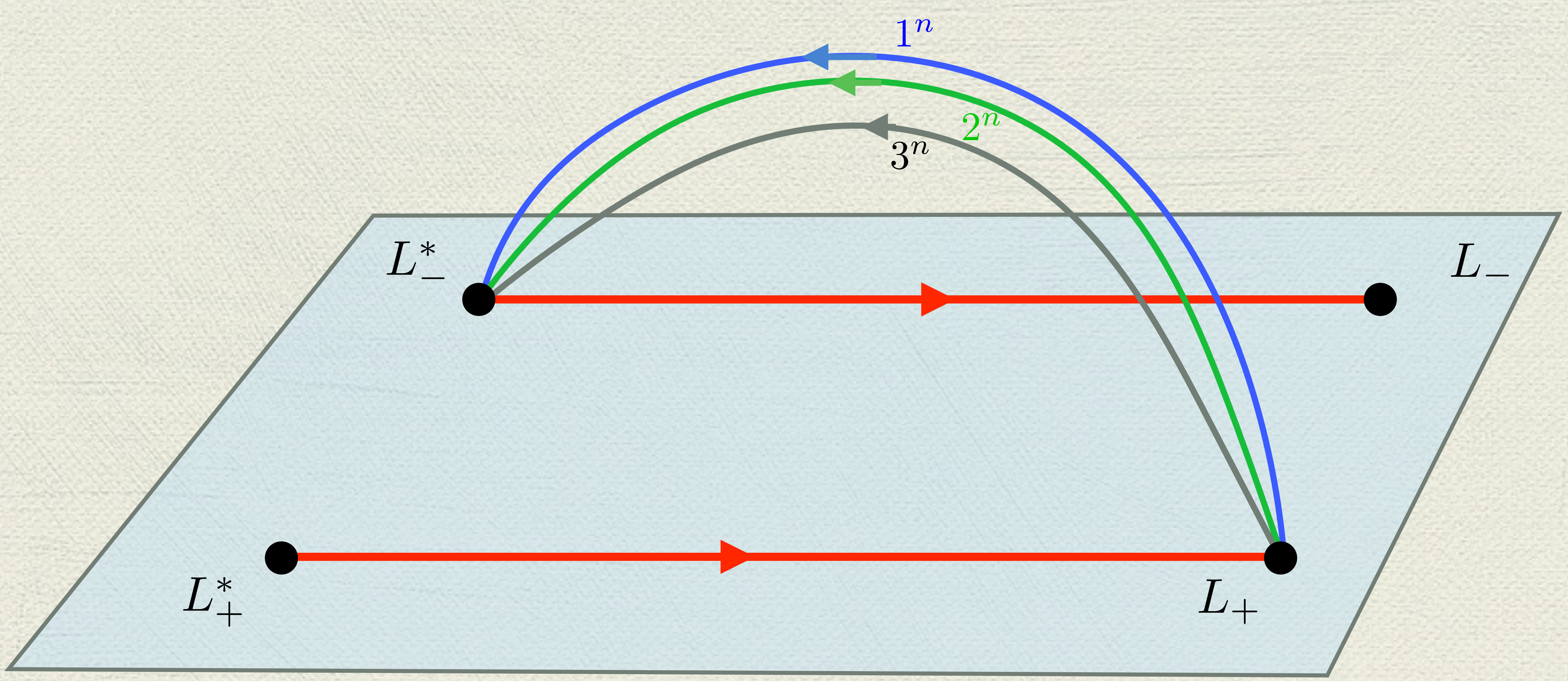
Restpoint connections $L_- \to L_+^*$ or $L_+ \to L_-^*$



Syzygies 1^n, n odd

Zero angular momentum — Framework of restpoint connections

Extra information here: for nearly equal masses, the restpoint connections can be chosen to be nearly isosceles — no collisions !

