CHAOS IN THE THREE-BODY PROBLEM

POINCARE´ 100 CONFERENCE

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Poincaré and Chaos

Poincaré's first encounter with chaos came in his revision of his Prize Memoir on celestial mechanics. He realized that the invariant curves associated to an unstable periodic motion could cross one another and that this would produce complicated dynamical behavior nearby. Later, near the end of the last volume of his treatise "Les Méthodes Nouvelles de la Mécanique Celeste," he wrote the famous description:

If one tries to imagine the figure formed by these two curves with an infinite number of intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind of trellis, a fabric, a network of infinitely tight mesh; each of the two curves must not cross itself but it must fold on itself in a very complicated way to intersect all of the meshes of the fabric infinitely many times.

One will be struck by the complexity of this picture, which I will not even attempt to draw.

Homoclinic tangle, homoclinic chaos

Examples of Chaos in the Three-Body Problem

Since Poincaré there has been a lot of work on proving that such chaotic behavior occurs in the three-body problem. One goal of the talk is to survey a small part of this work.

- ☐ Poincaré's chaos -- PCR3BP
- ☐ Chaos near infinity in the Sitnikov problem
- ☐ Chaos near triple collisions in the Sitnikov problem

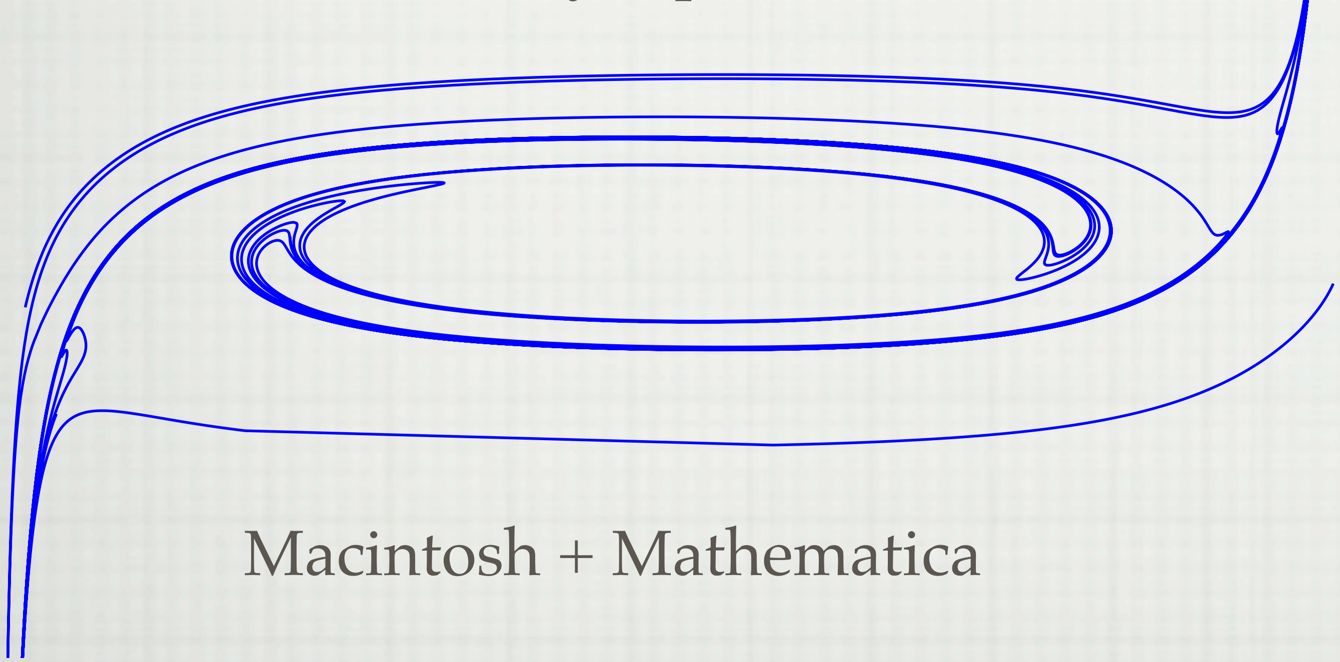
(Other examples: near Lagrange points, near double collisions, planar three-body problem, etc.)

Work of Poincaré, Sitnikov, Alexeev, Moser, McGehee, Conley, Easton, Simo, Llibre, Xia, Bolotin, Mackay, Gorodetski, Kaloshin, R.M.

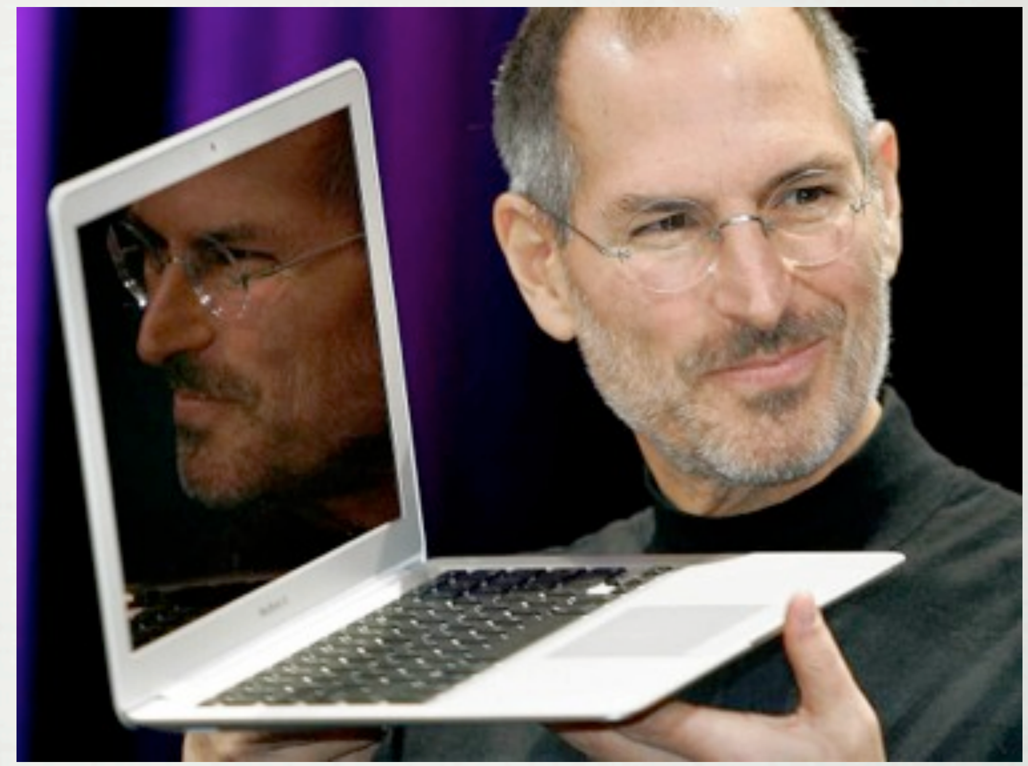
Attempting To Draw

Another goal of the talk is to show some pictures that Poincaré could only imagine.

With a little help from our friends (the computers) we can visualize Poincaré's trellises and also simulate the chaotic motions of the three bodies that they represent.



Macintosh + Mathematica



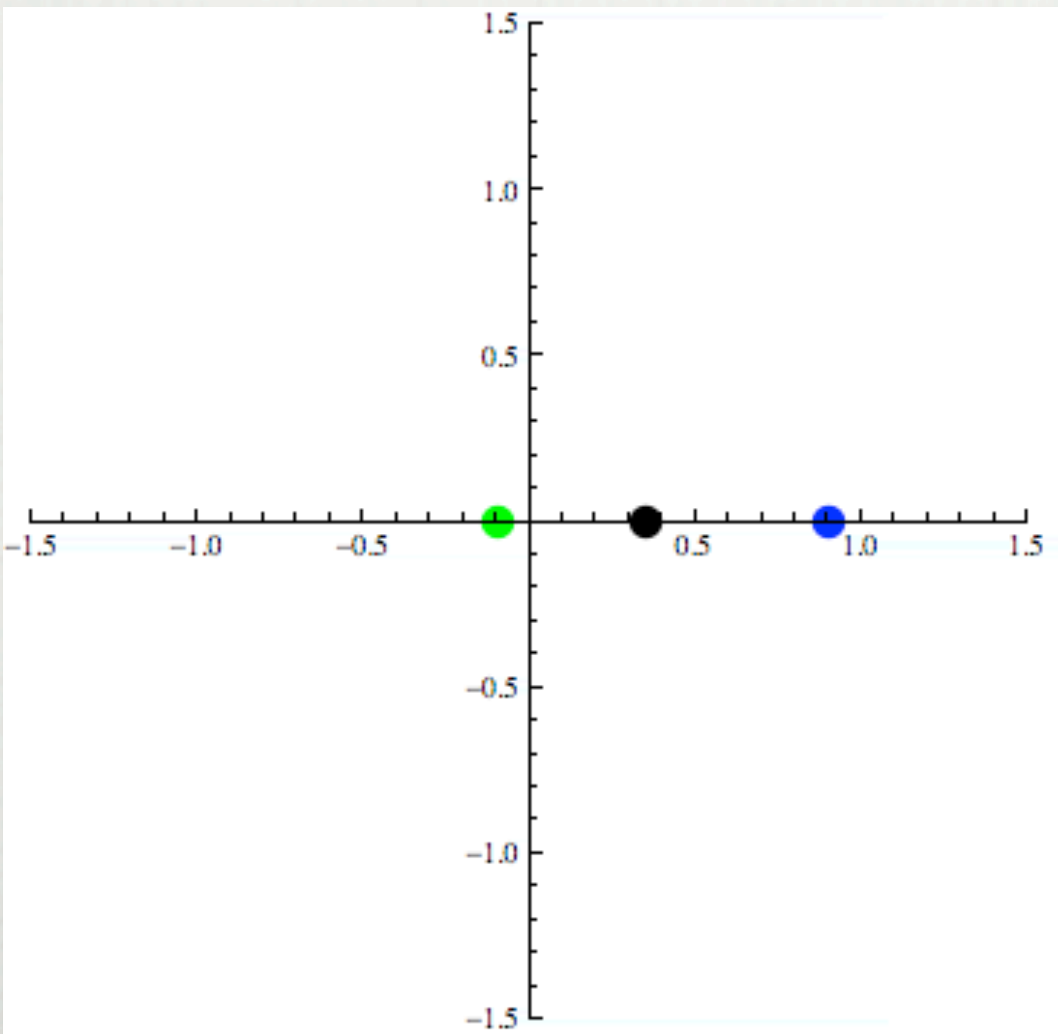
It turns out that Poincaré had a really good imagination -- even with computers some of the things he imagined are hard to see.

Poincaré's Three-Body Problem

Poincaré worked mostly on the Planar, Circular, Restricted Three-Body Problem or PCR3BP. We have two *primary* bodies with masses

$$m_1 = 1 - \mu \qquad m_2 = \mu$$

These move on a circular orbit of the two-body problem. A third body of negligible mass moves in the plane under the influence of the gravitational forces of the primaries.



In this example,
the Green body has mass 0.9
the Blue body has mass 0.1
the Black body has mass 0

The green and blue masses are moving around the origin in circles at constant angular velocity 1. The motion of the black mass is more interesting.

Rotating Coordinates

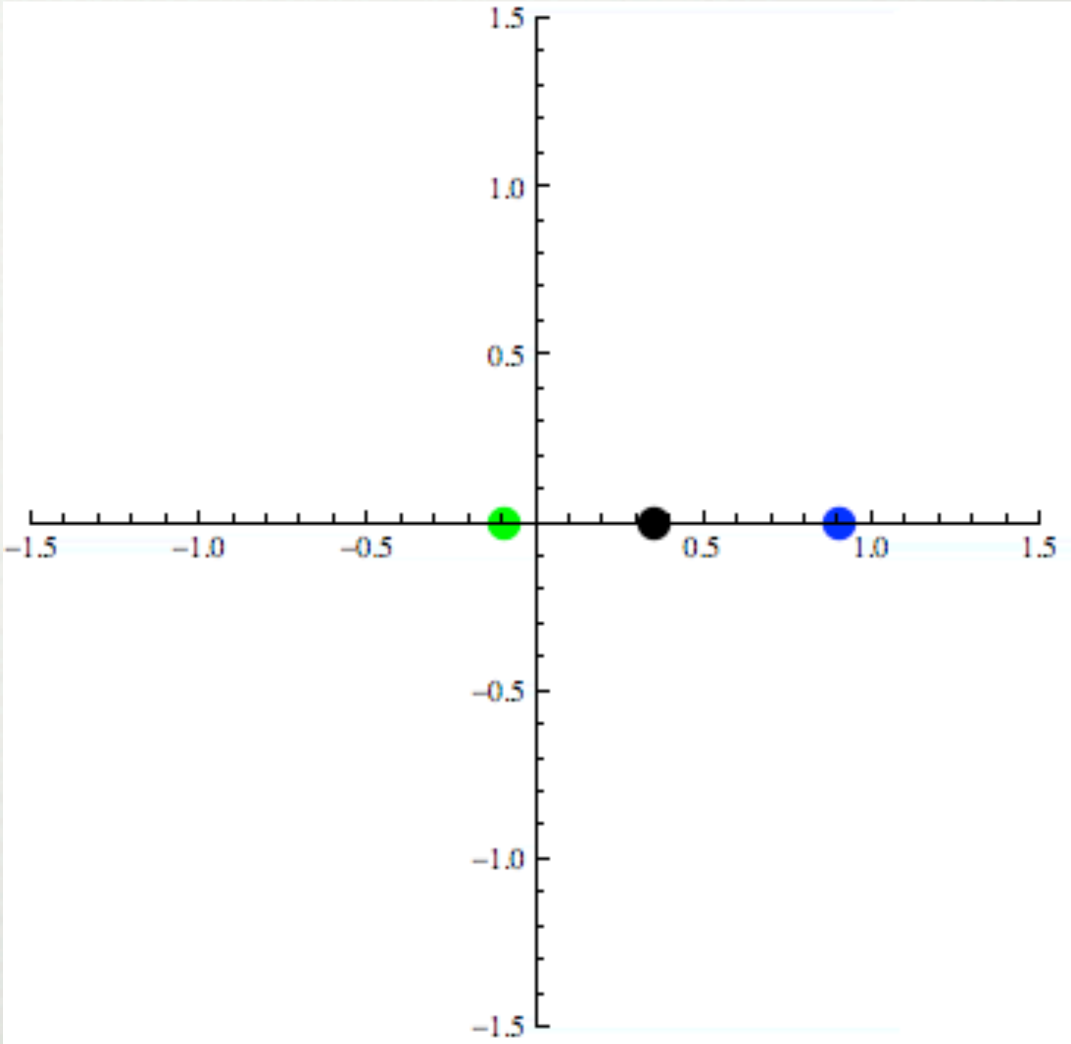
In a uniformly rotating coordinate system, the primaries remain fixed on the x-axis and we can concentrate on the third body. Here is the same solution as on the last slide. We can see that in the rotating frame, the black mass moves on a very simple periodic orbit. Poincaré called this a ***periodic solution of the first sort***.

The PCR3BP in rotating coordinates is a dynamical system of *two degree of freedom*. The state of the black body is determined by two position variables (x,y) and the corresponding velocity variables (u,v). Thus the phase space has dimension 4. However, there is a conserved quantity which we will call the "energy":

$$\frac{1}{2}(u^2+v^2) - V(x,y) = h = const$$

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2+y^2}}$$

Thus the motion takes place on a three-dimensional submanifold in the four-dimensional phase space



Hill's Regions

The conservation of "energy" implies that the third body moves in the region

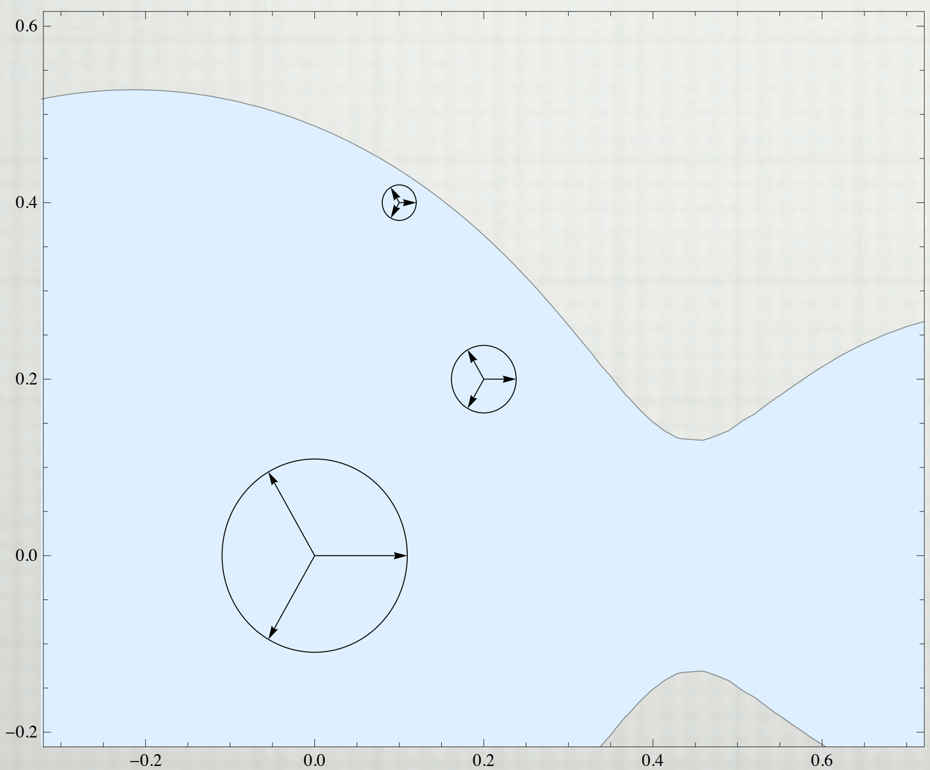
$$\mathcal{H}(h) = \{(x,y) : V(x,y) + h \geq 0\}$$

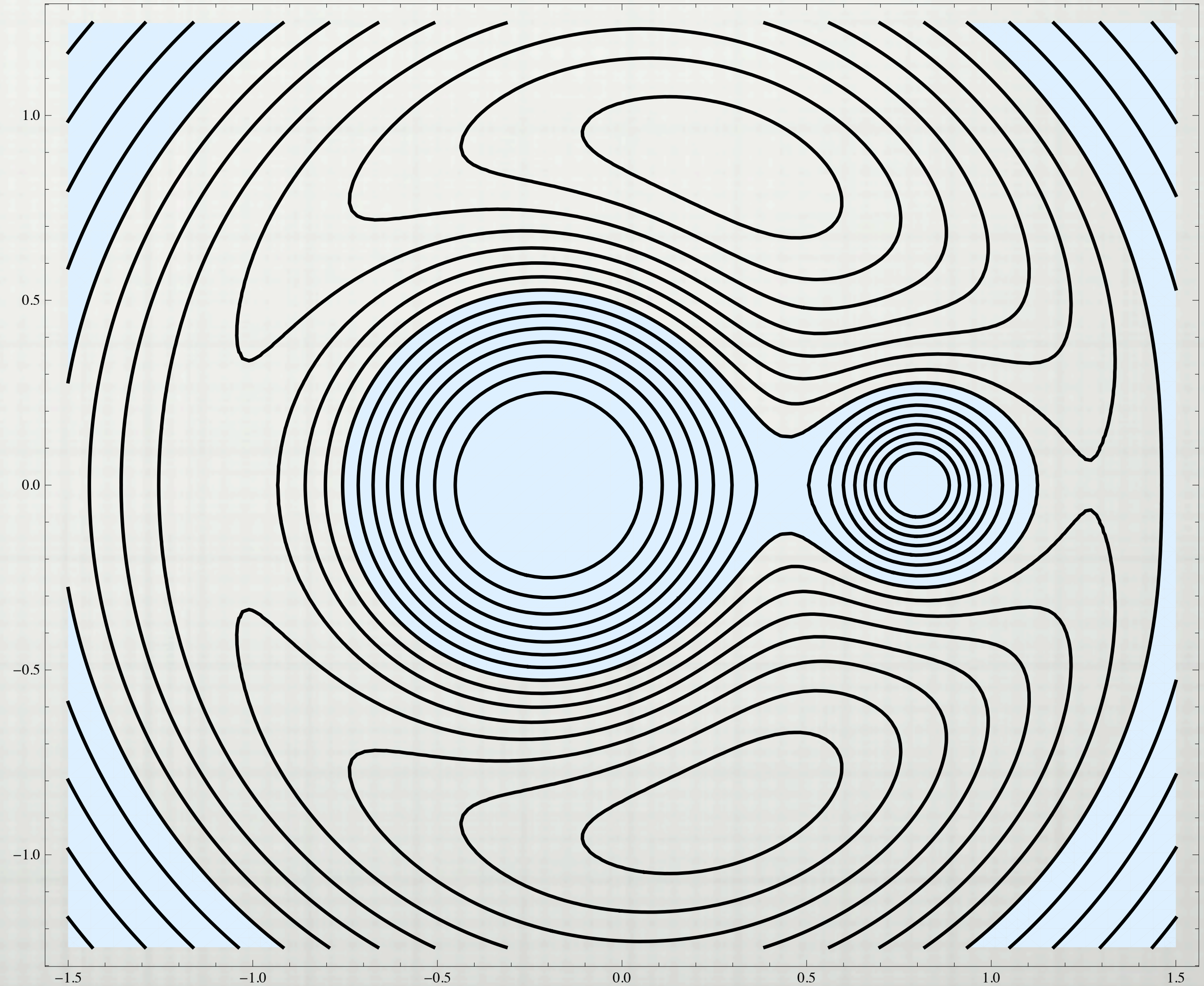
Here is a contour plot of V(x,y). A *Hill's region* for a fixed energy h is shaded in light blue. For a motion with this energy, (x(t),y(t)) must remain in the blue region.

At each point (x,y) there is a circle of admissible velocities (u,v). On the boundary curve, only the zero velocity vector

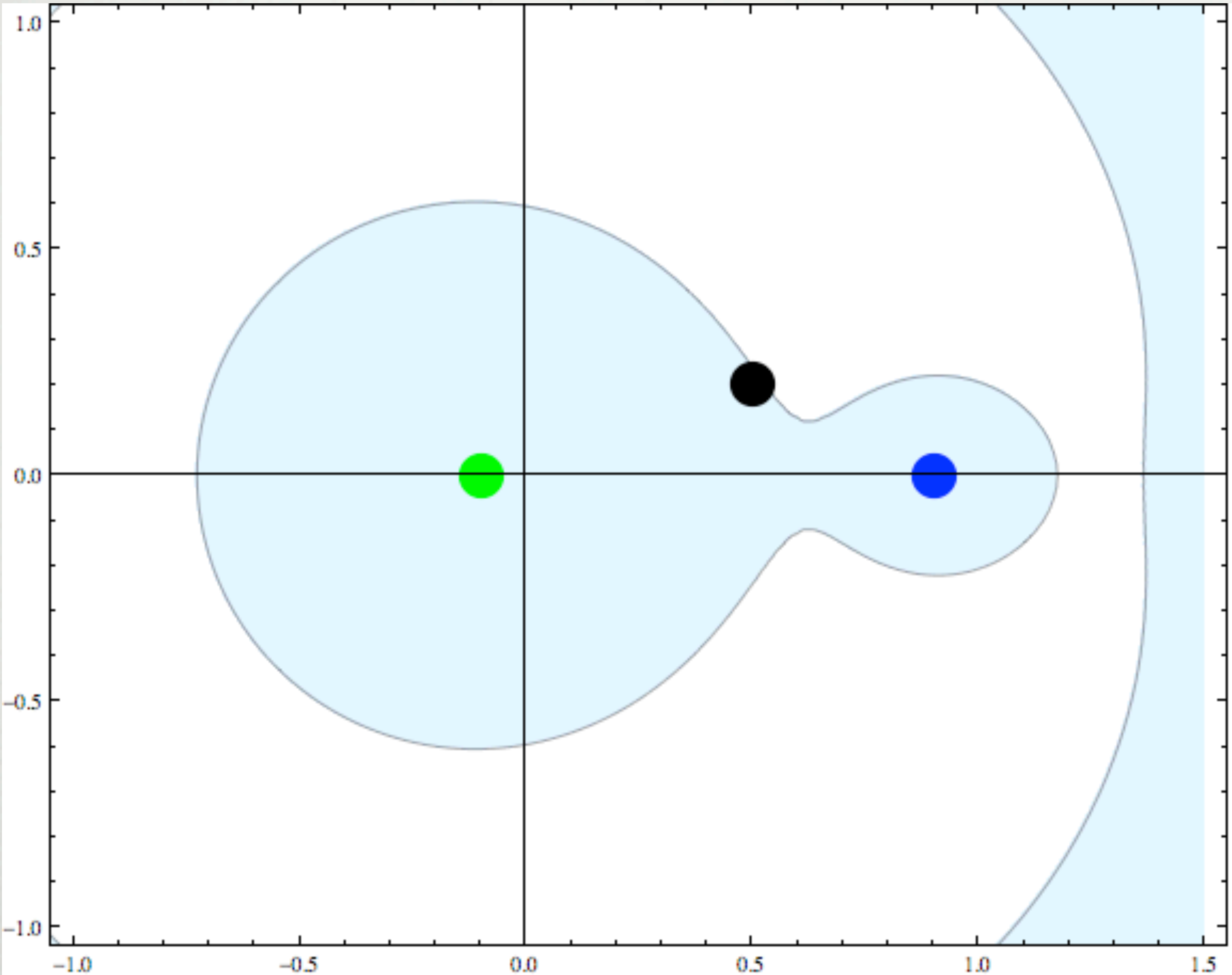
$$(u,v)= (0,0)$$

is admissible.





Example of a PCR3BP Orbit in its Hill's Region



This is a "transit" orbit which makes (possibly chaotic) excursions from one primary mass to the other. Imagine a spacecraft moving back and forth between the earth and the moon.

Poincaré Continuation -- First Sort Orbits

When $\mu = 0$ we have primary masses $m_1 = 1$ and $m_2 = 0$. The third body is influenced only by m_1 so the PCR3BP reduces to the two-body problem (but in rotating coordinates). Following Poincaré, we can let m_3 move on a *circular* two-body orbit and then ask what happens when $\mu > 0$. Using the implicit function theorem, one can *continue* these orbits to simple periodic solutions of the PCR3BP – the periodic orbits of the *first sort*.

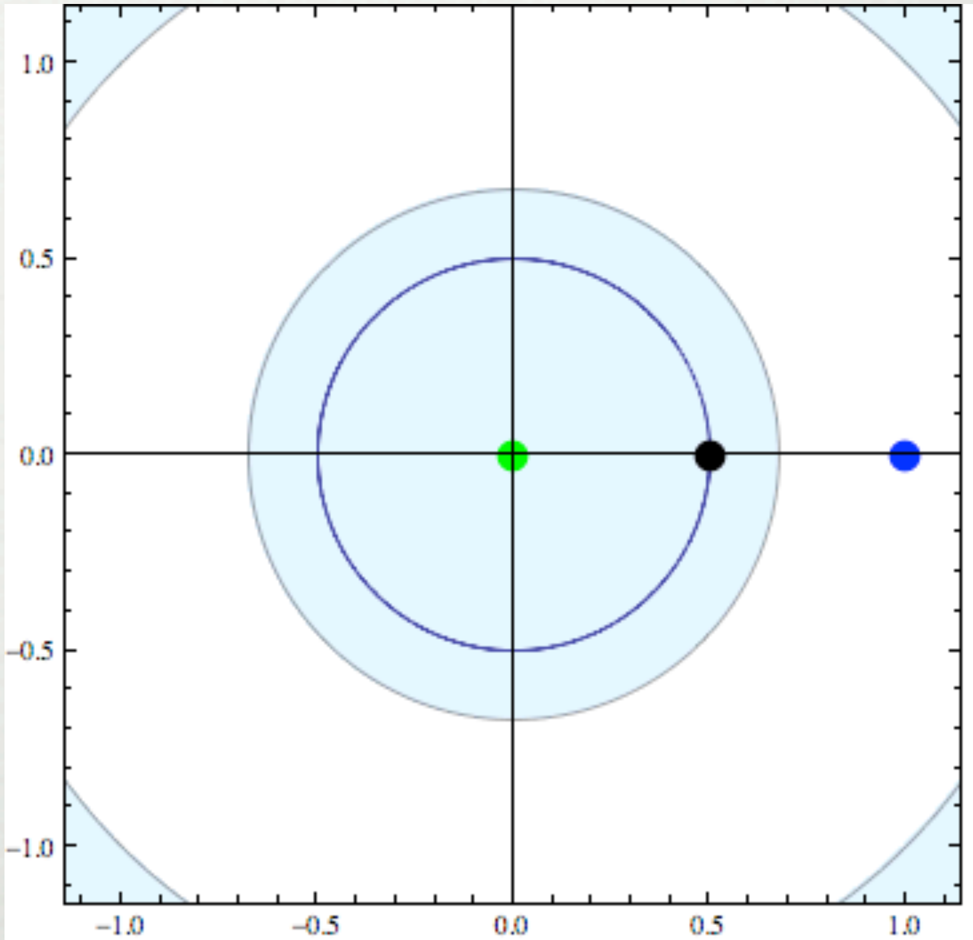
Here is a sequence of first-sort orbits, starting from the circular orbit with

$$\mu = 0$$

and continuing it over the range

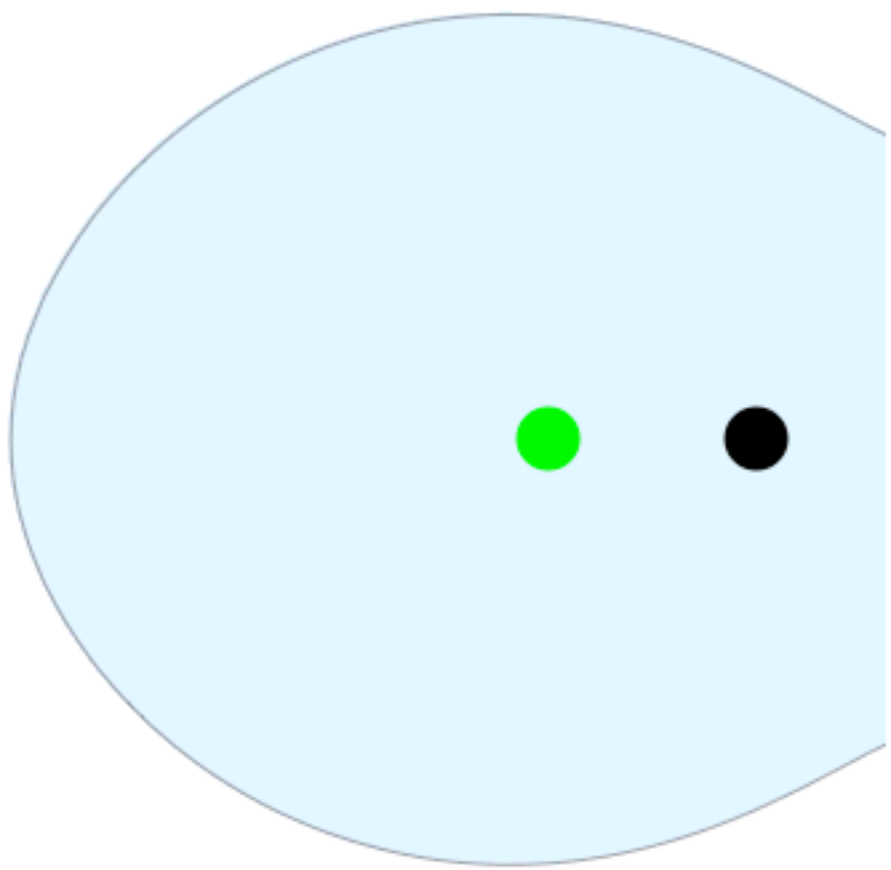
$$0 \leq \mu \leq \frac{1}{2}$$

The shapes change but we always have a simple, periodic orbit which closes up after one revolution (in rotating coordinates).

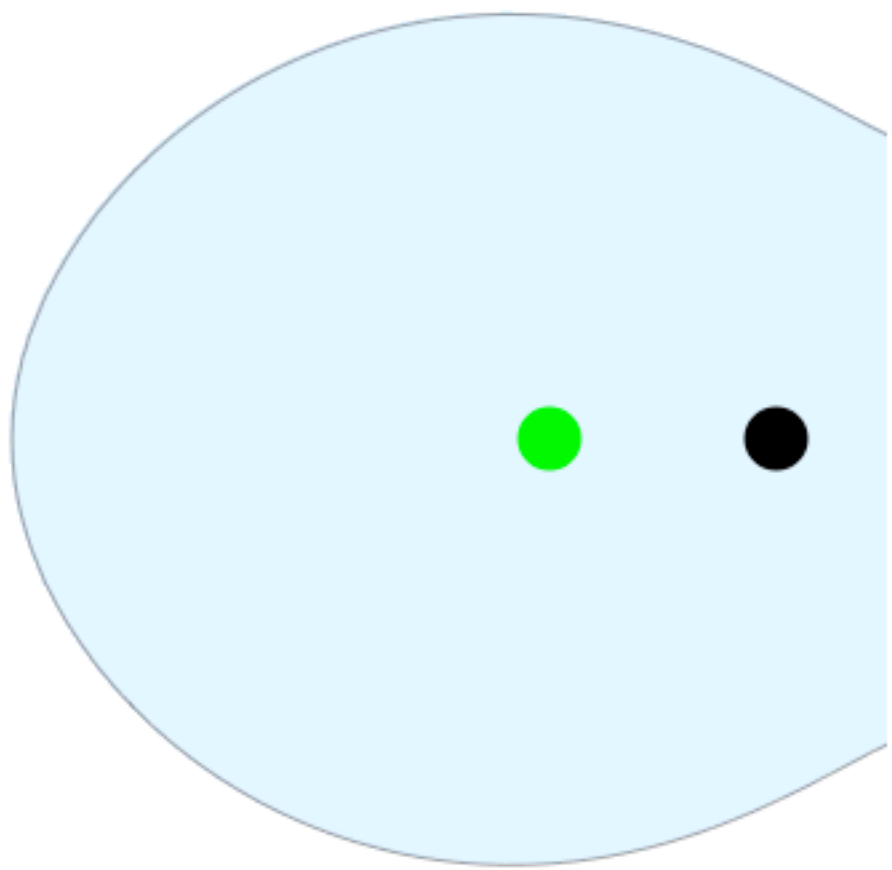


Second-Genre Orbits (Harmonics)

Choosing one of these first-sort orbits (here we choose one with $\mu = 0.4$), we can look for nearby periodic motions which close up after n revolutions. Their periods will be close to nT where T is the period of the first sort orbit. Poincaré devoted many pages of his "Méthodes nouvelles .." to the study of these orbits and it is here where he found chaos.



First-sort orbit



Nearby second-genre orbit with n=3

Poincaré Sections

Another of Poincaré's big ideas in dynamical systems is that the behavior of dynamical systems of two degrees of freedom (i.e., the continuous-time motion of a phase point moving on a three-dimensional energy manifold) can often be reduced to the behavior of a discrete-time *Poincaré return map* of a two-dimensional surface, a *Poincaré section* of the flow.

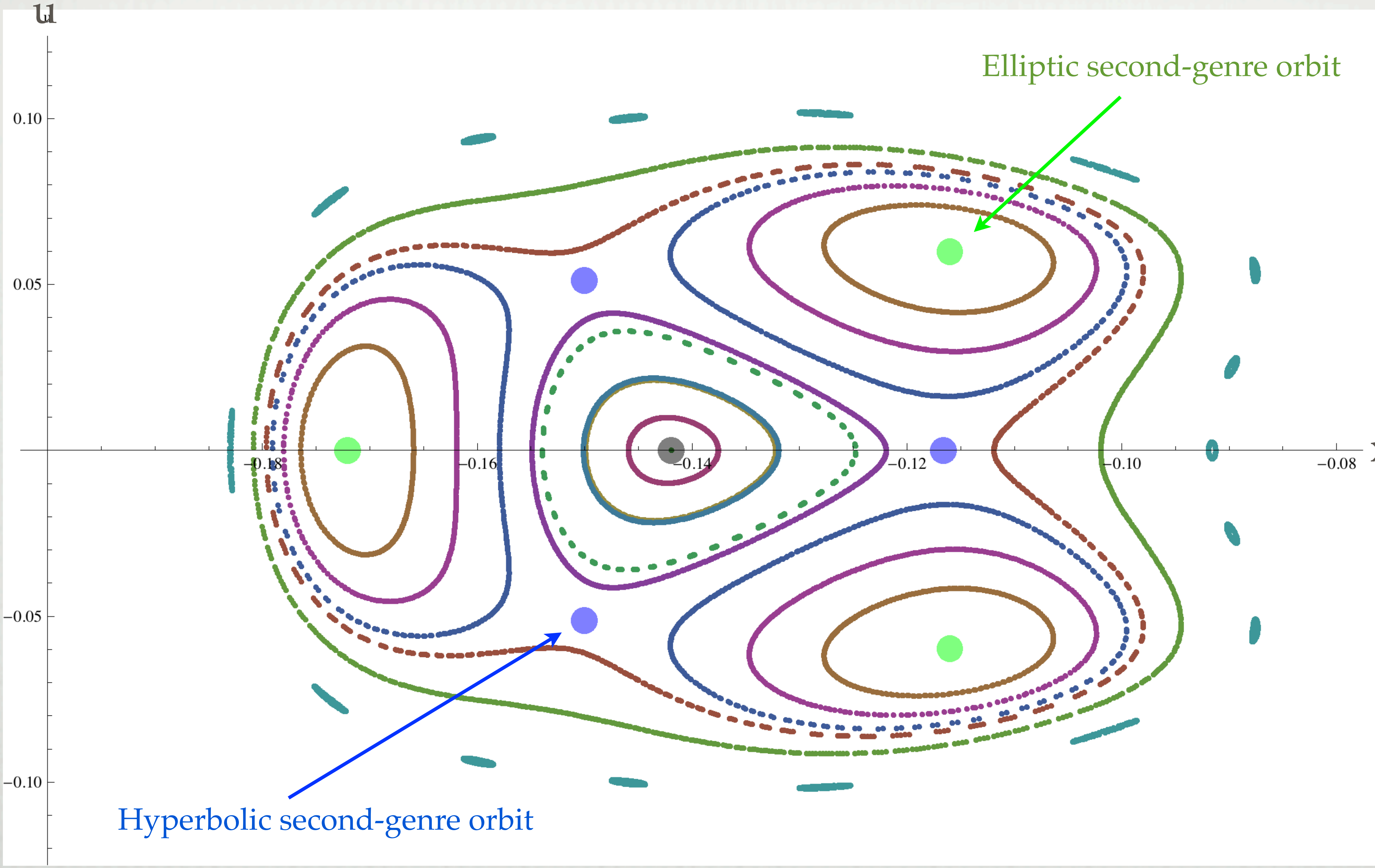
For example, in the PCR3BP we can record the state of the third body every time it crosses the x-axis in the rotating system. The periodic orbit of the first sort which we have been using as an example is a fixed point of the corresponding Poincaré map since after one revolution of the rotating plane, it return to its initial state.

In both the rotating and inertial frames, we could say that we are recording the *syzygies* or *eclipses* of the orbit -- states where the three bodies are collinear. In the Poincaré section, the state of the third body is of the form $(x,y,u,v) = (x,0,u,v)$ where (u,v) is on the circle of velocities corresponding to the position $(x,0)$.

We will eliminate v and use (x,u) as two coordinates on the Poincaré section.

Poincaré Section for $\mu = 0.4,\ h = -1.70711\ldots$

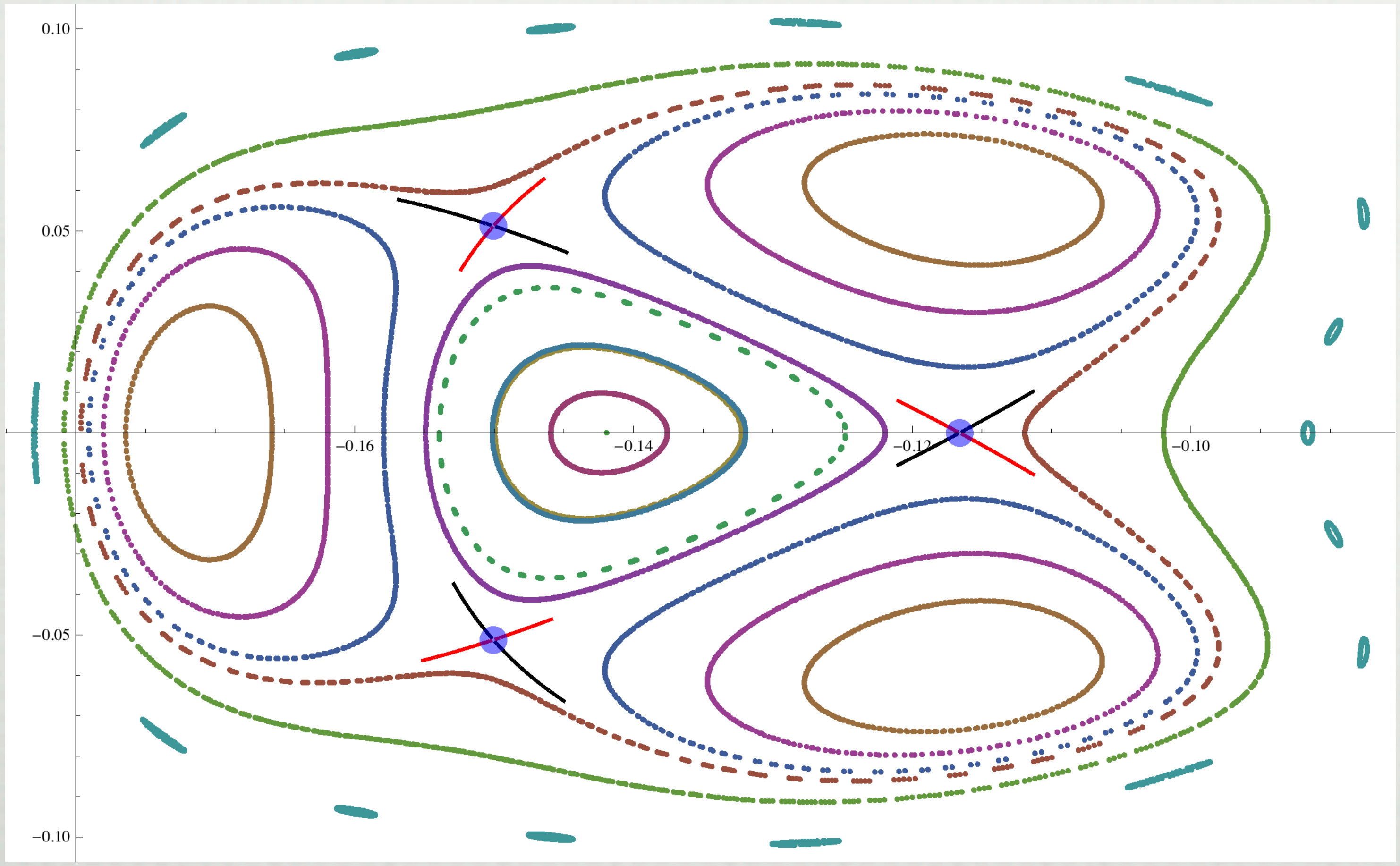
Poincare map of (x,u)-plane. Gray fixed point represents the first-sort orbit. Green and blue period three points represent second-genre orbits.



Asymptotic and bi-asymptotic solutions

Hyperbolic periodic solutions have stable and unstable manifolds consisting of solutions which converge to the periodic solution as $t \to \pm\infty$. Poincaré called these *asymptotic solutions*. A solution which lies in the both the stable and unstable manifolds approaches the periodic solution in both time directions and is called *bi-asymptotic*.

For our Poincaré map, the stable and unstable manifolds are curves in the (x, u)-plane. Orbits in the stable curve (red) approach the period-3 point in forward time; orbits in the unstable curve (black) approach in backward time.



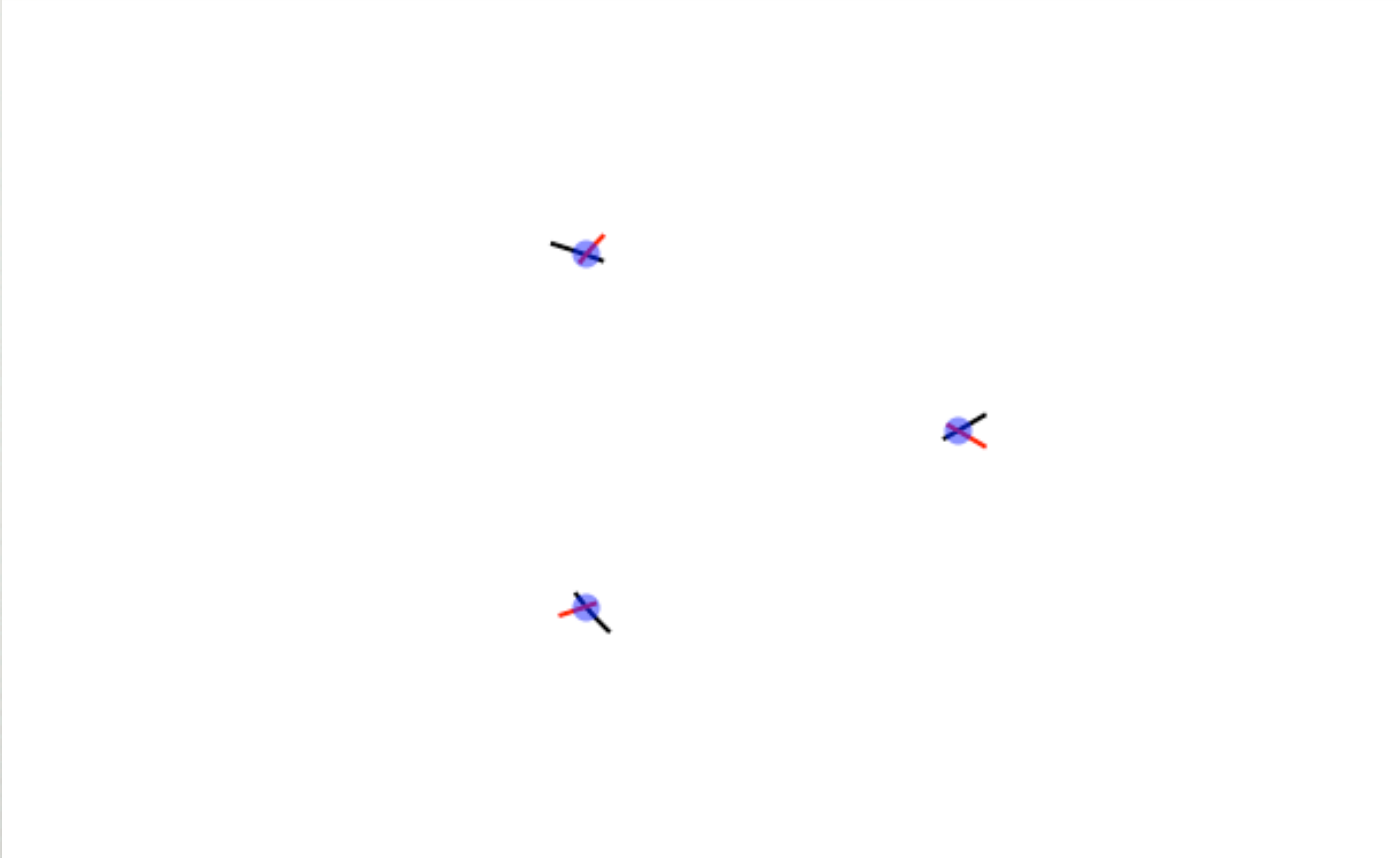
This picture shows short pieces of the stable and unstable manifolds of the 2nd-genre orbit.

The curves are invariant under the dynamics so they can be extended by repeatedly applying the Poincaré map and its inverse.