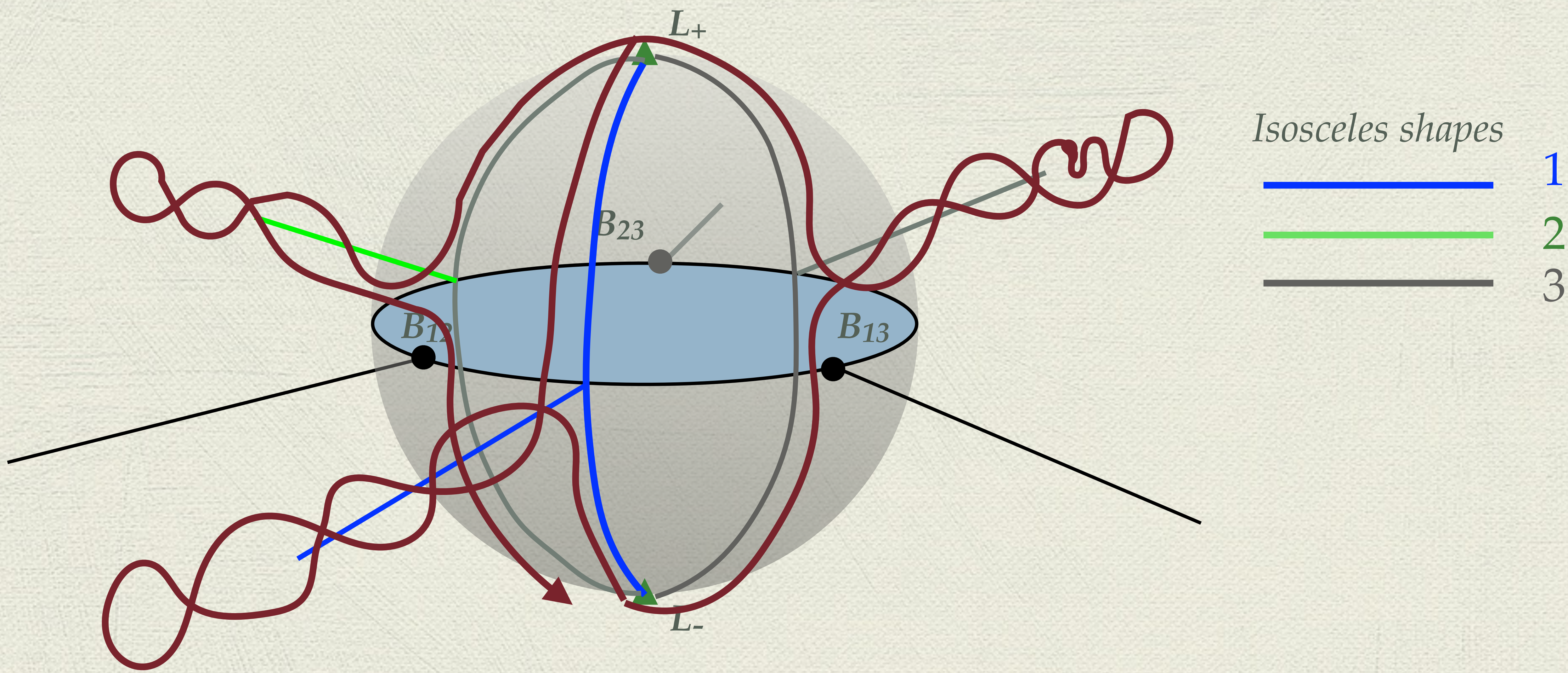


Spinning near collision: $L_+^* \to L_+$ and $L_-^* \to L_-$

Three kinds of syzygy block orbits: $L_+ \to L_-^*$ and $L_- \to L_+^*$ (not shown)

As we know, we can shadow any given sequence of these by making transitions near the the equilateral collision restpoints.

Our Orbits Viewed in the Reduced Configuration Space



Sequence of near-isosceles motions with transitions near triple collision. Clearly we can get any homotopy class this way.