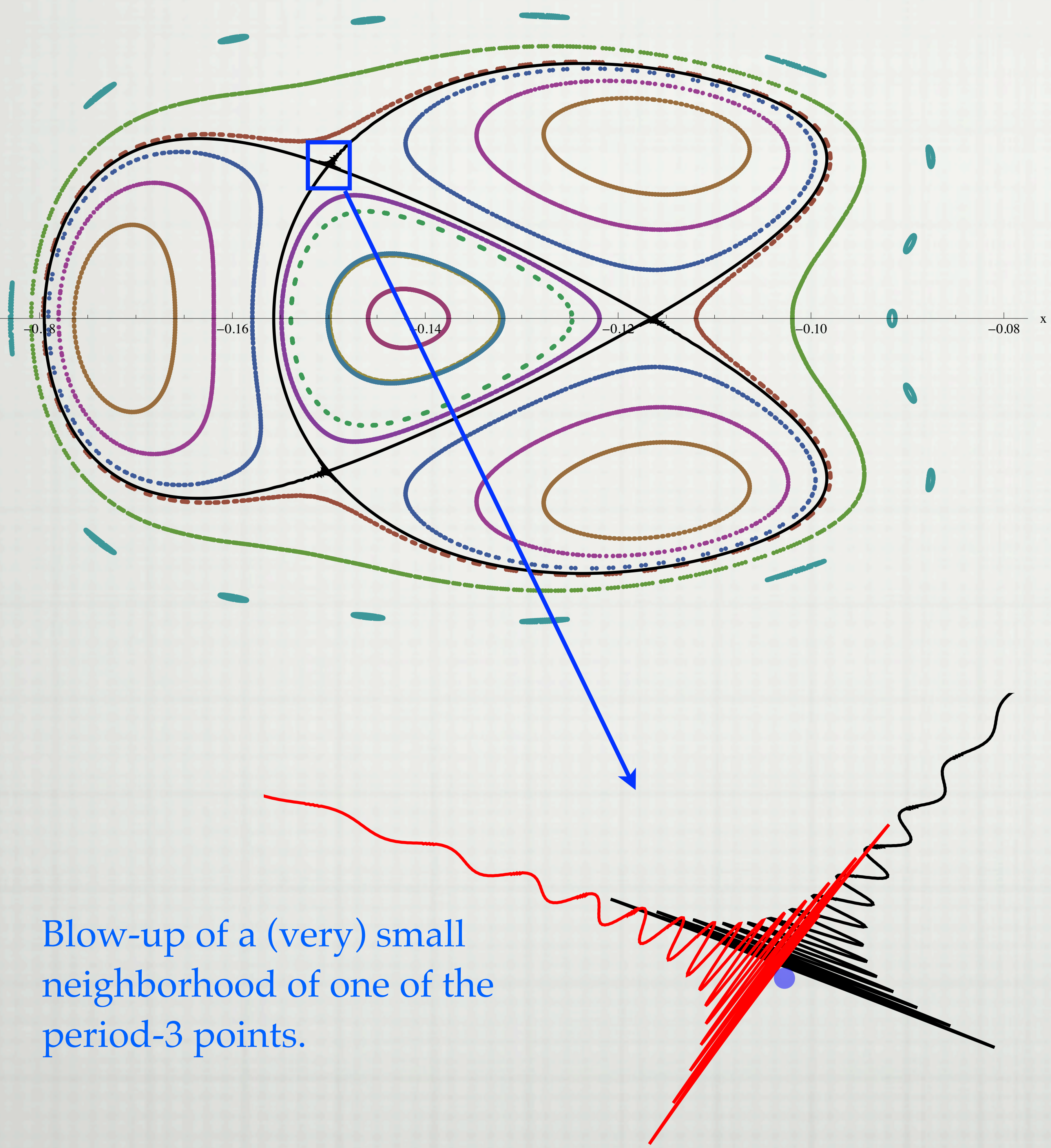
To find bi-asymptotic solutions we want the extended curves to intersect.

Extending the Invariant Manifolds

Here is what happens as we extend the stable and unstable manifolds. It appears that the stable and unstable manifolds overlap to produce curves of bi-asymptotic solutions. These seem to converge to the periodic orbit in both time directions but with a phase shift of 1/3 period.



Poincaré's Chaos-- Transversality of the Curves



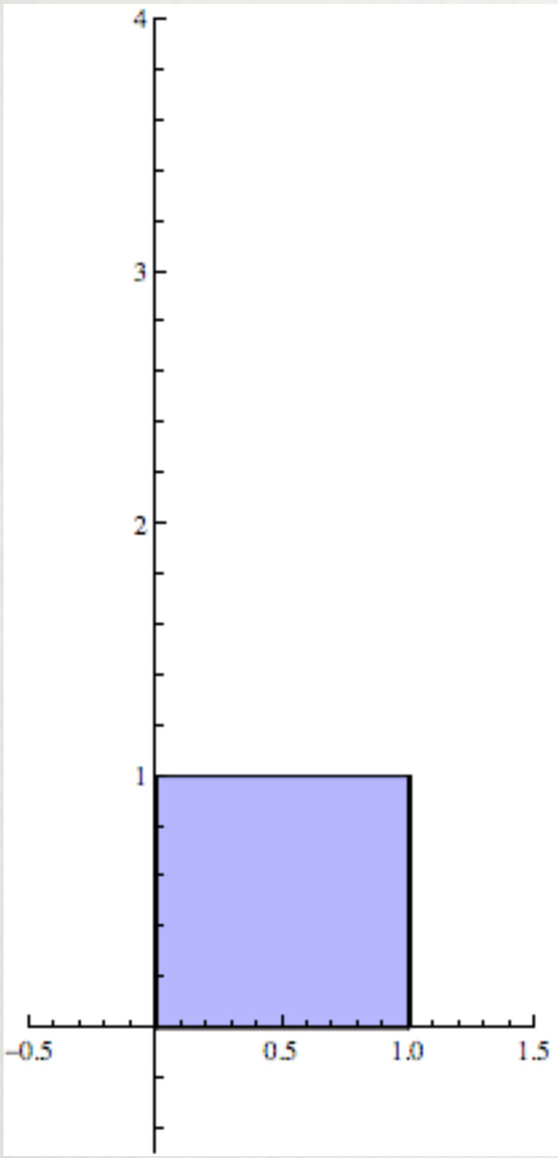
Blow-up of a (very) small neighborhood of one of the period-3 points.

Poincaré's famous paragraph from the beginning of the talk arose from his realization (while pondering his errors in his original prize manuscript) that while the stable and unstable curve had to intersect, they did not have to overlap exactly. In fact, more likely, they could intersect *transversely*.

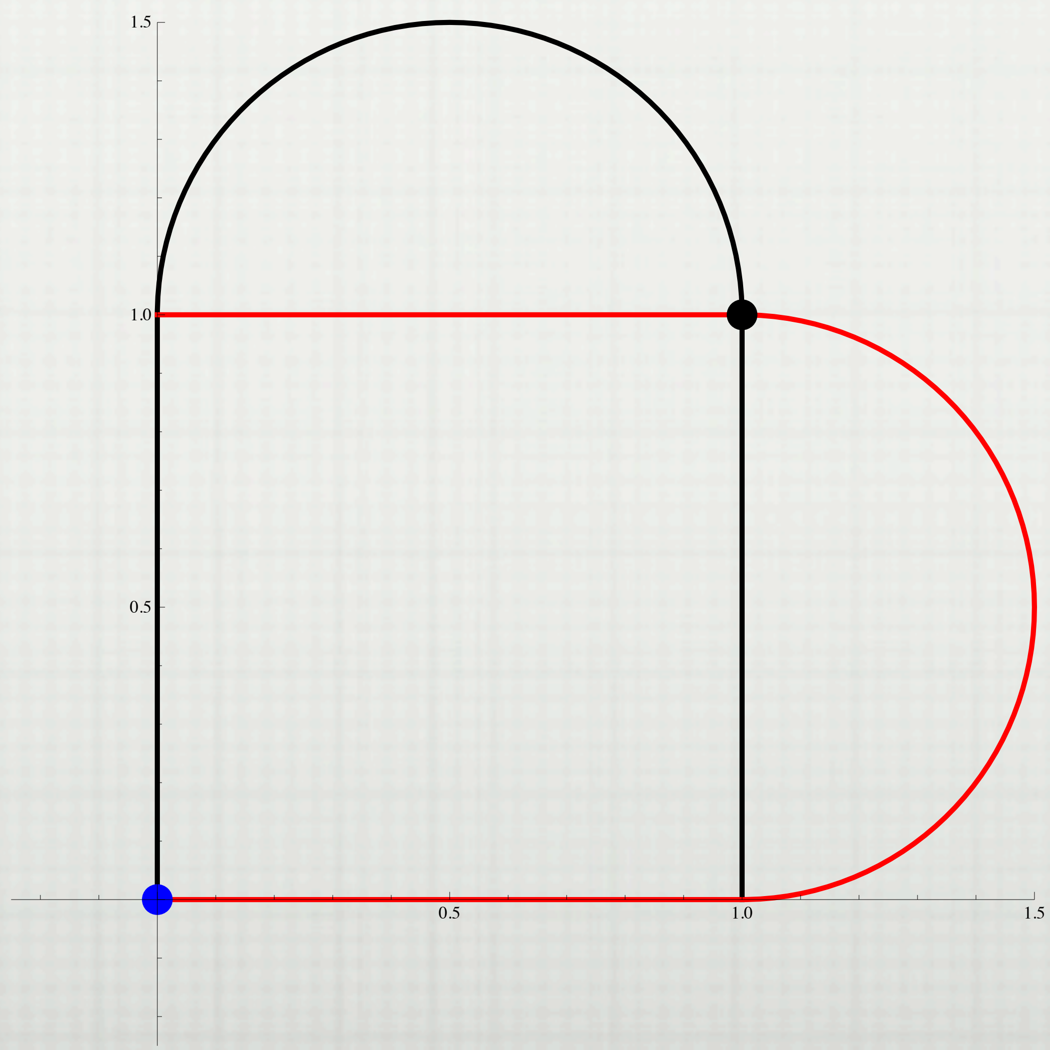
Then because they are invariant the hyperbolic dynamics forces them to fold and cross in a very complicated manner.

This seems to be true here, but the trellises are very small and hard to see, even with the computer.

The Smale Horseshoe

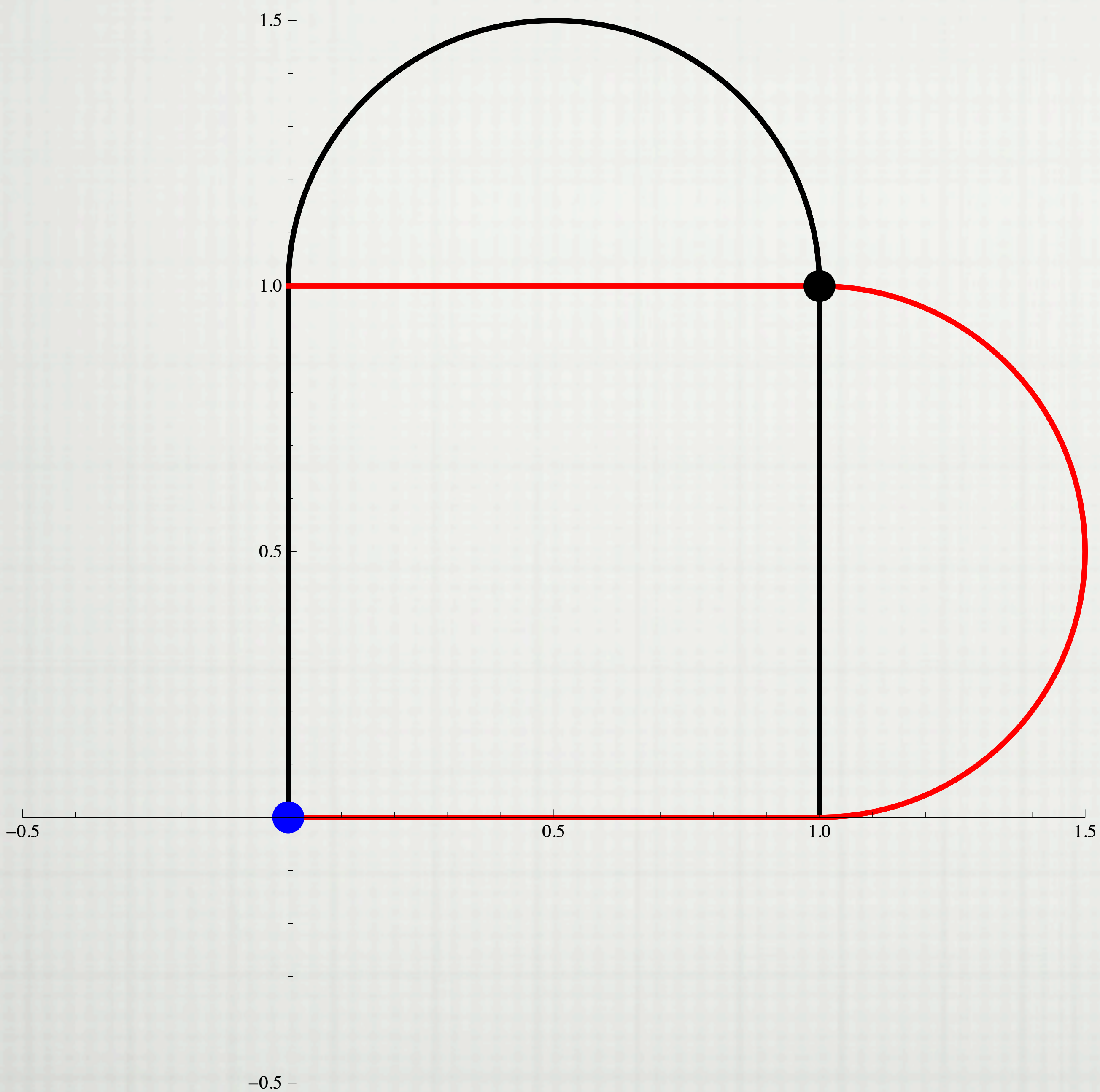


Smale's horseshoe map is a simple model of the dynamics produced when stable and unstable manifolds cross transversely. Repeated hyperbolic stretching followed by folding applied to the unit square produces a fractal shape.



There is a hyperbolic fixed point at the origin with stable manifold (red) and unstable manifold (black) along the axes. Folding produces a transverse homoclinic point.

A Simple Trellise



Repeated folding and stretching of the invariant manifolds produces a beautiful trellise just as Poincaré imagined:

each of the two curves must not cross itself but it must fold on itself in a very complicated way to intersect all of the meshes of the fabric infinitely many times.

In particular we get infinitely many other intersections (bi-asymptotic orbits) nearby.

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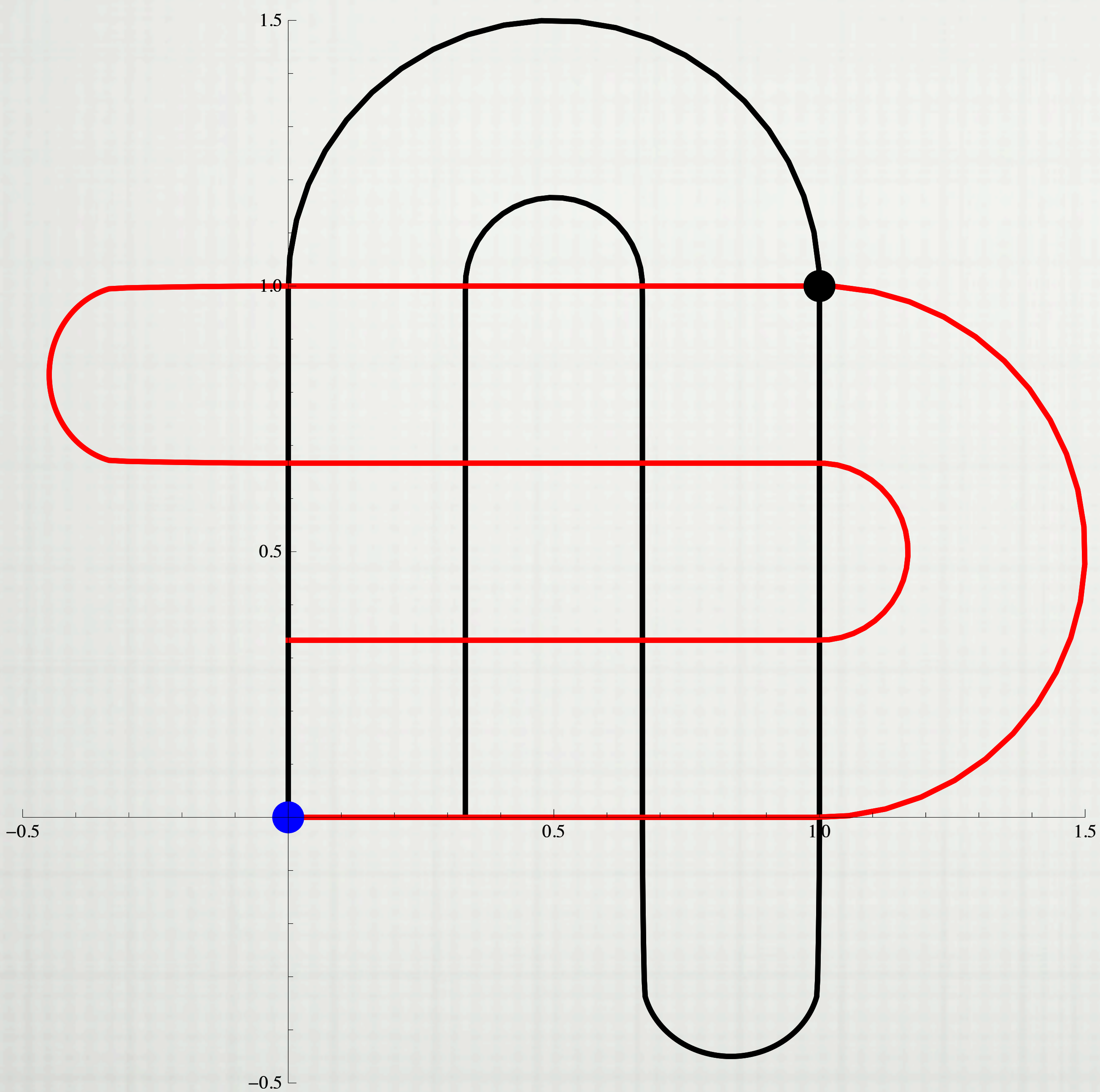

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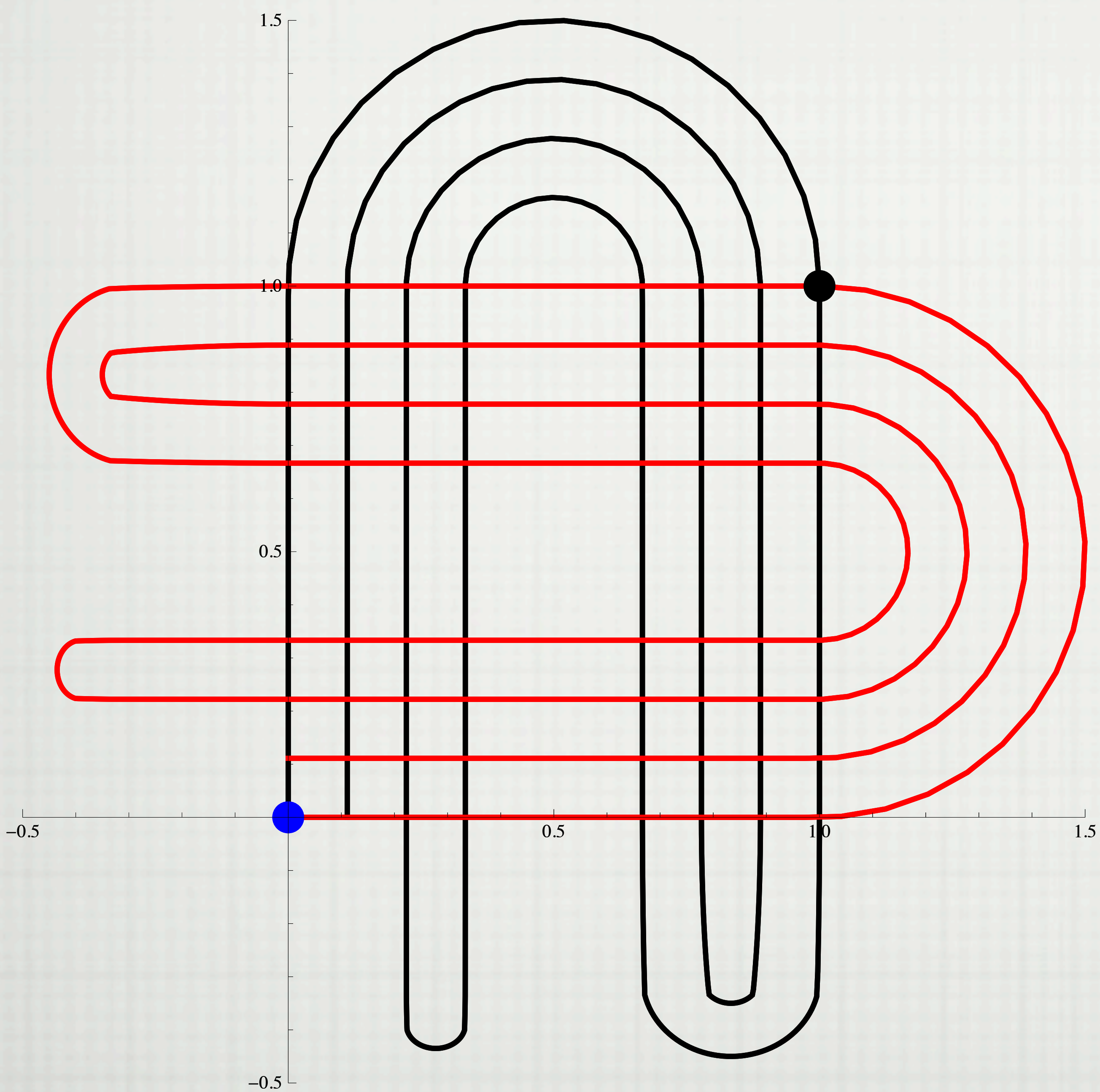

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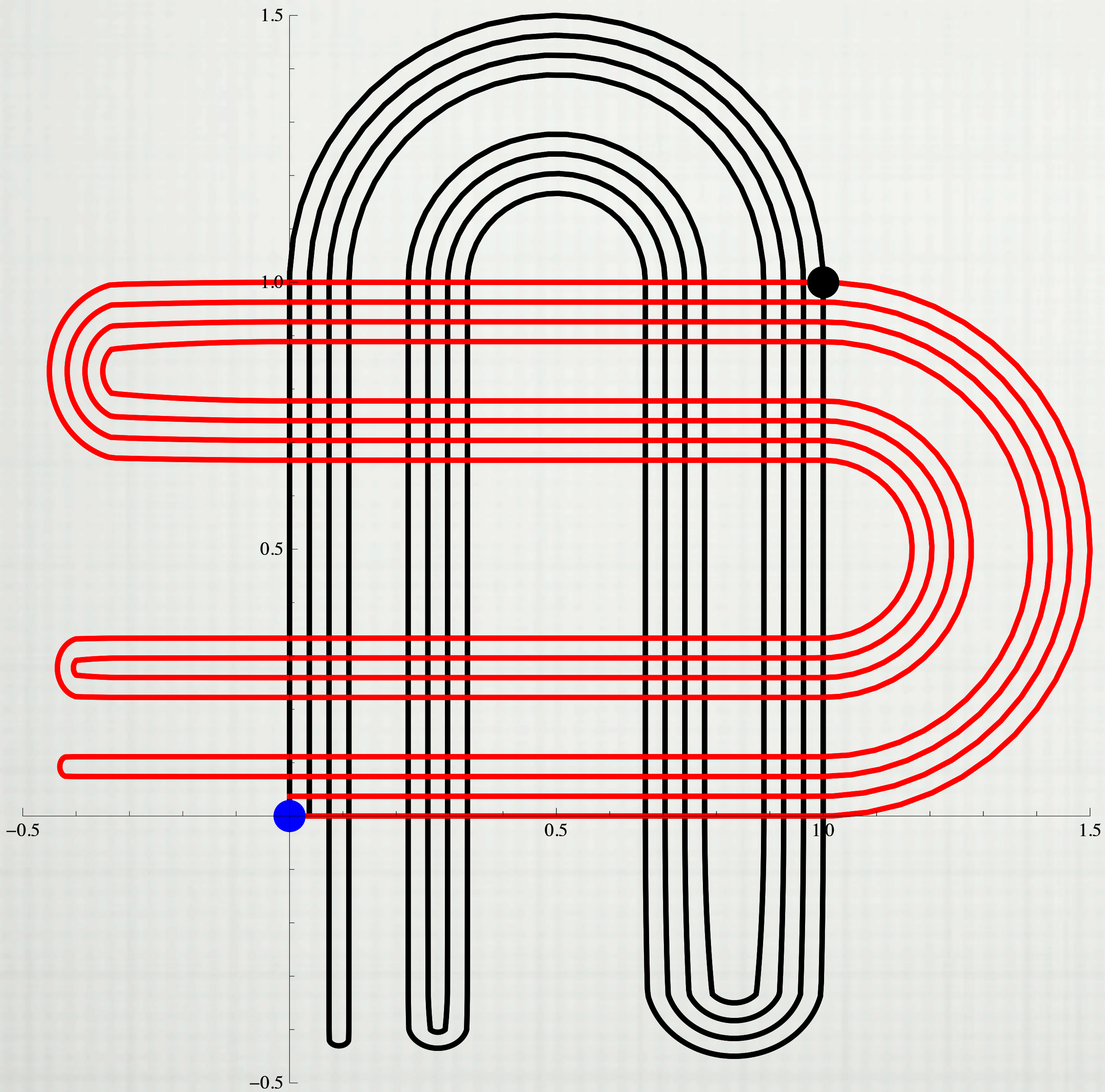

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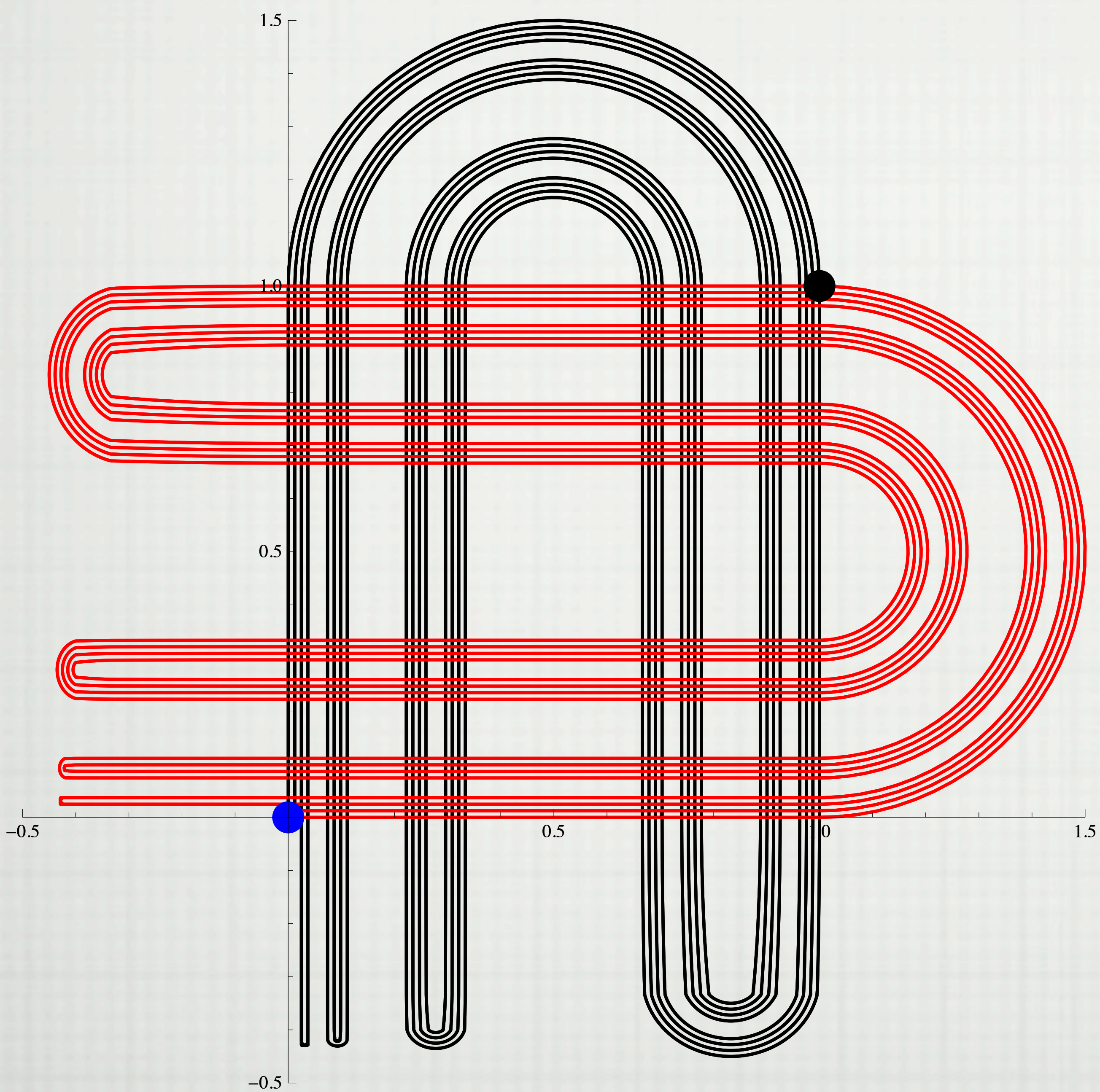

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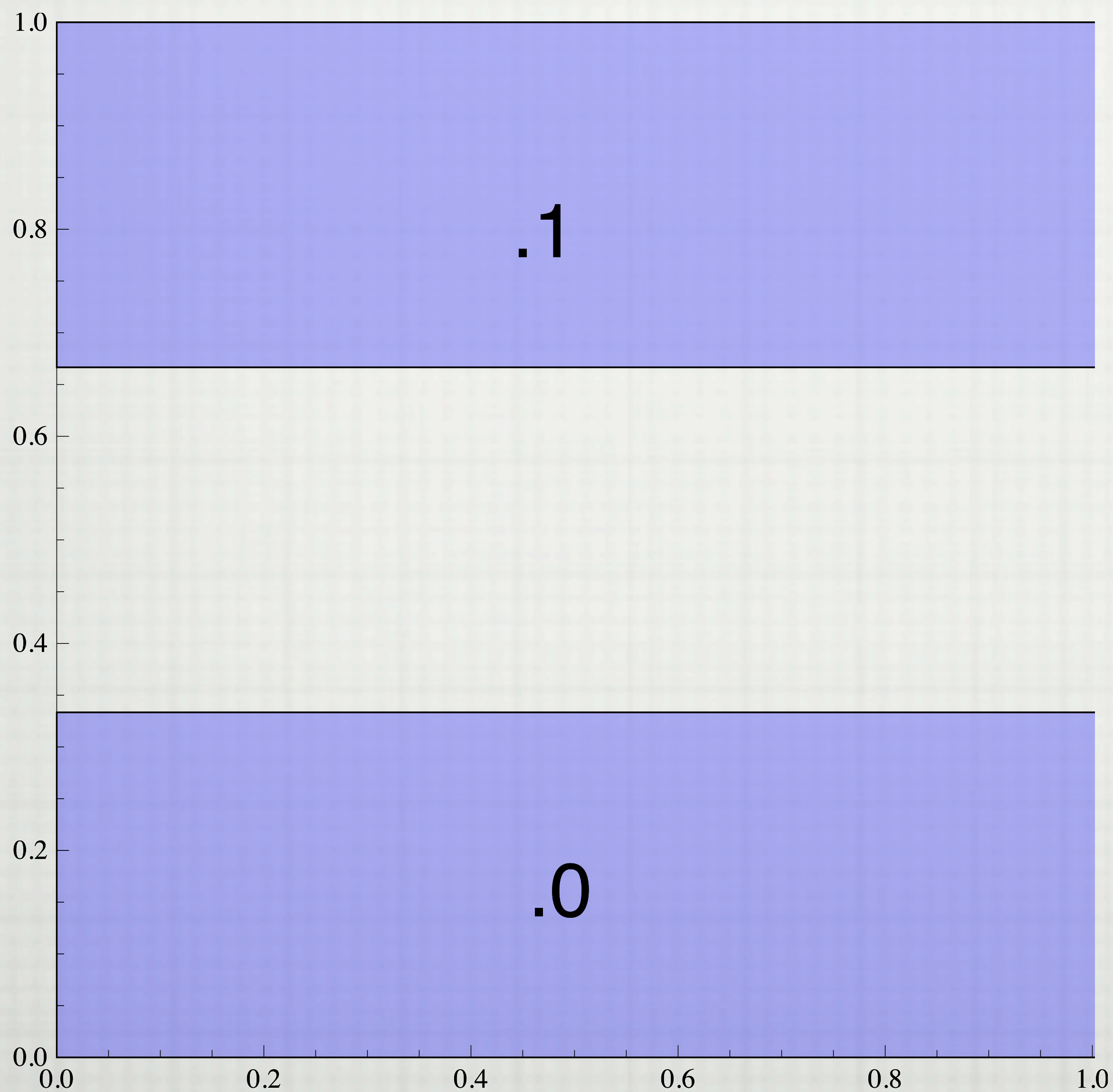
Symbolic Dynamics and Chaos

Poincaré may not have known the full story about the nearby dynamics. Nowadays we use symbolic sequences (itineraries) to code the trajectories.

$$\ldots \epsilon_{-2}\,\epsilon_{-1}\,.\epsilon_0\,\epsilon_1\,\epsilon_2 \ldots \qquad \epsilon_n = 0, 1$$



We have a one-to-one correspondence between orbits of the horseshoe map which remain in the unit square and bi-infinite sequences of 0's and 1's which describe how the orbit "hops" between box 0 and box 1.

This is an (uncountably) infinite set of orbits which includes all of the (countably infinite) intersection points of the stable and unstable curves but many other orbits as well.

All itineraries are realized by an orbit, even "random" ones produced, say, by a coin toss.

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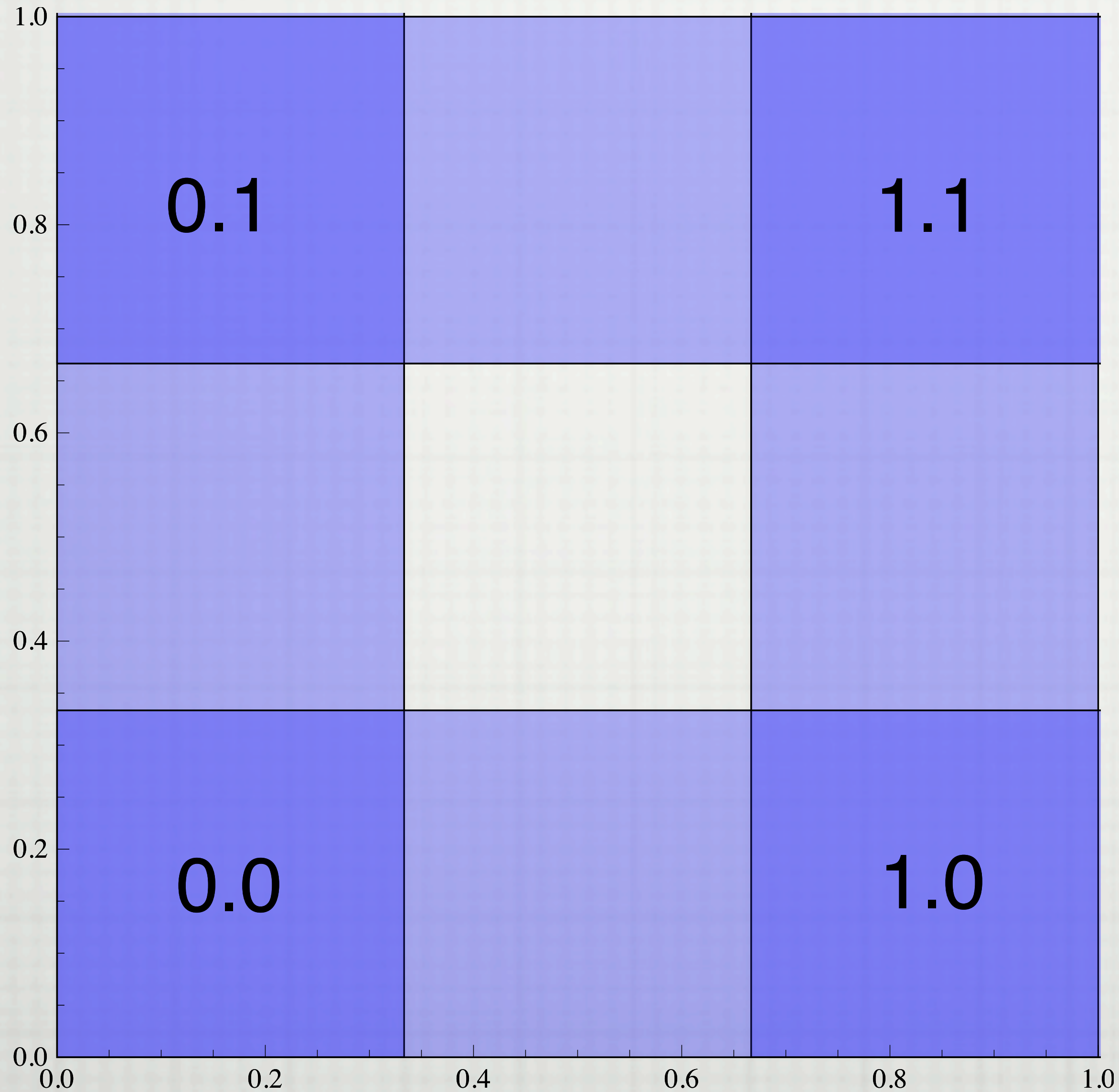

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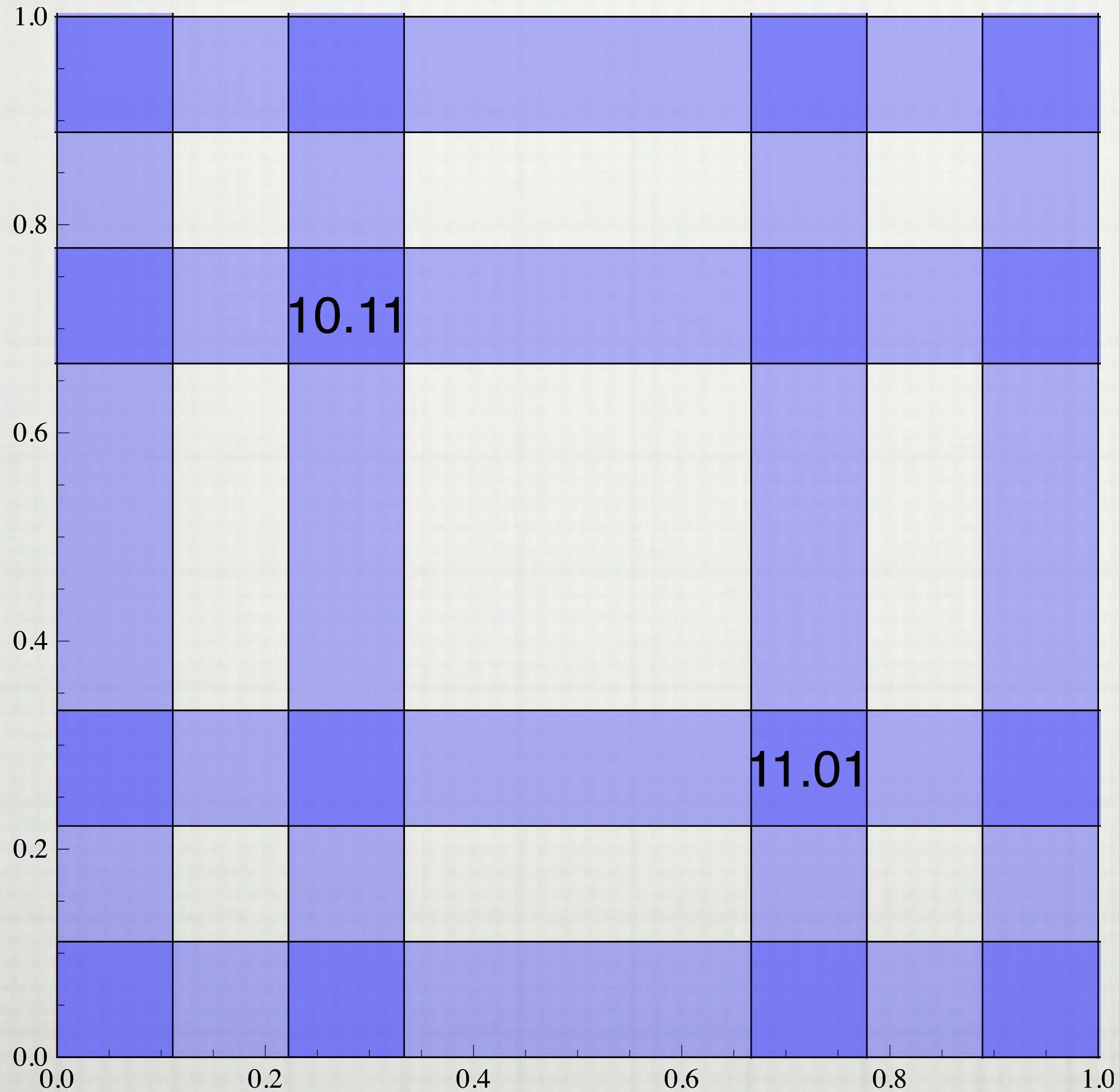

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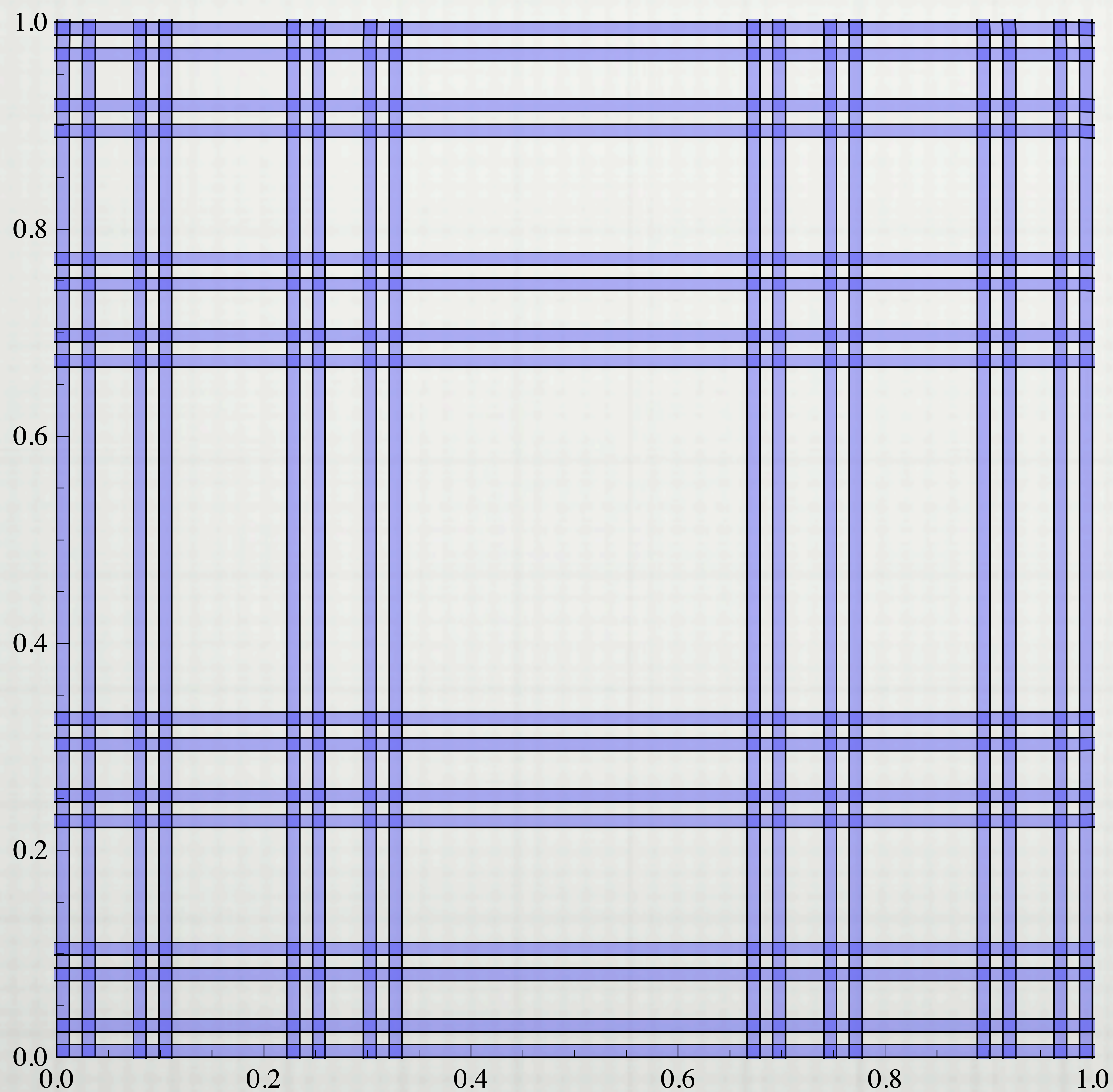

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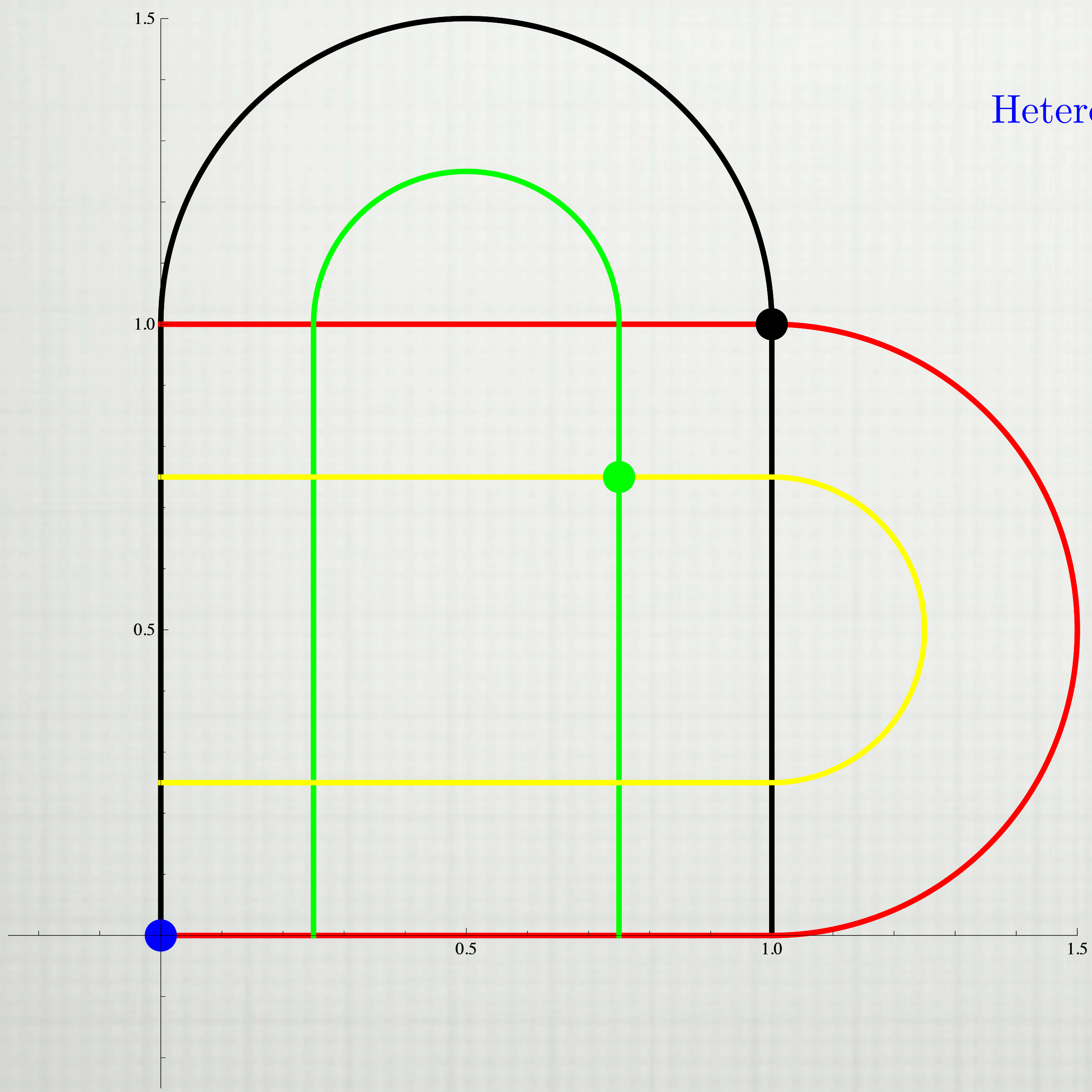
Itineraries related to the fixed points

Fixed Point $(0,0)$	$\ldots 00000.00000\ldots$
Homoclinic Point $(1,1)$	$\ldots 00001.10000\ldots$
Other Homoclinic Points	$\ldots 00000\epsilon_m \ldots \epsilon_n 00000\ldots$
Fixed Point $(\frac{3}{4},\frac{3}{4})$	$\ldots 11111.11111\ldots$
Heteroclinic $(0,0)\to(\frac{3}{4},\frac{3}{4})$	$\ldots 00000\epsilon_m \ldots \epsilon_n 11111\ldots$



The fixed point (3/4, 3/4) has its own stable and unstable manifolds which form a separate trellise which interlaces with the trellise of (0,0). For example the yellow and red curves must fold in such a way that they never cross.