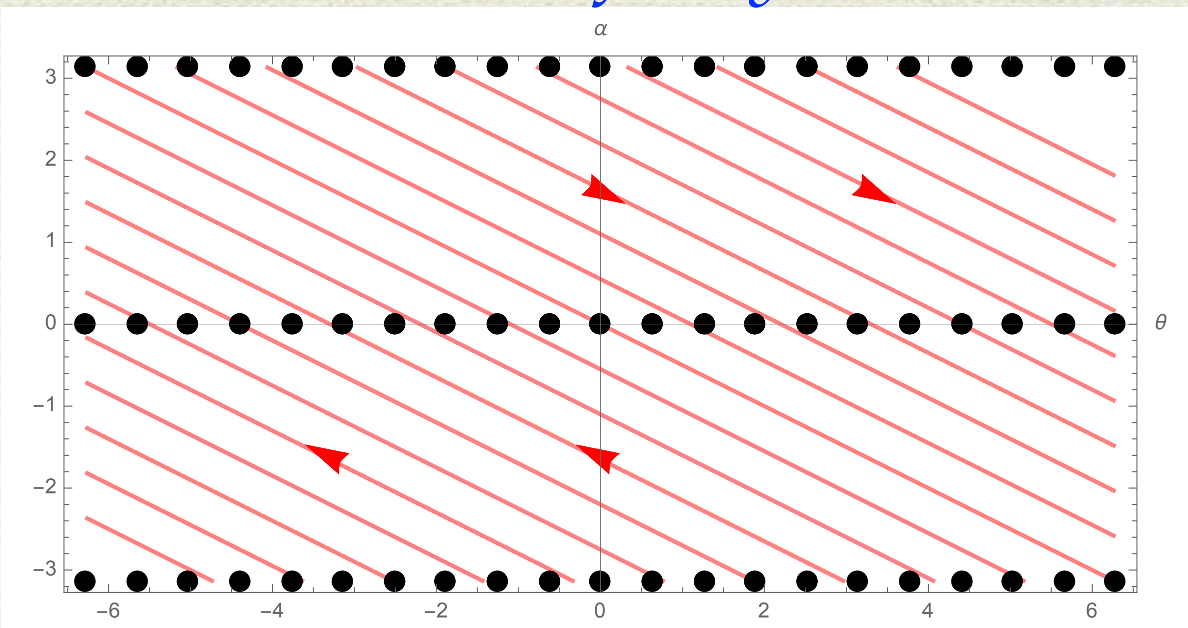
More about Parabolic Infinity

We have been focussing on solutions near triple collision but the blow-up method is also useful for studying parabolic solutions. Recall that these are solutions with energy $h = 0$ such that all bodies tend to infinity with zero asymptotic velocity.

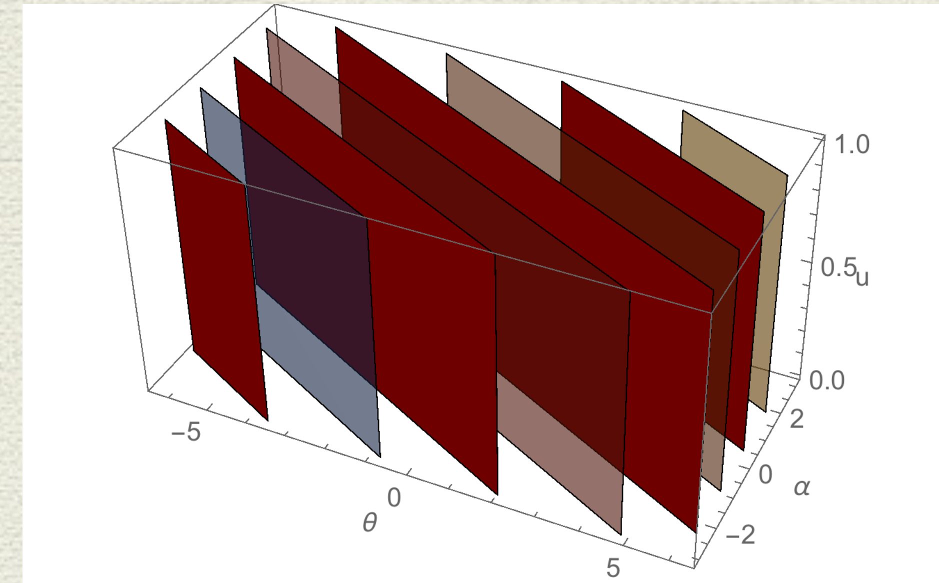
Recall the results for the zero energy 2BP