But the itineraries of all these point account for only countably many itineraries. The point is that all of the uncountably many itineraries are realized.

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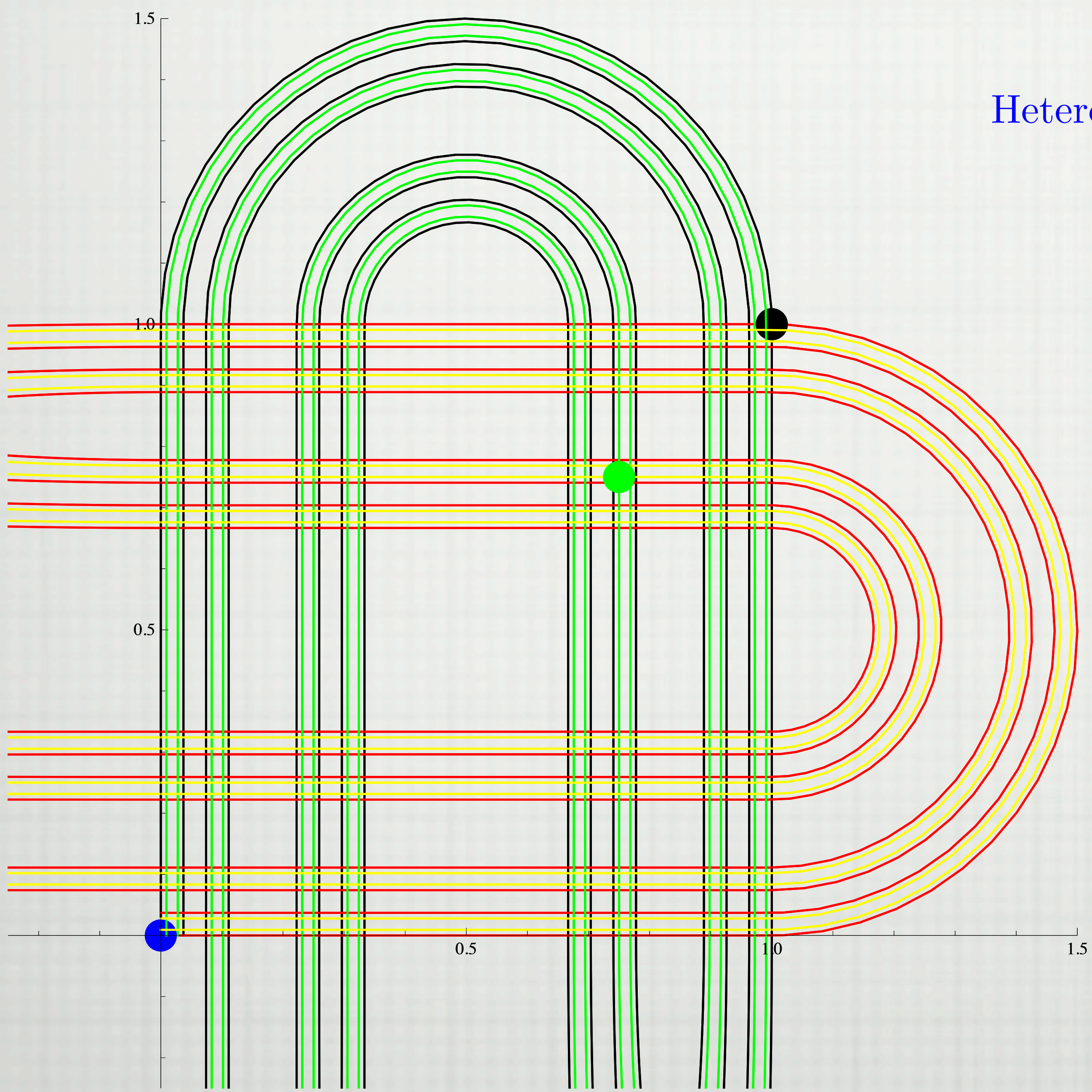

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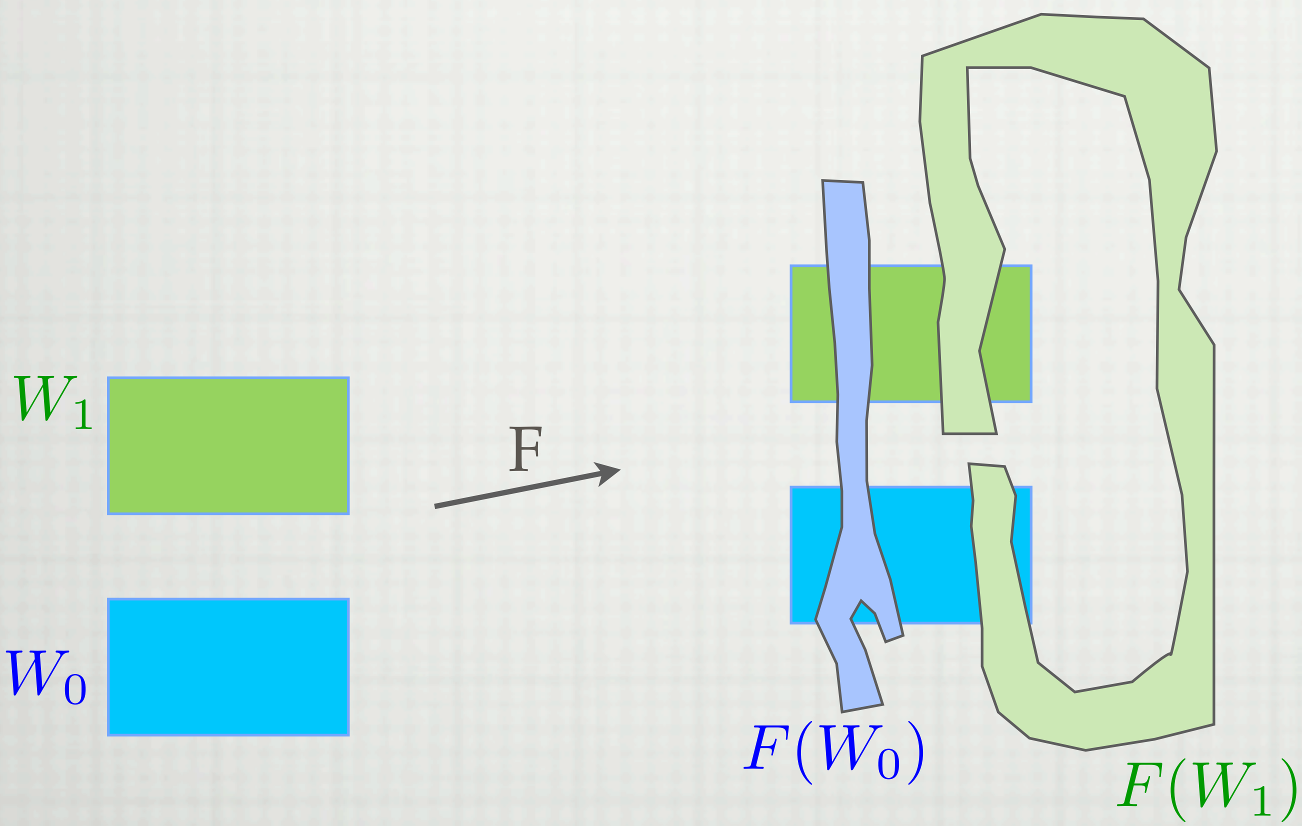
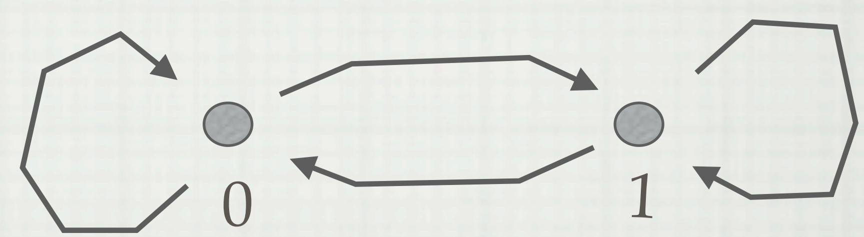
Windows and Connection Graphs

We can summarize the implications of homoclinic chaos in the horseshoe map by saying that there are two boxes or "windows" which are stretched across one another by the map. For other problems, the stretching will not be as nice as in the horseshoe map. For example, we can set up windows when stable and unstable curves intersect non-transversely as long as they still "cross".

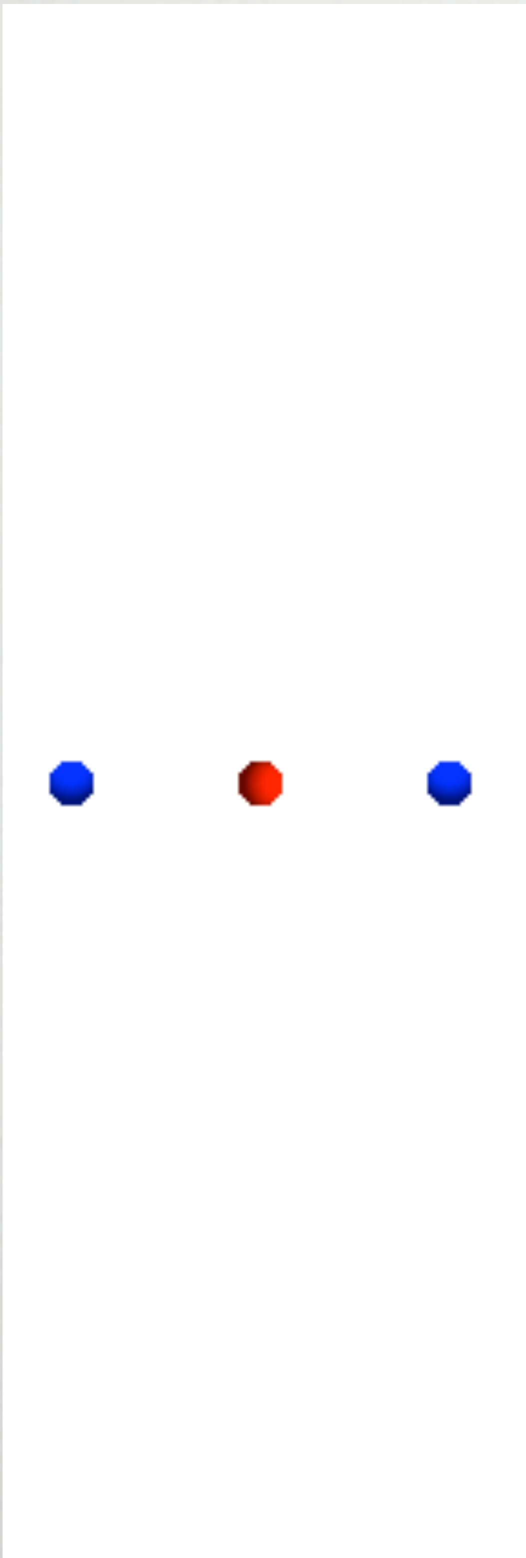


The situation is represented by a directed graph which describes which itineraries are realized by orbits:



Later will describe chaotic behavior in other situations using windows and connection graphs.

Chaos near Infinity in Sitnikov's Problem

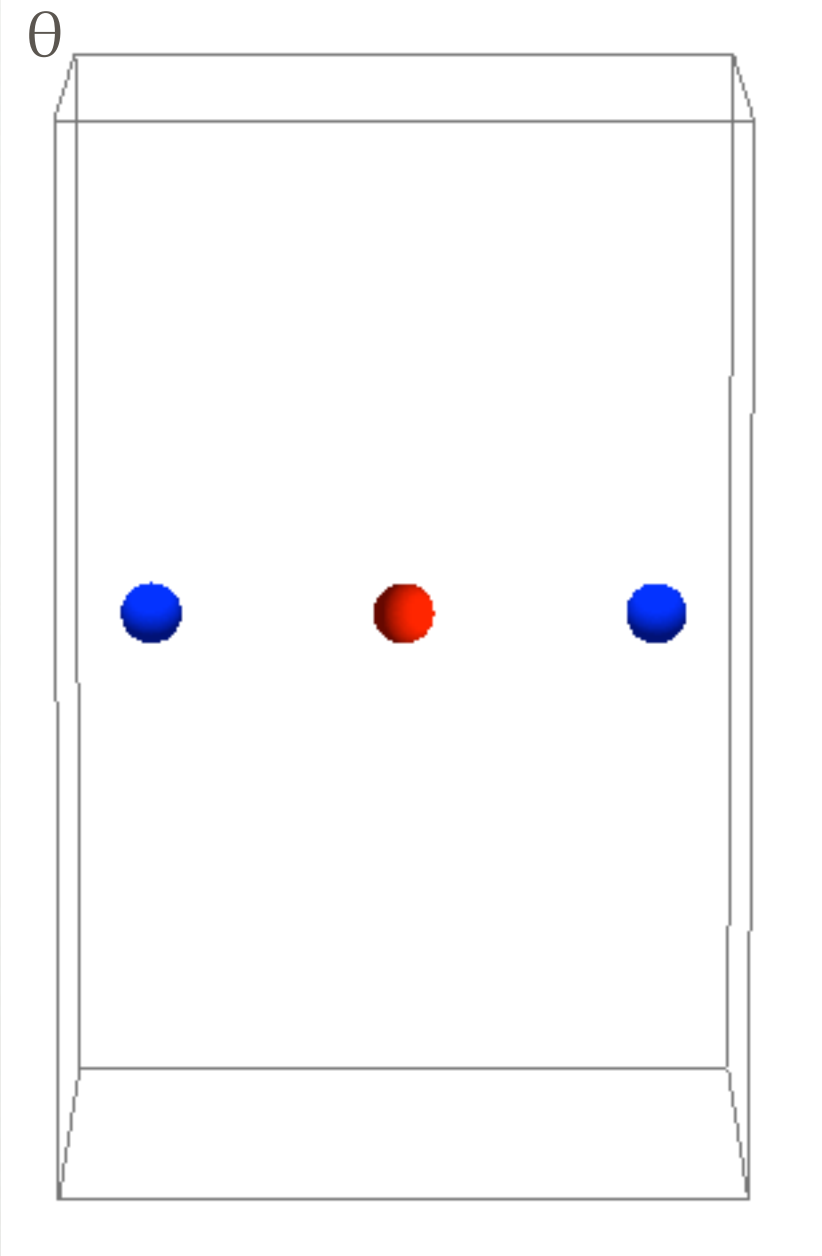


We will now focus on an even simpler three-body system where the homoclinic tangles are easier to see. In the 3D isosceles three-body problem we have two equal masses, say $m_1 = m_2 = 1$, moving symmetrically around the z-axis which a third body of mass m_3 moves up and down on the axis. The shape of is always an isosceles triangle.

The special case $m_3 = 0$ is the *Sitnikov* problem. Then m_1, m_2 move on symmetrical elliptical orbits in the (x, y)-plane. It is a dynamical system with $1\frac{1}{2}$ degrees of freedom. The state of the third body is determined by one position z and velocity $\dot{z}$ but there is a time-periodic forcing. We have a three-dimensional flow on $\mathbf{R}^2 \times \mathbf{S}^1 = \{(z, \dot{z}, t \bmod 2\pi)\}$.

Here is a typical orbit. We will be especially interested in orbits where z tends to infinity and then the third mass flies off the screen. So it is convenient of replace z by a bounded variable.

Sitnikov problem in θ coordinates



Setting $z = \frac{1}{2}\tan\theta$, Newton's laws give the following system of three ODE's:

$$\dot{\theta} = 2\cos^2\theta\, v$$

$$\dot{v} = -\frac{4\sin\theta\cos^2\theta}{(\cos^2\theta(1+\epsilon\cos u(t))^2 + \sin^2\theta)^{\frac{3}{2}}}$$

$$\dot{t} = 1$$

where ϵ is the eccentricity and $u(t)$ is the eccentric anomaly of the two-body motion of the primaries. $u(t)$ satisfies Kepler's equation

$$t = u(t) + \epsilon\sin u(t).$$

θ remains in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Orbits with $z \to \pm\infty$ now converge to the top or bottom of the box with $\theta \to \pm\frac{\pi}{2}$.

Integrable Limit

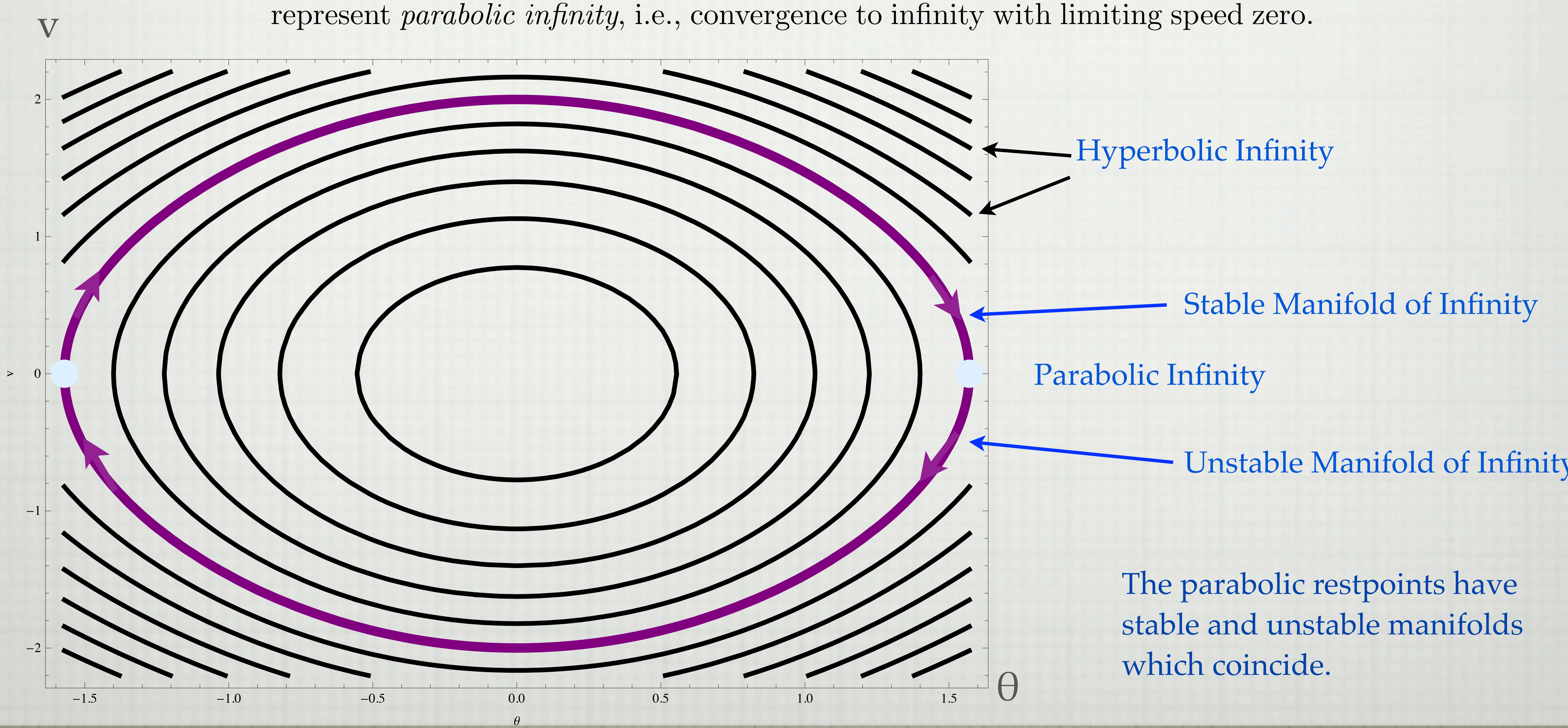
When the eccentricity $\epsilon = 0$ the problem becomes a time-independent system of one degree of freedom. The orbits move on curves of constant energy

$$H = \frac{1}{2}v^2 - 2\cos\theta = h.$$

Orbits with $z \to \pm\infty$ converge to restpoints at $\theta = \pm\frac{\pi}{2}$. In particular

$$\infty_{\pm} : \qquad (\theta, v) = (\frac{\pi}{2}, 0)$$

represent *parabolic infinity*, i.e., convergence to infinity with limiting speed zero.

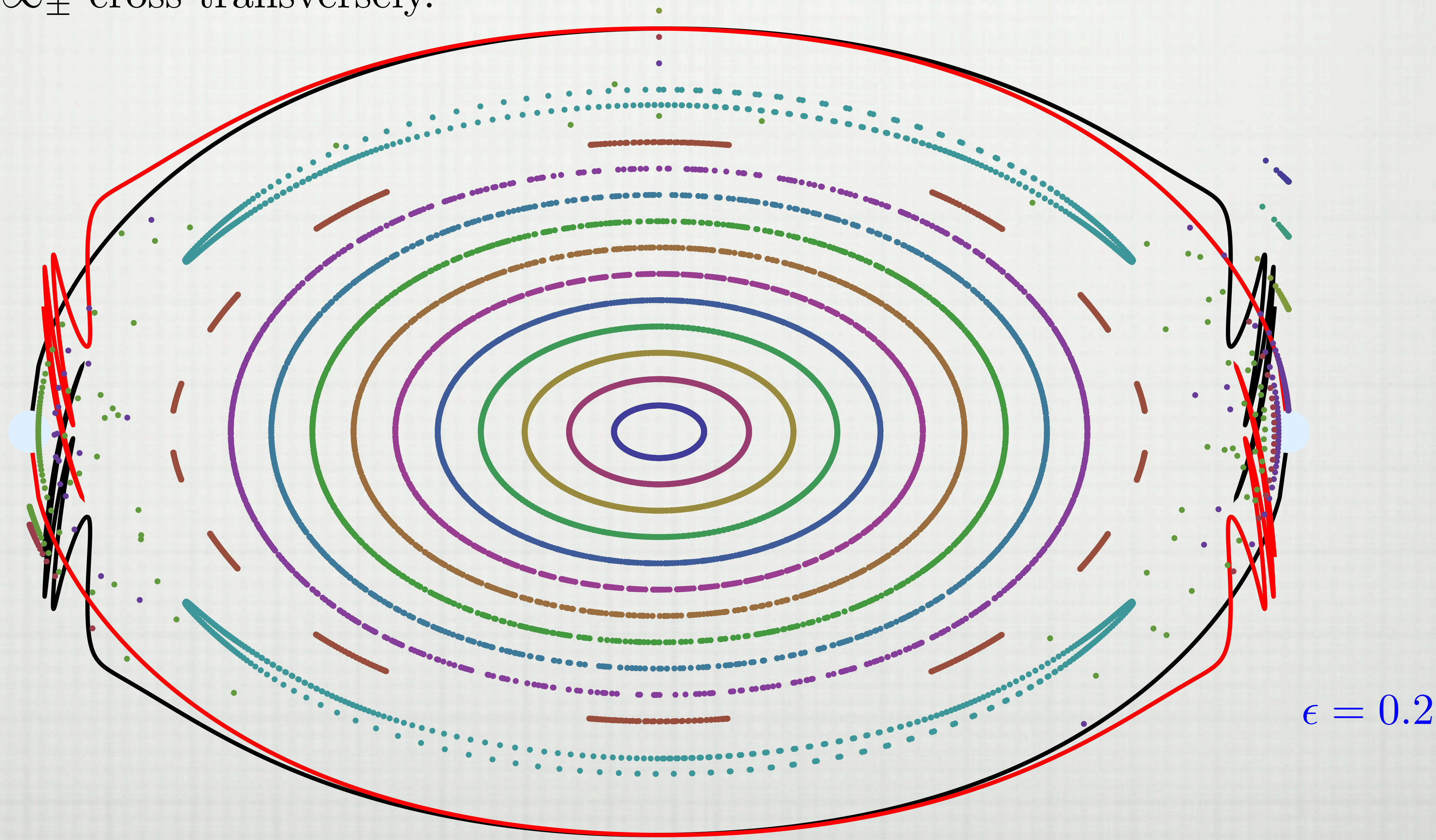


The parabolic restpoints have stable and unstable manifolds which coincide.

Poincaré Map for $\epsilon > 0$

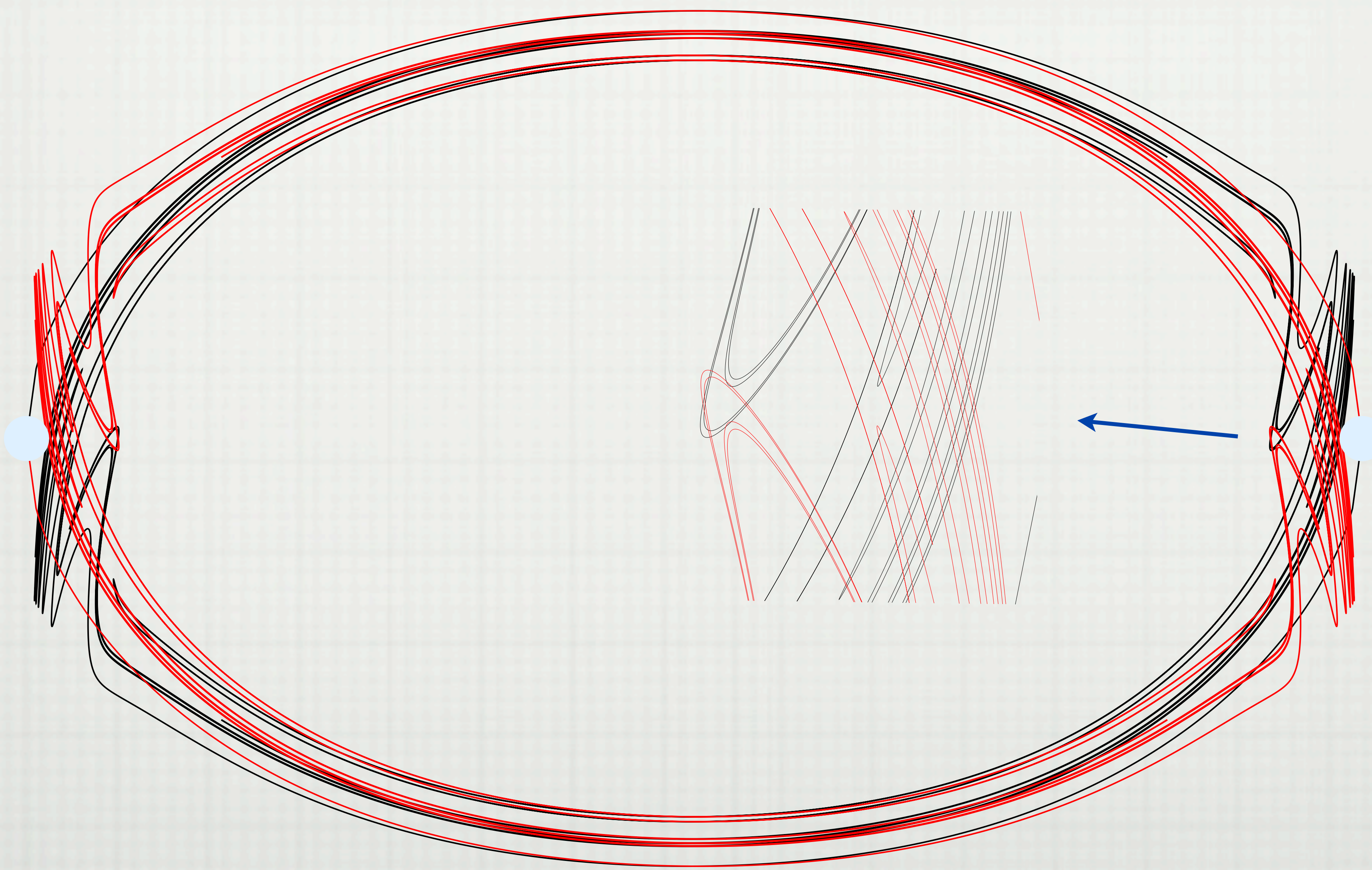
When $\epsilon > 0$ we have a Poincaré map of the (θ, v)-plane representing the state of the third mass at times $t = 0 \bmod 2\pi$ (when the elliptical bodies are at maximal distance).

There are fixed points at infinity and now the stable and unstable manfolds of $\infty_\pm$ cross transversely.



Homoclinic Tangle for $\epsilon = 0.2$

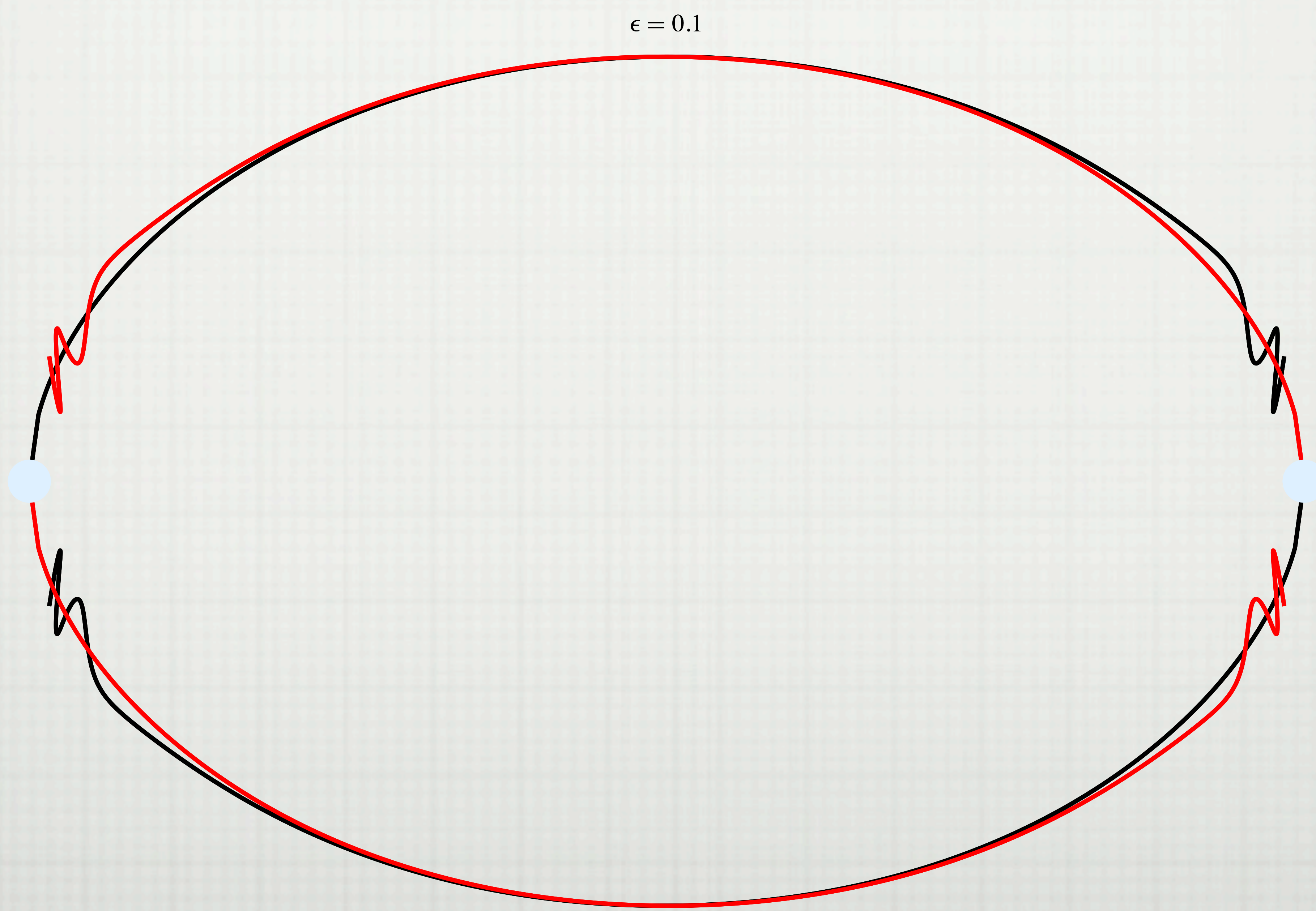
Extending by iteration of the Poincaré map, the two (red) stable manifolds fold without intersecting one another but they cross the two (black) unstable manifolds to produce homoclinic and heteroclinic orbits as in the horseshoe.



Even more complicated than the horseshoe -- homoclinic tangencies

Increasing eccentricity

As the eccentricity of the primaries increases, the stable and unstable manifolds of infinity invade more and more of the phase space. We will say more about the large eccentricity case later.



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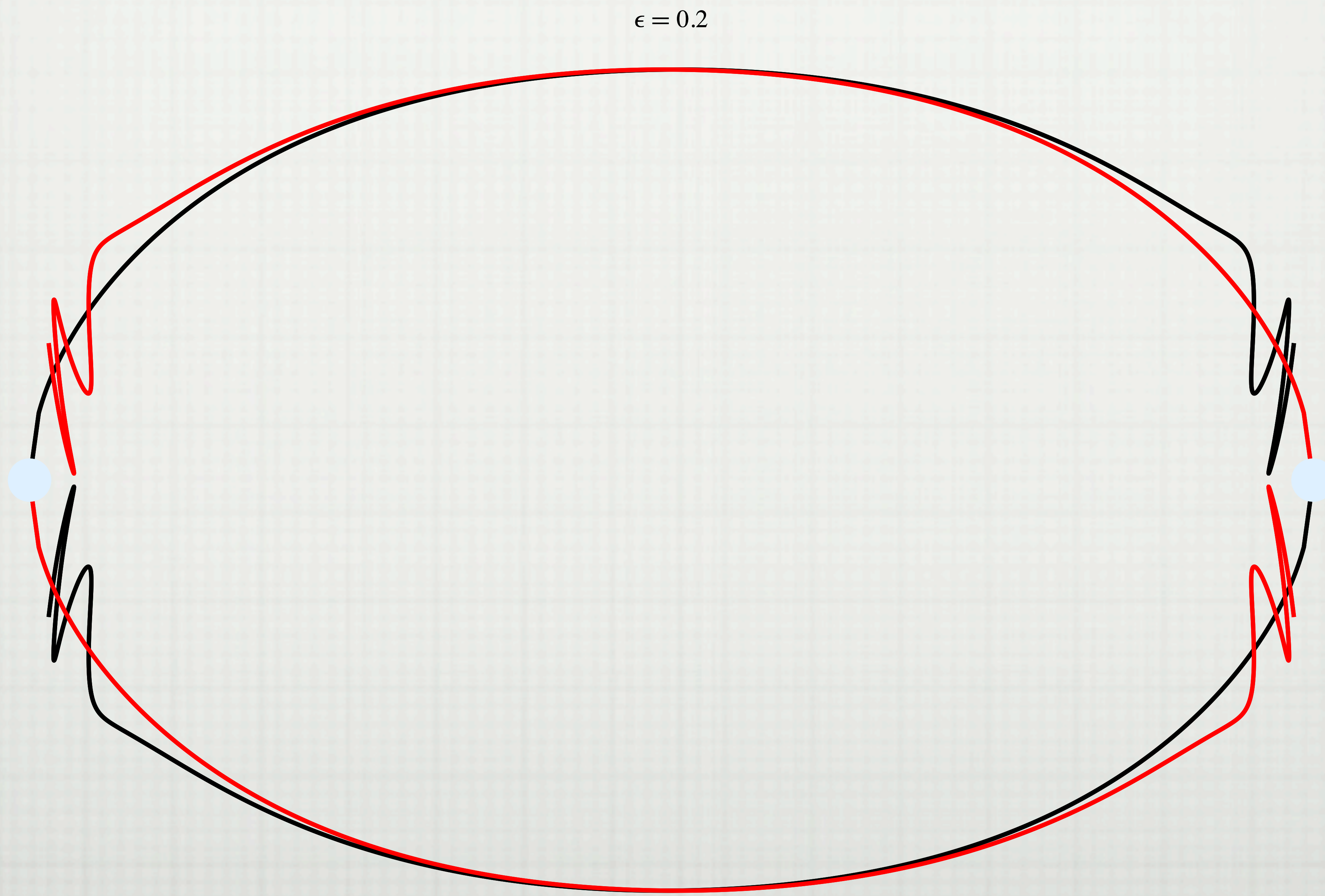

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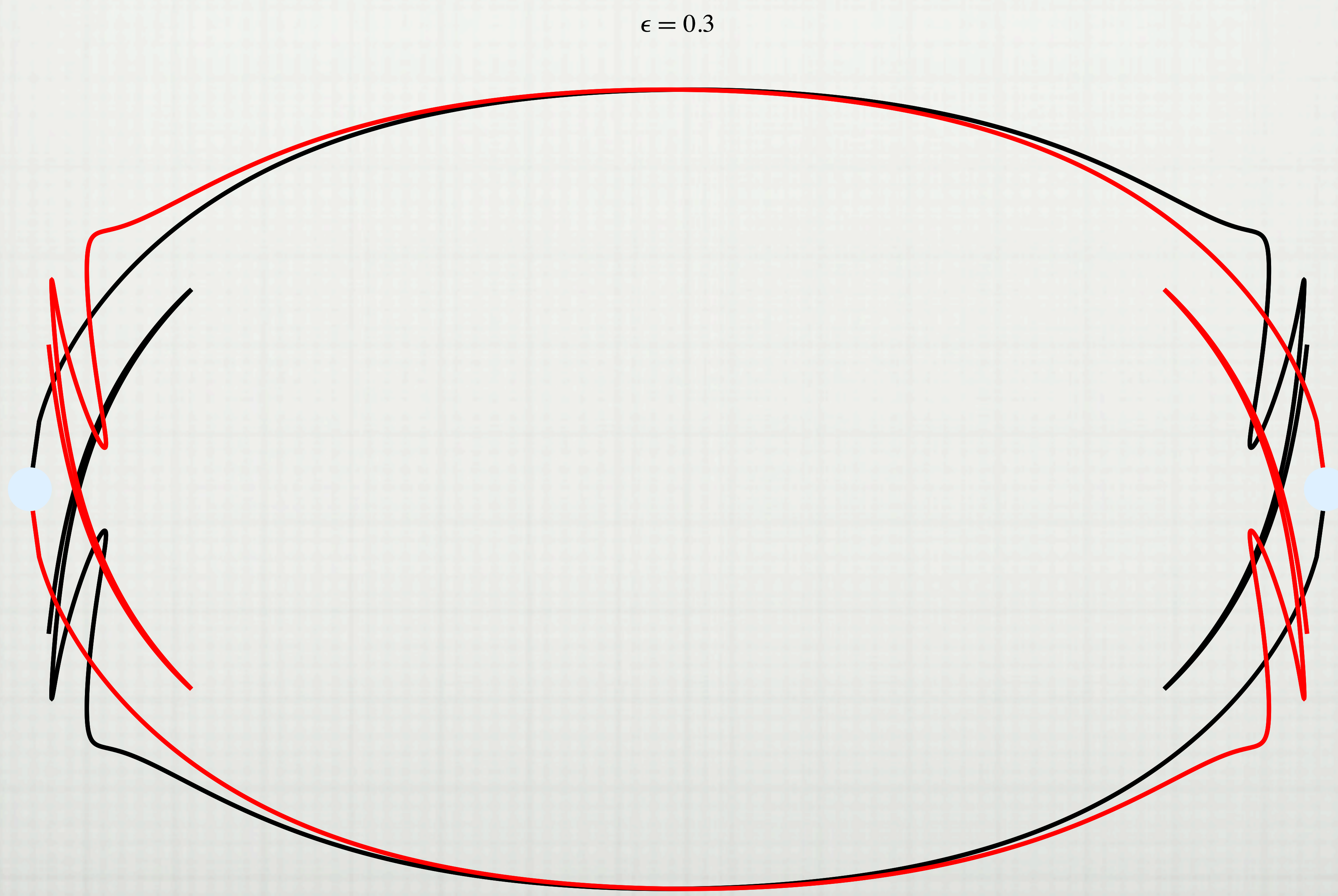

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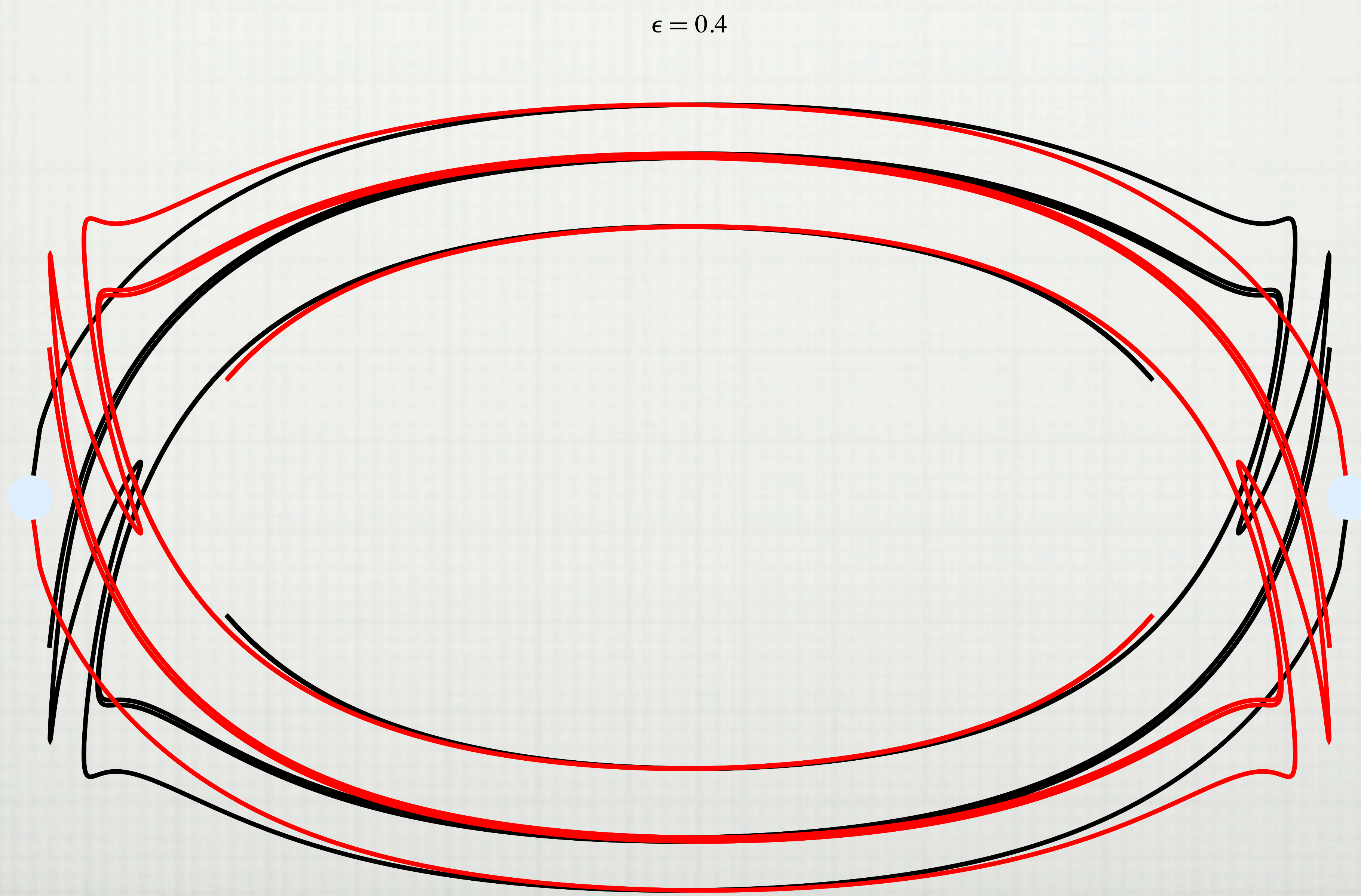

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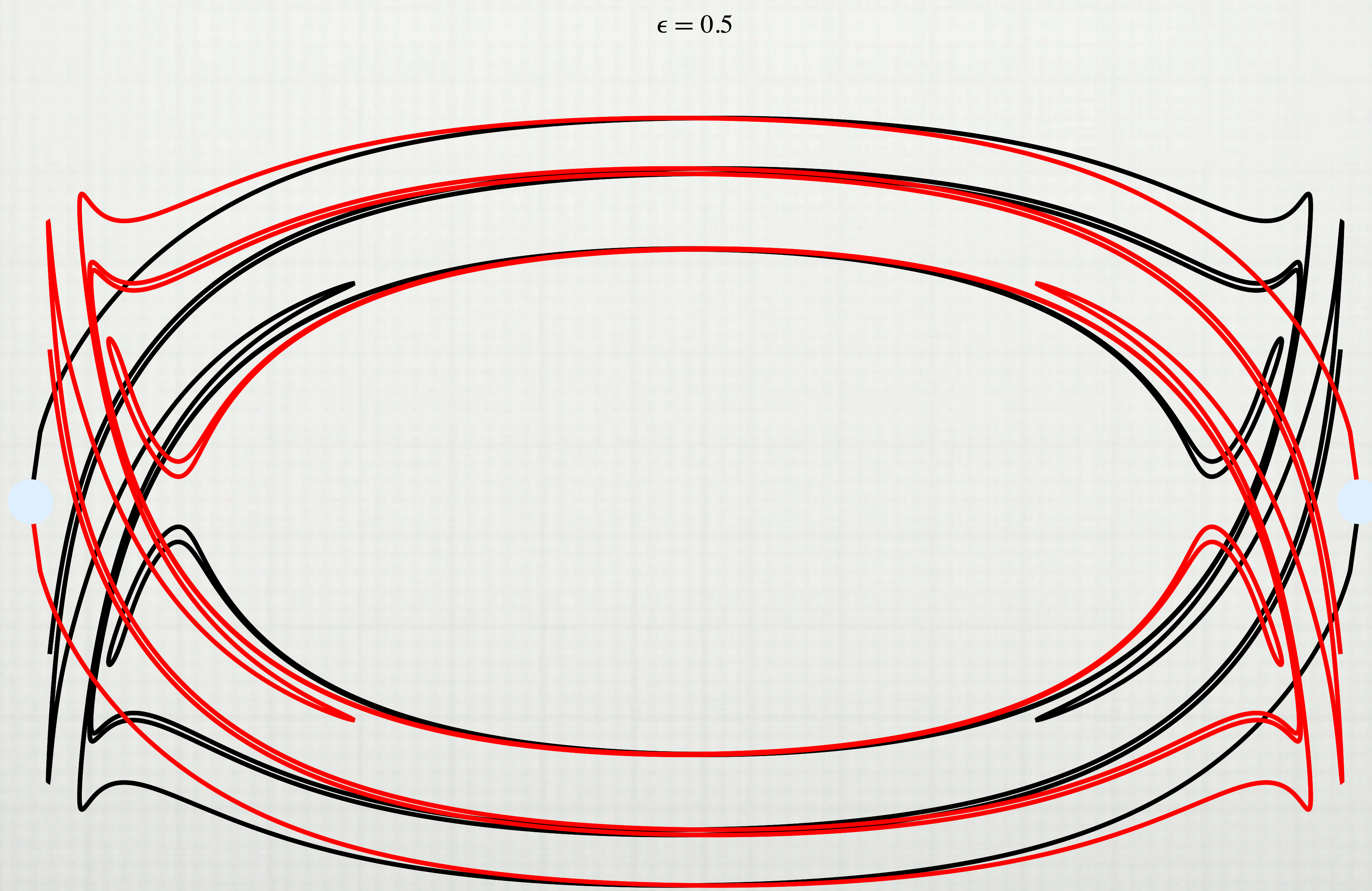