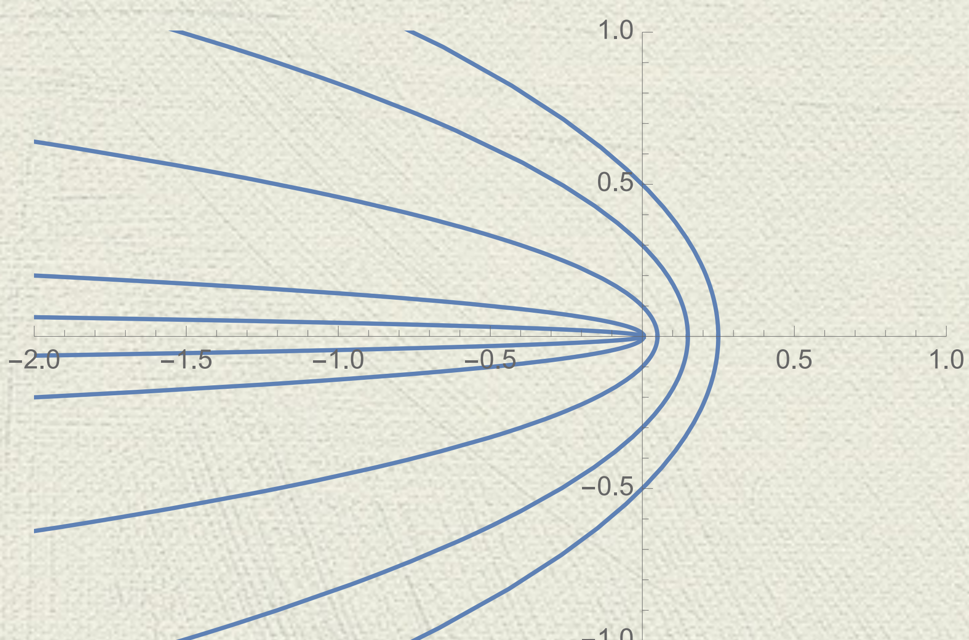
- 4D phase space, 3D energy manifold (not reduced by SO(2))
- Parabolic infinity manifold $u = 1/r = 0$ is a torus T^2
- Circles of restpoints which are limits of parabolic orbits $u \to 0$
- Each restpoint with $v>0$ (limits of forward time parabolics) has a 2D stable manifold; a Lagrangian submanifold of phase space
- The whole energy manifold is foliated by these