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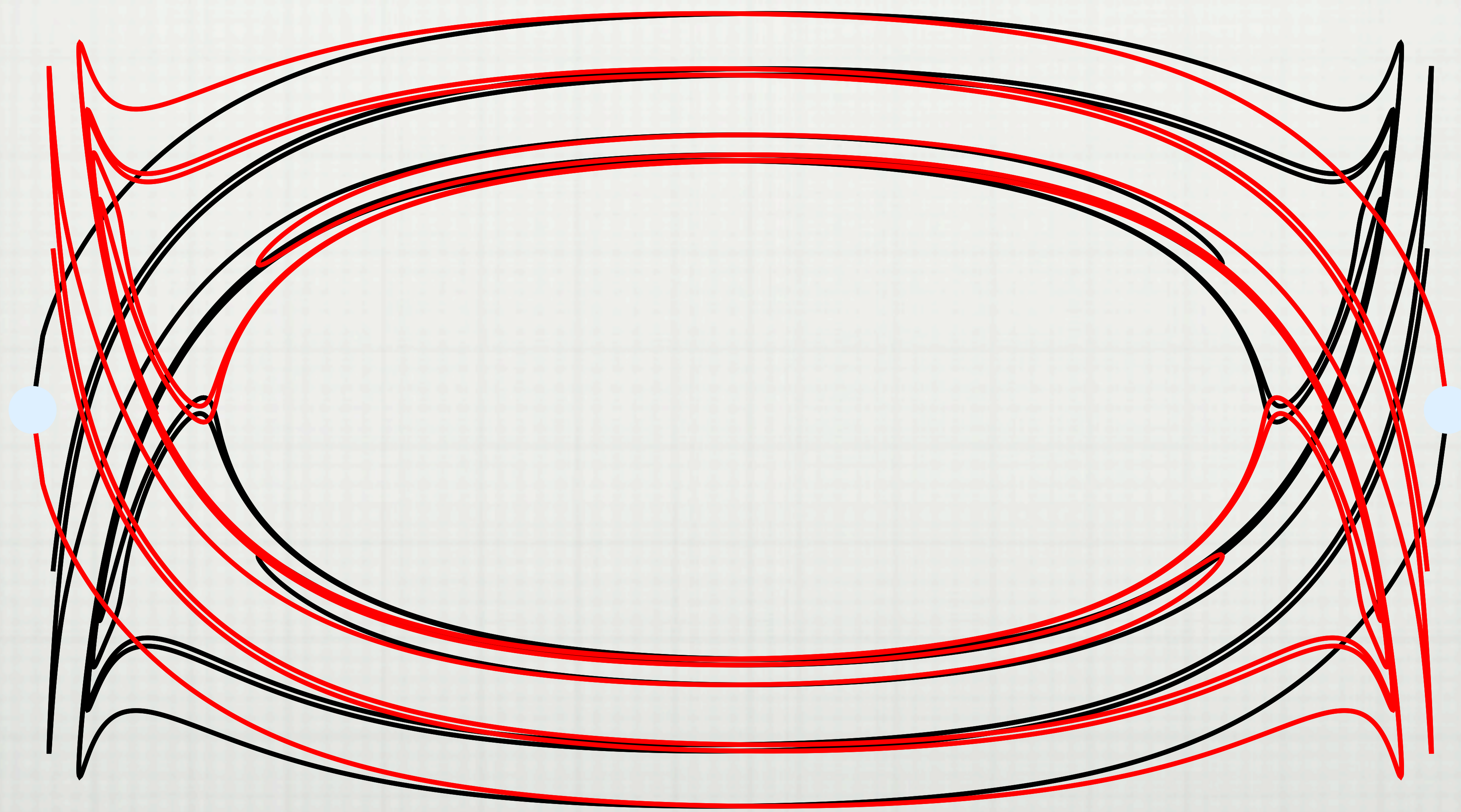
$\epsilon = 0.6$



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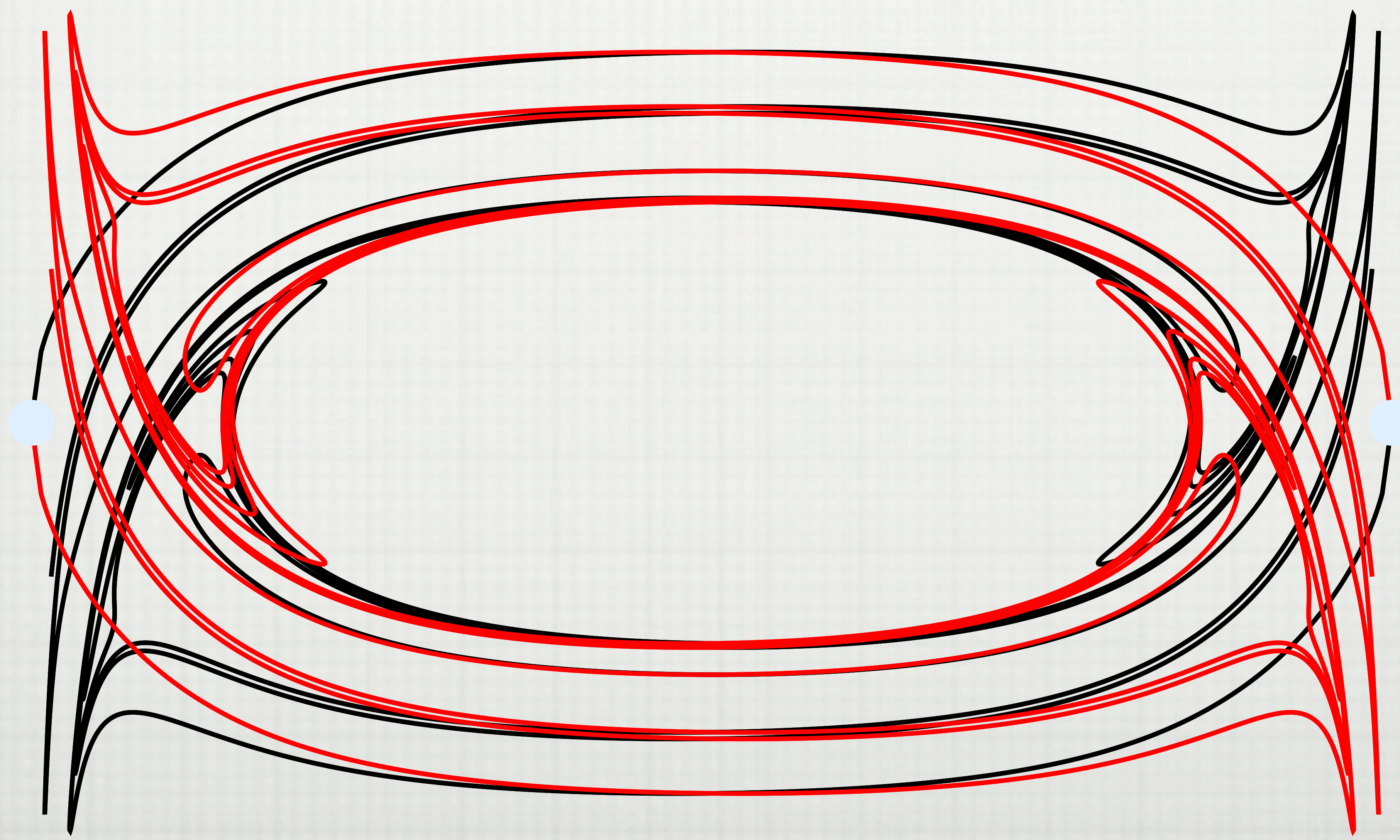
$\epsilon = 0.7$



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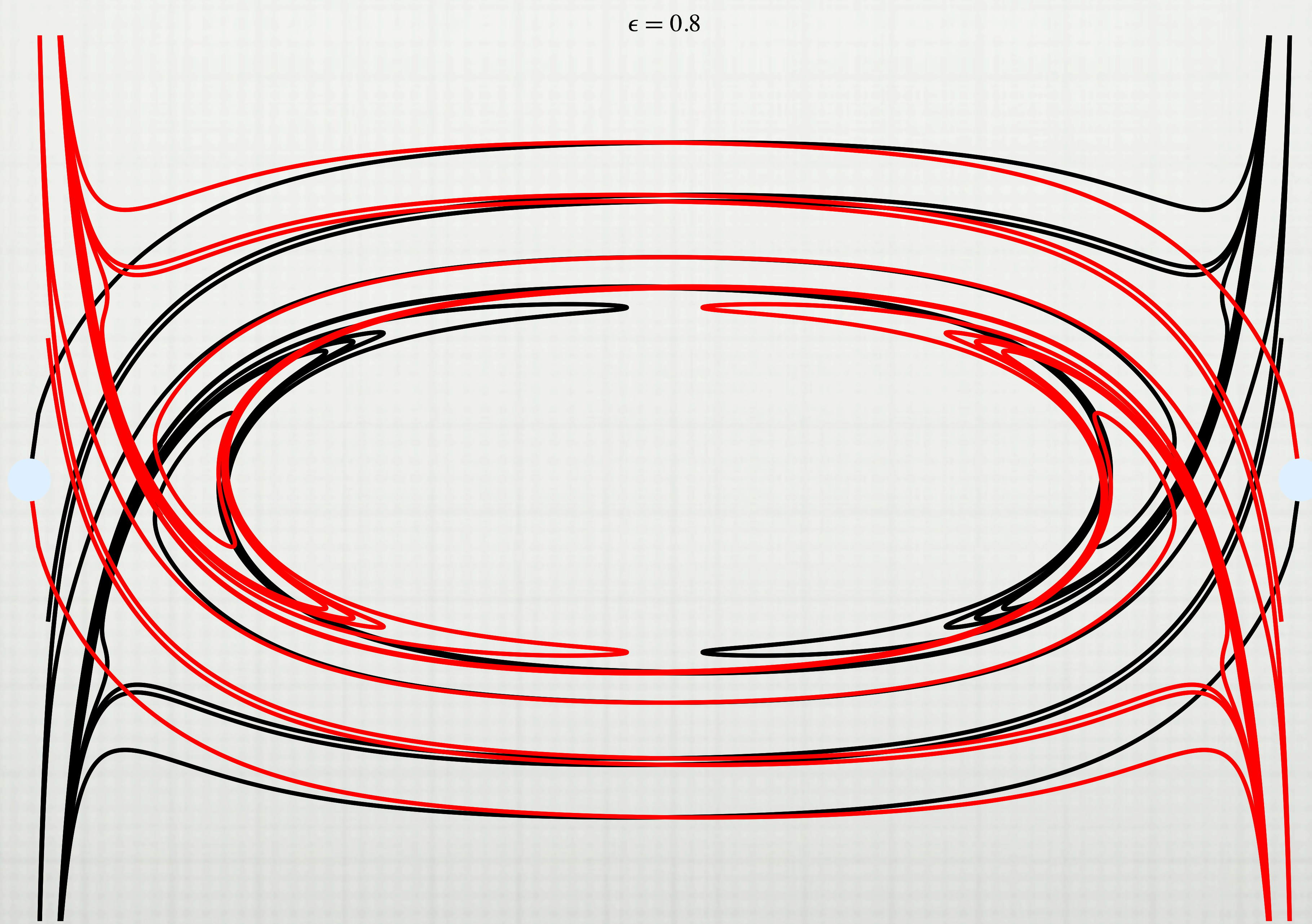

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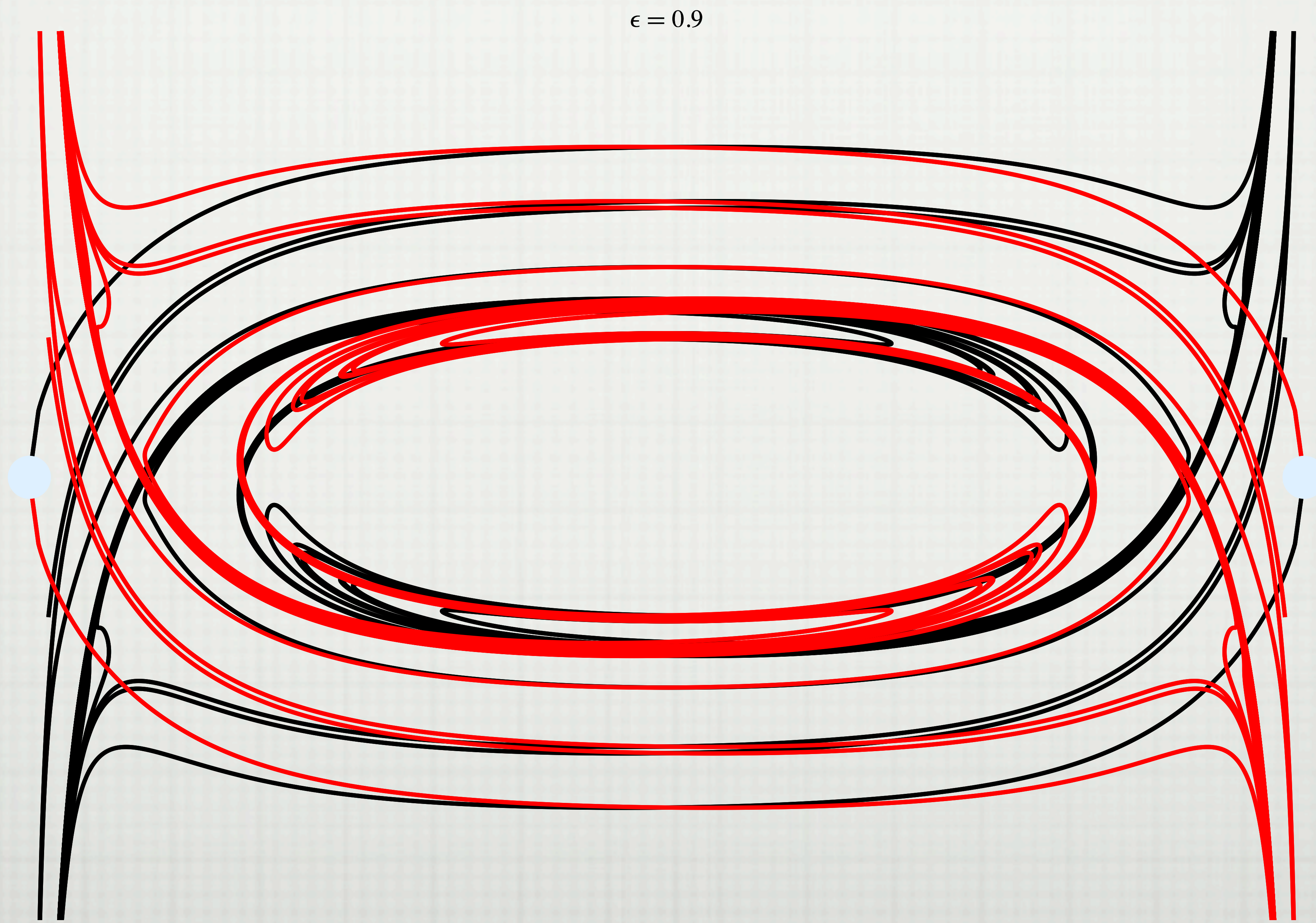
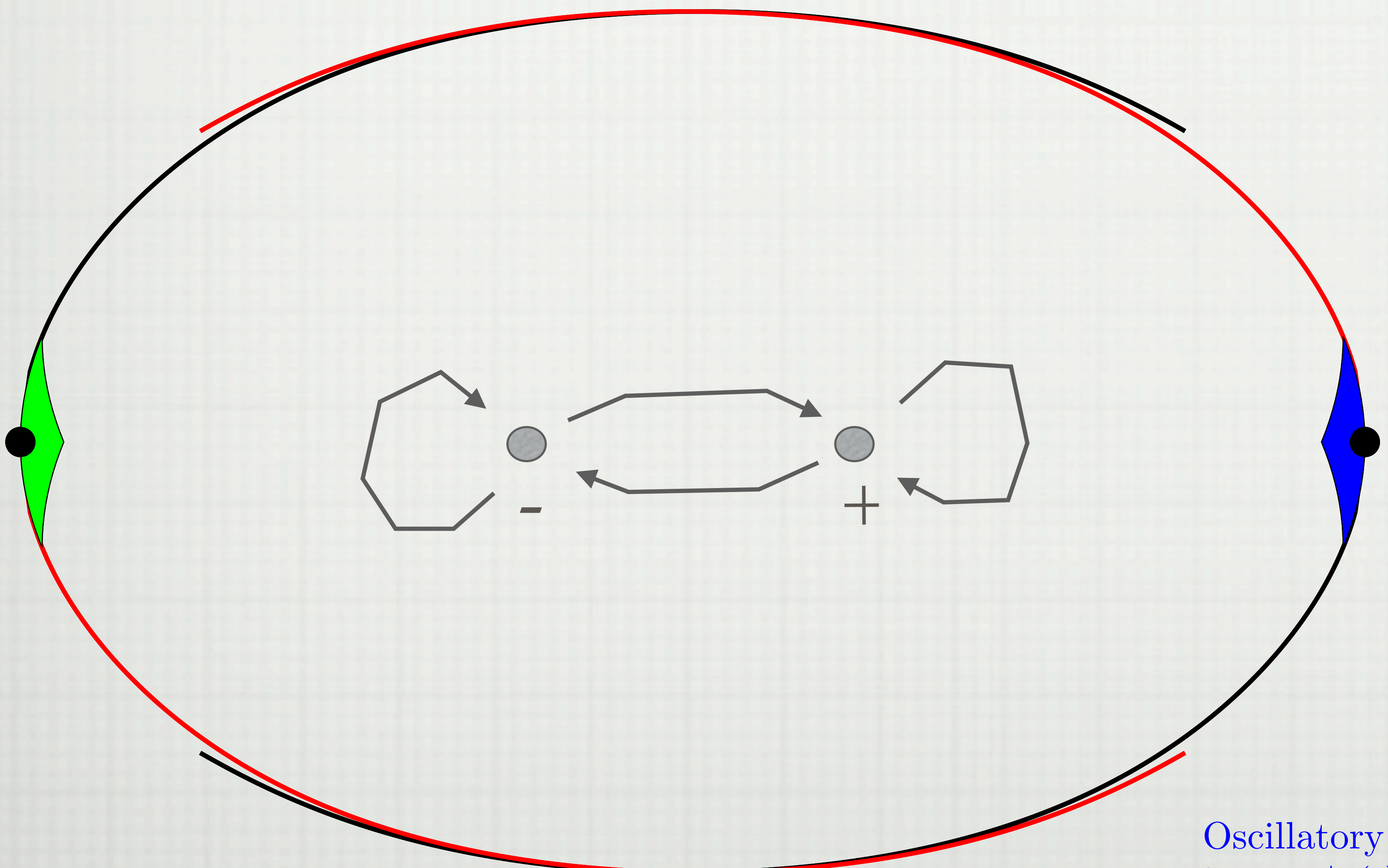


Symbolic Dynamics

We can set up a window near each of $\pm\infty$. These are stretched across one another by a suitable iterate of the Poincaré map. We can describe orbits with sequences of +'s and -'s. At each step one can choose to stay near the same parabolic fixed point or else transfer to the other. Long sequences of the same symbol represent orbits which spend a long time near infinity.



Oscillatory Orbits:

$\limsup |z(t)| = \infty \qquad \liminf |z(t)| = 0$

$\ldots + - + + - + + + - + + + + + + + + - \ldots$

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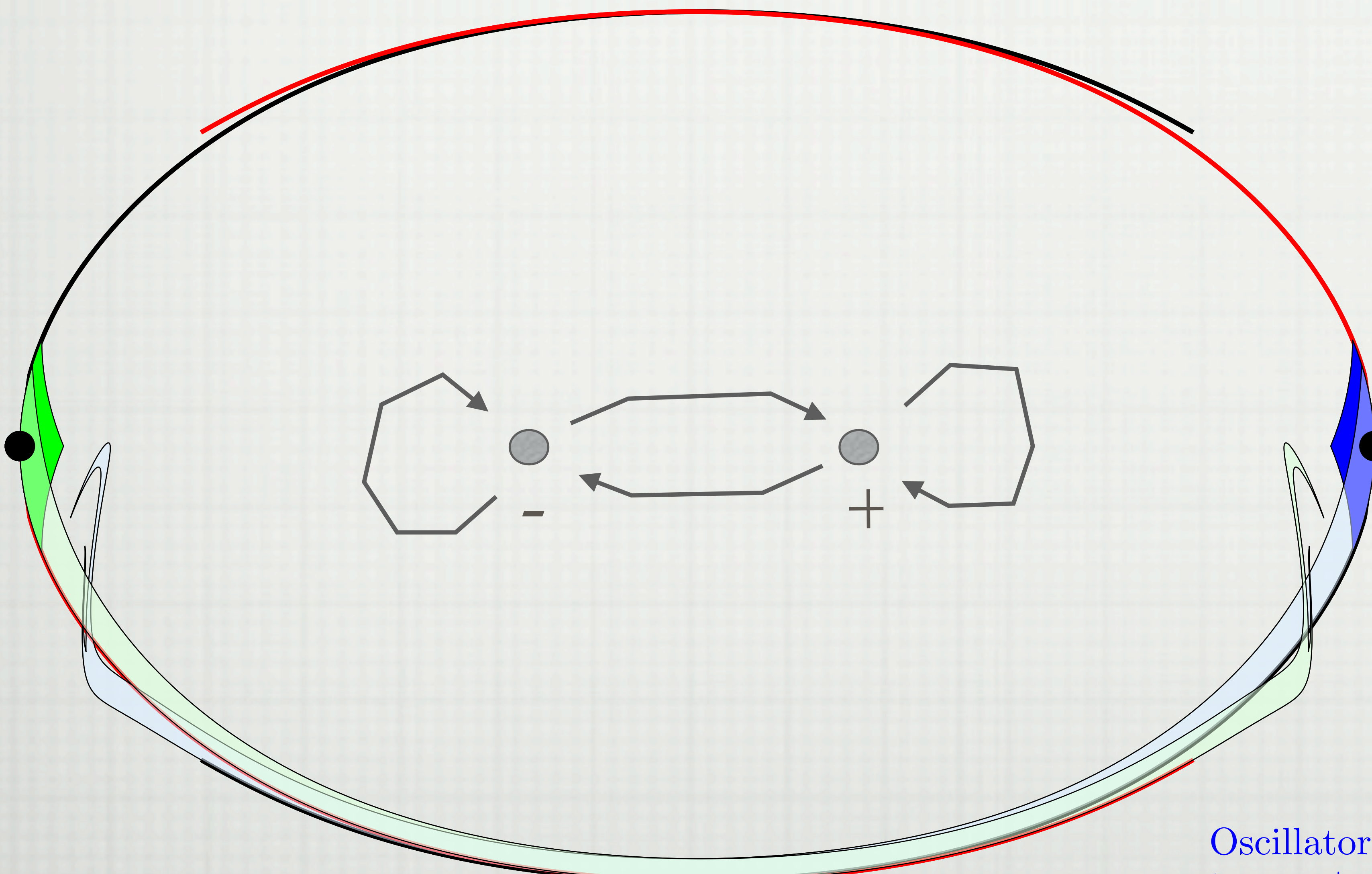

Oscillatory Orbits:

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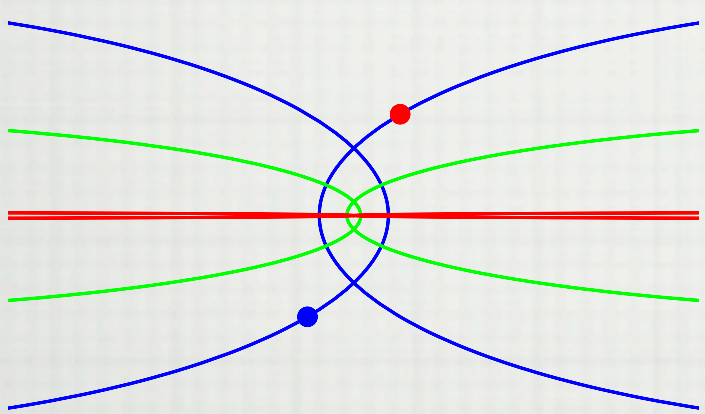
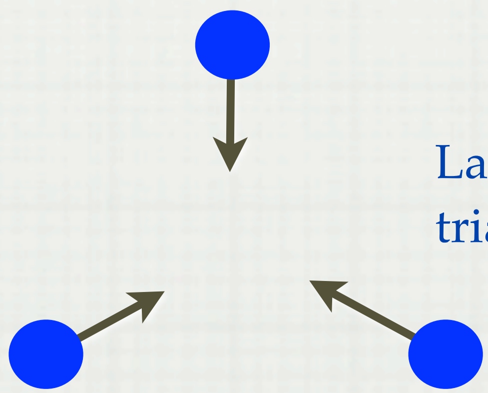
Chaos near Triple Collisions

We can get chaotic behavior by finding “windows” which are stretched across one another by the Poincaré map. In previous slides, the stretching has been associated with stable and unstable curves of fixed or periodic points. Another source of stretching in the three-body problem is close approaches to collision.