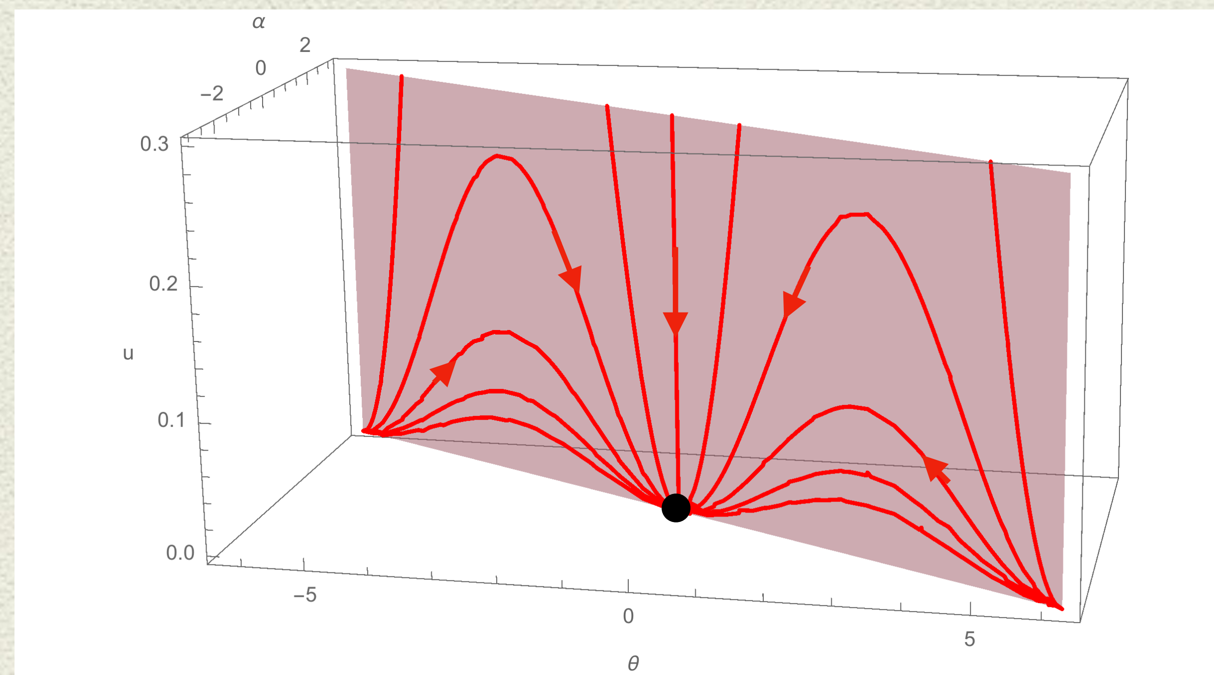
Torus at infinity u=0



Foliation of h=0 manifold



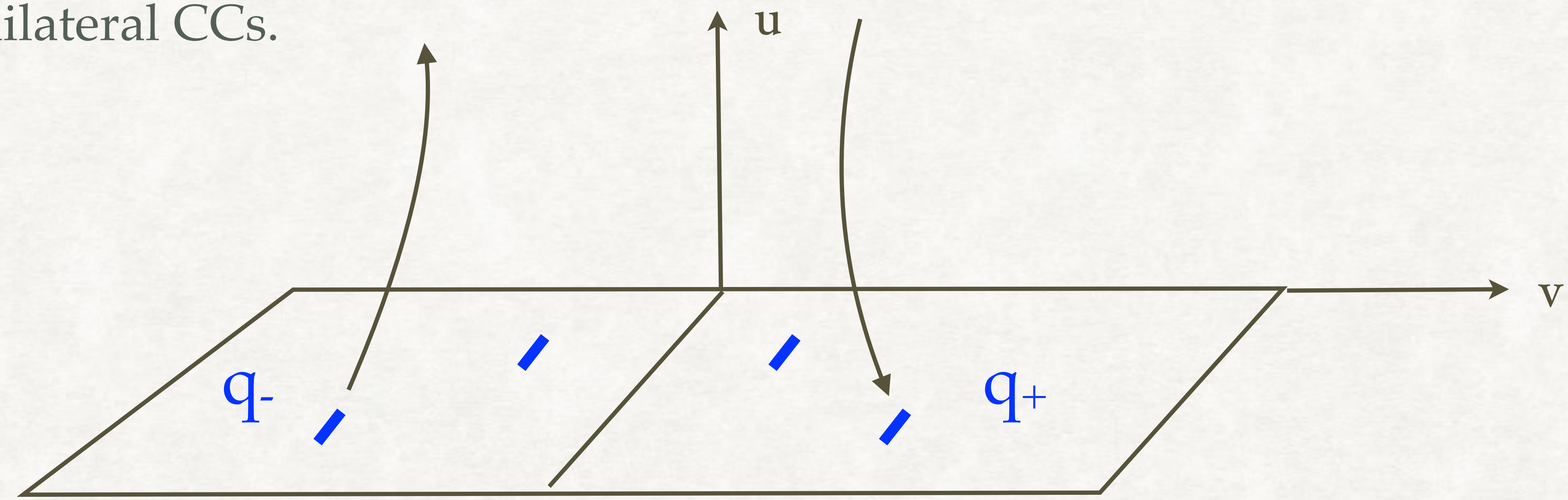




One of the stable manifolds

Planar 3BP —-Equilateral Restpoints at Parabolic Infinity

Recall the each CC determines two restpoints at parabolic infinity. The Lyapunov function v has opposite signs at these points. From the differential equation u ‘ = -vu, the restpoint with v > 0 is the limit for orbits with u —> 0 in forward time. Focus on the restpoints coming from the equilateral CCs.



Parabolic infinity restpoints: $q_\pm = (u, s, v, w) = (0, s, v, 0), v = \pm\sqrt{2U(s)}$
The eight eigenvalues are the same except for the sign of the first one

$$\lambda = -v, v, 0, -\frac{v}{2}, \frac{-v}{4}\left(1 \pm \sqrt{13 \pm 12\sqrt{k}}\right)$$

Forward time parabolic:	$q_+ : -, +, 0, -$	$-, -, +, +$	$\dim W^s(q_+) = 4$
Backward time parabolic:	$q_- : +, -, 0, +$	$-, -, +, +$	$\dim W^u(q_-) = 4$

The other manifolds are inside u=0

Generalization — Minimal Central Configurations

Return for a moment to the N-body problem in $\mathbb{R}^d$ with $\mathbb{SO}(d)$ symmetry with phase space $T(X-\Delta)$. Suppose s is a nondegenerate minimum of U on $\mathcal{E}$. Then the Hessian will have $\dim O(s)$ eigenvalues $\alpha = 0$ with all other eigenvalues $\alpha_i > 0$. The eigenvalues will be

$$\lambda = 0, \ldots, 0, -\frac{v}{2}, \ldots, -\frac{v}{2}, \frac{-v \pm \sqrt{v^2 + 16\alpha_i}}{4}$$

Since $\alpha_i > 0$, all eigenvalues are real and half of the eigenvalues involving the α_i are of each sign. Hence

$$\text{Forward time parabolic: } q_+ : \dim W^s(q_+) = \frac{1}{2}\dim T(X-\Delta)$$

$$\text{Backward time parabolic: } q_- : \dim W^u(q_-) = \frac{1}{2}\dim T(X-\Delta)$$

Examples:

- *equilateral CCs in the planar 3BP*
- *CCs of the 2Bp*
- *collinear CCs viewed in the collinear 3BP but in the planar 3BP they are saddles*

Theorem: Let s be a nondegenerate minimum CC of the N-body problem in R^d and let q_+ and q_- be the corresponding restpoints at parabolic infinity. Then $W^s(q_+)$ and $W^u(q_-)$ are Lagrangian submanifolds of the $u>0$ phase space. Moreover, the parts near infinity are Lagrangian graphs over the configuration space.

Sketch of proof: Recall that for the 2BP we had explicit formulas for the stable and unstable manifolds. We calculated the symplectic structure in blown-up coordinates and just checked that it vanished on the tangent spaces to these manifolds. That approach will not work here. Instead the proof uses estimates based on the eigenvalues and eigenvectors at the equilibrium points.

Why is it a graph near infinity ?

This follows from the nature of the stable eigenvectors at $q_\pm$. For example at q_+ (the restpoint with $v_0 > 0$) there is an eigenvector

$$(\delta u, \delta v, \delta s, \delta w) = (\delta u, 0, 0, 0) \qquad \lambda = -v_0$$

in the u direction, several eigenvectors

$$(\delta u, \delta v, \delta s, \delta w) = (0, 0, \delta s, \lambda \delta s) \qquad \lambda - -\frac{v_0}{2}$$

with δs tangent to the $\mathbb{SO}(2)$ orbit and eigenvectors

$$(\delta u, \delta v, \delta s, \delta w) = (0, 0, \delta s, \lambda \delta s) \qquad \lambda = \frac{-v_0 - \sqrt{v_0^2 + 16\alpha_i}}{4}$$

with δs transverse to the $\mathbb{SO}(2)$ orbit. The eigenspace projects isomorphically onto to the configuration space $(\delta u, \delta s)$.

Why is it Lagrangian ?

Consider a forward-time parabolic solution

$$\gamma(\tau) = (u(\tau), v(\tau), s(\tau), w(\tau)) \to q_+ \qquad \tau \to \infty.$$

Let $a(\tau), b(\tau)$ be solutions to the variational equations along $\gamma(\tau)$ which are tangent to $W^s(q_+)$. The symplectic form $\Omega(\tau) = \Omega(\gamma(\tau))(a(\tau), b(\tau))$ is constant, so it suffices to show that $\Omega(\tau) \to 0$ as $\tau \to \infty$.
The symplectic form in blown-up coordinates (u, v, s, w) can be written

$$\Omega = u^{-\frac{3}{2}}\, dv \wedge du + \sum m_i \left(u^{-\frac{1}{2}} ds_i \wedge dw_i + \frac{1}{2} u^{-\frac{3}{2}} s_i \cdot dw_i \wedge du \right)$$

and we will just estimate each term in this sum. Since $u(\tau) \to 0$ the factors $u^{-\frac{1}{2}}, u^{-\frac{3}{2}} \to \infty$. But since $u' = -vu$ and $v(\tau) \to v_0$ and $u(0) > 0$ we can derive upper bounds of the form

$$u^{-\frac{1}{2}} \le c_1 \exp(v_0 \tau) \quad u^{-\frac{3}{2}} \le c_2 \exp(v_0 \tau)$$

The other factors tend to zero exponentially at rates determined by the stable eigenvalues at q_+:

$$-\frac{v_0}{2}, \ldots, \frac{v_0}{2}, \lambda_i = \frac{-v_0 - \sqrt{v_0^2 + 16\alpha_i}}{4}.$$

Since we are at a nondegenerate minimum, the eigenvalues not associated to the $\mathbb{SO}(d)$ symmetry satisfy

$$\lambda_i < -\frac{v_0}{2}$$

Using these estimates one can show that the exponential decay rates are sufficient to overcome the growth of $u^{-\frac{1}{2}}, u^{-\frac{3}{2}}$ to give $\Omega(\tau) \to 0$.

Minimality Properties of Parabolic Orbits

If $(q(t), v(t)), t \in D$ is a solution of the N-body problem then for every $[a,b] \subset D$, $q(t)$ is a critical curve of the action functional

$$A(q) = \int_a^b L(q(t), \dot{q}(t))\, dt \qquad L(q,v) = \frac{1}{2}\|v\|^2 + U(s)$$

- q is ***minimizer*** *on [a,b]* if A(q) is minimal among all absolutely continuous curves γ with γ(a) = q(a) and γ(b) = q(b)
- q is a ***global minimizer*** if this is true for every [a,b] in D
- q is a (global) ***free time minimizer (FTM)*** if for every [a,b], A(q) is minimal among all absolutely continuous curves γ on some time interval [c,d] with γ(c) = q(a) and γ(d) = q(b)

FTMs for the N-body problem have been studied recently by Maderna, Venturelli, Da Luz, Percino, Sanchez Morgado, Barutello, Secchi, ….
and in a joint paper by me, Richard Montgomery and Hector Sanchez Morgado

Some Result about FTMs in the N-body Problem

- (DaLuz/ Maderna) Every FTM is a parabolic solution. It follows that $h = 0$, all particles tend to infinity with zero asymptotic speed, the blown-up solution converges to the set of equilibria at $u=0$. For the 3BP this means converging to one of the 5 CC shapes.

- (Maderna / Venturelli) The equilateral homothetic solution starting at triple collision and tending parabolically to infinity is a FTM

- (Maderna / Venturelli, Percino / Sanchez) Let s_0 be a minimal CC of the NBP. Then given any initial configuration q_0, there exists a FTM beginning at q_0 and asymptotic to s_0. As a corollary: the stable manifold of the corresponding restpoint at infinity must cover the whole configuration space.

This existence result applies to the 2BP and to the equilateral CCs in the 3BP since these are minimal CCs

Relation with dynamics on the parabolic infinity manifold $u = 0$

What about the collinear CCs ? Could we have a FTM for the 3BP which is in the stable manifold of a collinear CC ?

Suppose that the mass ratios are chosen so that the collinear restpoint exhibits spiraling. Then it turns out that the corresponding parabolic orbits *cannot* be minimal, much less FTM. The intuitive explanation is that, due to the spiraling, nearby solutions oscillate around the parabolic solution producing "conjugate points". Then one gets nearby curves with lower action. If all three collinear restpoints are spiraling (true for most masses) then any FTM must converge to an equilateral CC shape. Open problem when no spiraling.

This is in the joint paper with R.Mont., H. Sanchez but also follows from work of Barutello and Secchi by a different proof.

Are all equilateral parabolic orbits FTMs ?

Theorem: If q(t) is a parabolic orbit asymptotic to an equilateral CC, then the part of the orbit sufficiently close to infinity is a FTM.

Idea of proof:

- Limiting restpoint q_+ at u=0 has a local stable manifold which is a Lagrangian graph
- Orbit in Lagrangian graphs are minimizers when compared with curves lying under the graph (lift the curves to the graph, then cleverly use the Lagrangian property to compare actions)
- The orbit segment close to infinity is a global minimizer since it would be too expensive to leave the region under the graph and then return
- Take q_0 close to q_+. There exists a FTM starting at q_0 and tending to q_+. It must be the the same as the global minimizer over q_0

Hamilton-Jacobi Equation

Hamiltonian of the N-body problem: Let $p_i = m_i v_i$

$$H(q,p) = \frac{1}{2}\|p\|^2 - U(q) \qquad \|p\|^2 = \sum \frac{|p_i|^2}{m_i}$$

Standard symplectic structure so Lagrangian graphs are of the form

$$p = df(q) \qquad \text{for some real valued function} \qquad f(q)$$

Apply this to the local stable manifold $W^s_{loc}(q_+)$. The energy equation gives

$$H(q, df(q)) = \frac{1}{2}\|df(q)\|^2 - U(q) = 0 \qquad \text{Hamilton-Jacobi equation}$$

On the other hand, Percino and Sanchez Morgado proved their existence result using a weak solution of the Hamilton-Jacobi equation. The results above show that this solution is actually smooth near infinity.

Fine (The End)