There are two kinds of collisions and near-collisions:



Binary collisions



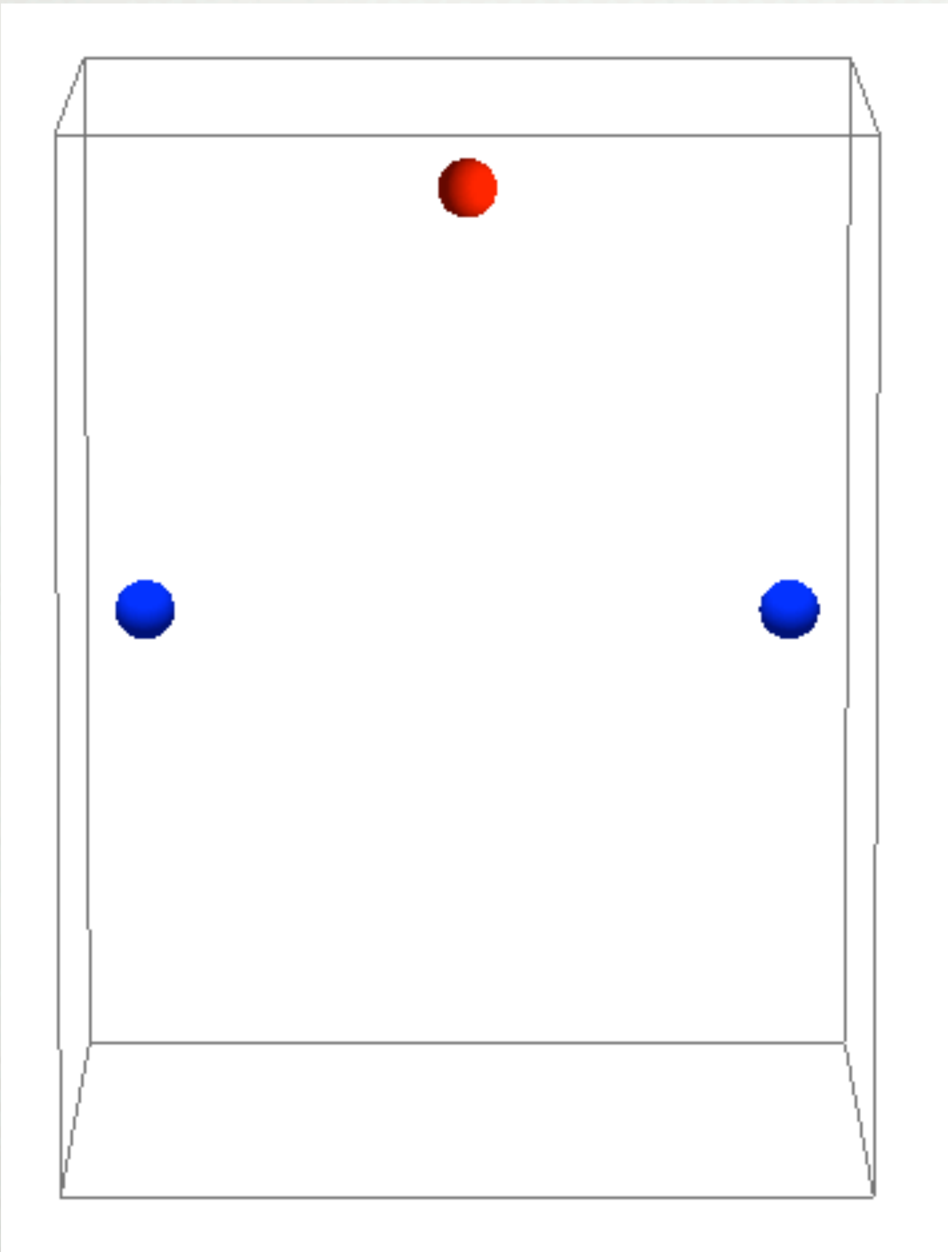
Lagrange equilateral triangle solution

Triple collisions

We will look at some orbits near triple collision in the Sitnikov problem

Near Triple Collision $\varepsilon = 0.96$

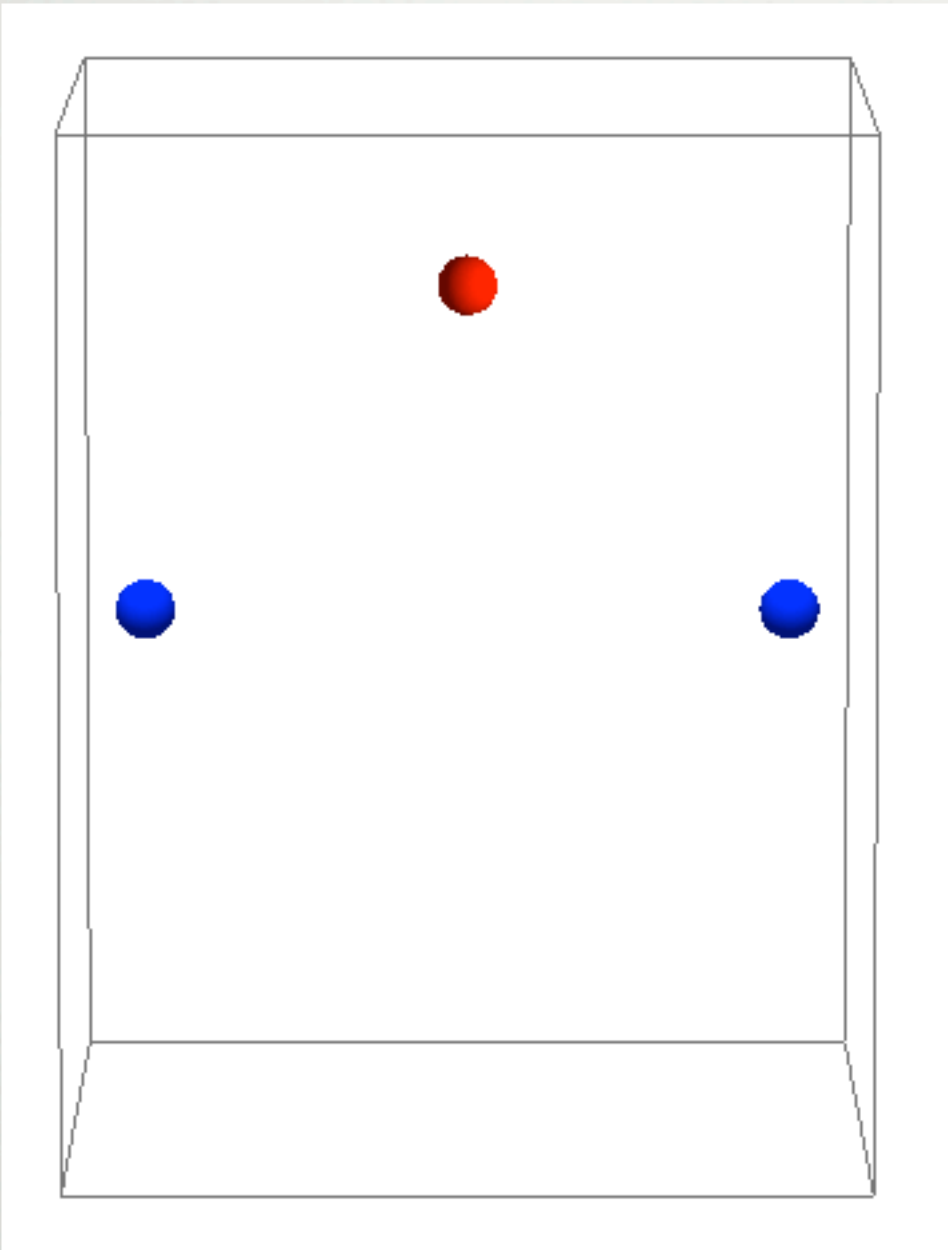
We can get near-triple collision orbits in Sitnikov's problem by choosing a high eccentricity for the orbit of the primary masses and then timing the third body to pass through the origin when the elliptical bodies are close.



This orbit is a highly unstable, hyperbolic periodic orbit with a close approach to triple collision (there is a corresponding hyperbolic fixed point of the Poincaré map).

It is close to the Lagrange equilateral collision solution in the planar problem.

More near-collision orbits



One can show that there are many more such near-triple-collision orbits. They differ in their detailed behavior while approaching triple collision.

Here is another orbit for ε=0.96 which "wobbles" above and below the plane of the primaries near triple collision.

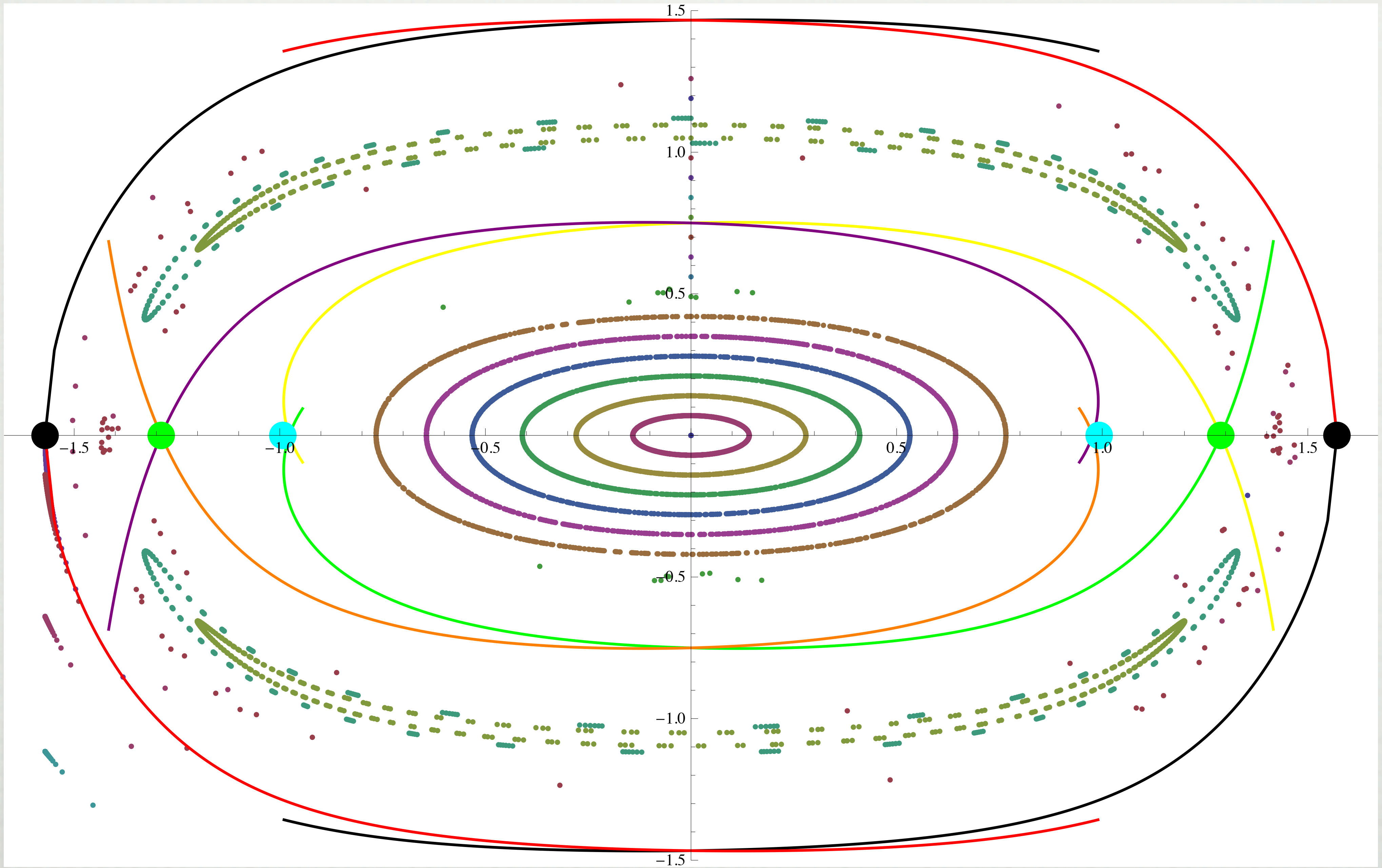
An infinite sequence of such hyperbolic fixed points of the Poincaré map is created as the eccentricity ε --> 1 (though for any fixed ε < 1, there will only be finitely many).

Also, the reflections of these orbit are distinct solutions.

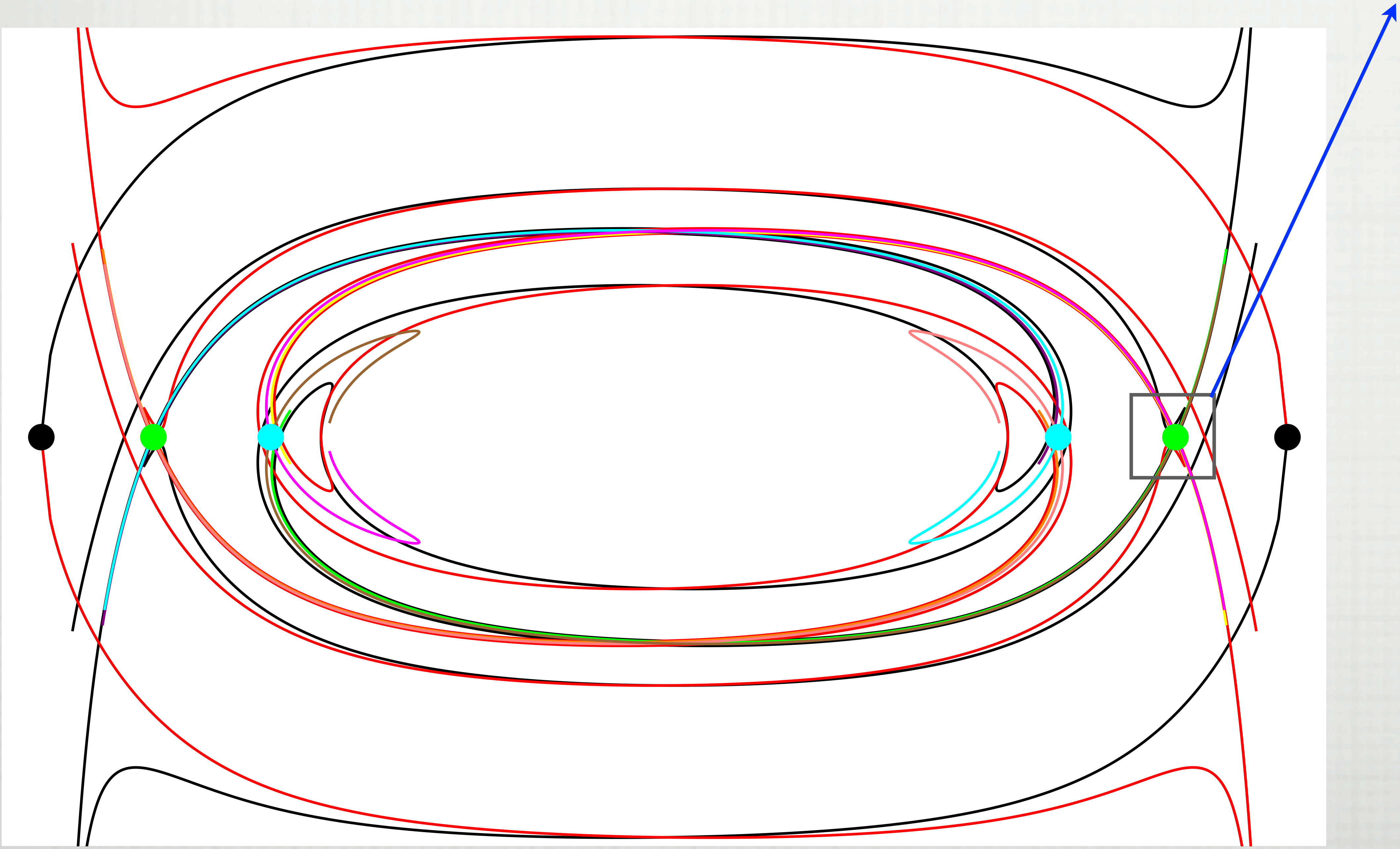
Poincaré map for $\varepsilon = 0.96$

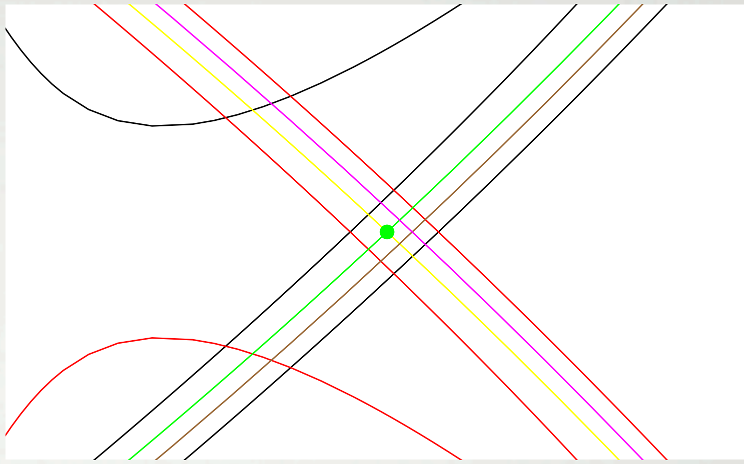
We have (at least) six hyperbolic fixed points all with stable and unstable curves.

● (black)	∞_+	∞_-	Parabolic Infinity
● (green)	P_+	P_-	Near triple collision
● (cyan)	Q_+	Q_-	Near triple collision with wobbles



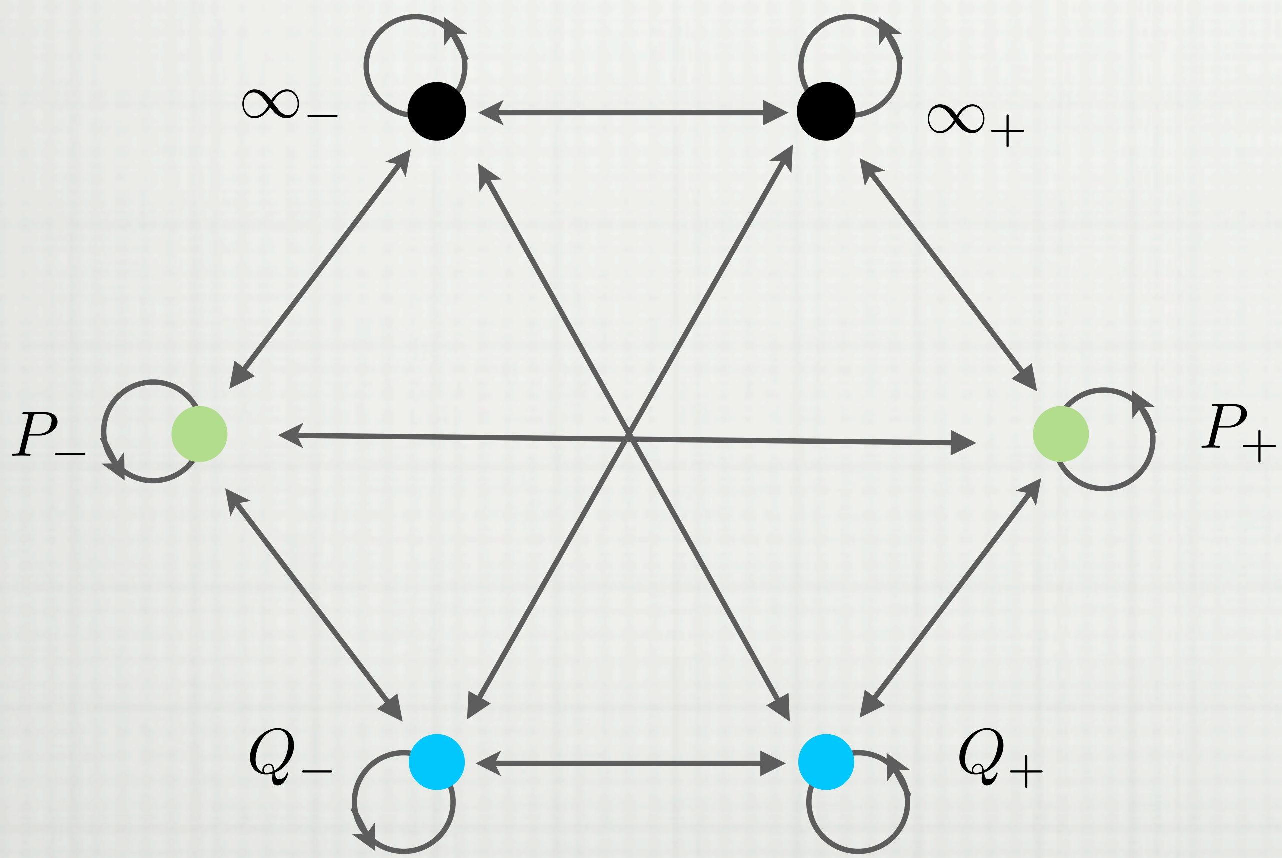
There is a rich network of transverse homoclinic connections between the six fixed points





Symbolic Dynamics

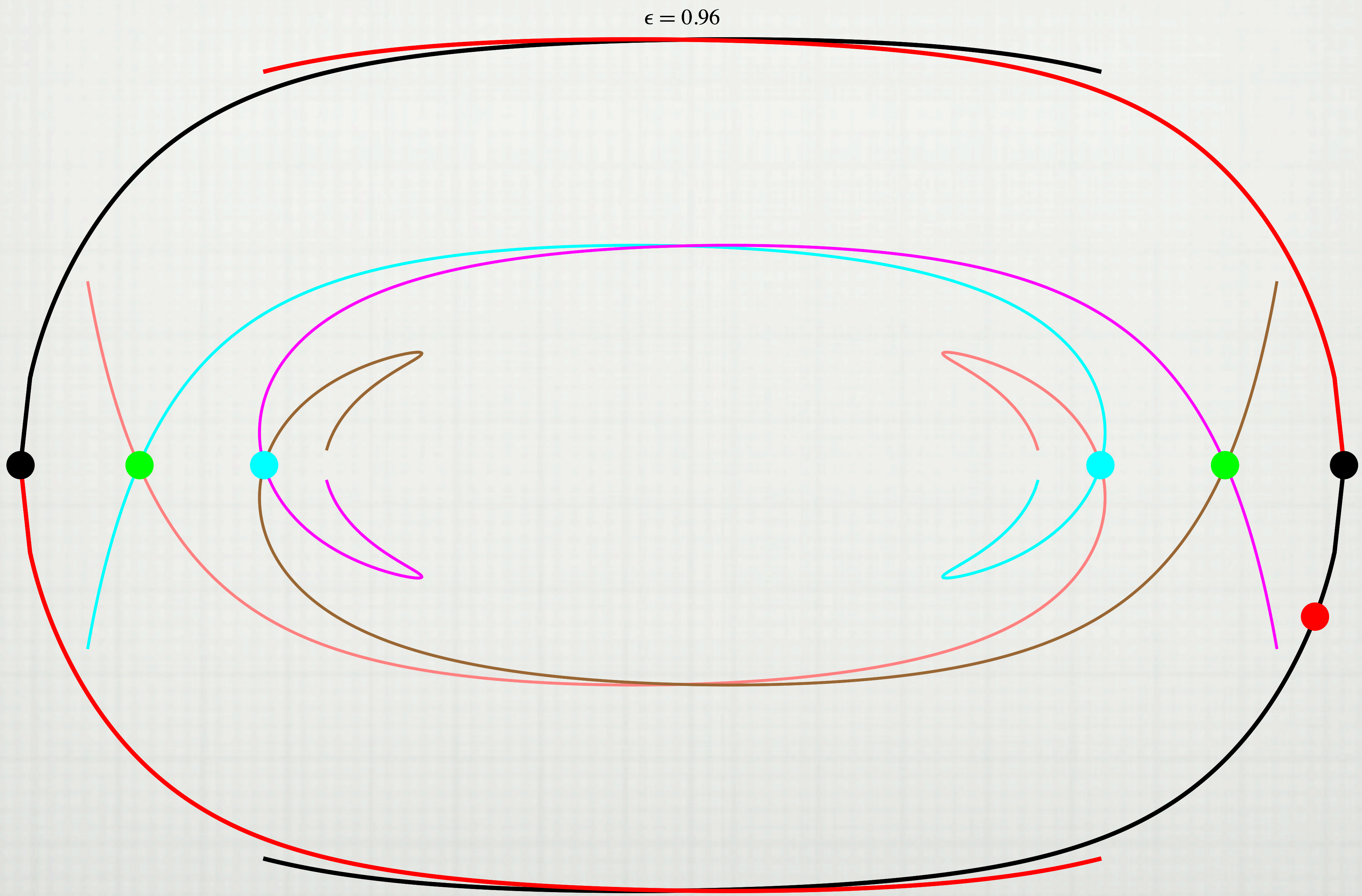
As a result, there will be orbits realizing every path in the graph below. In other words one can choose any sequence of the six symbols and find an orbit which exhibits that sequence of behaviors.



To realize a sequence means that the orbit passes through the corresponding sequence of windows for the Poincaré map.

This is rather delicate in practice since the behavior is sensitive changes in initial conditions.

Realizing the sequence $\ldots, \infty_+, Q+-, P_+, \infty_+, \ldots$



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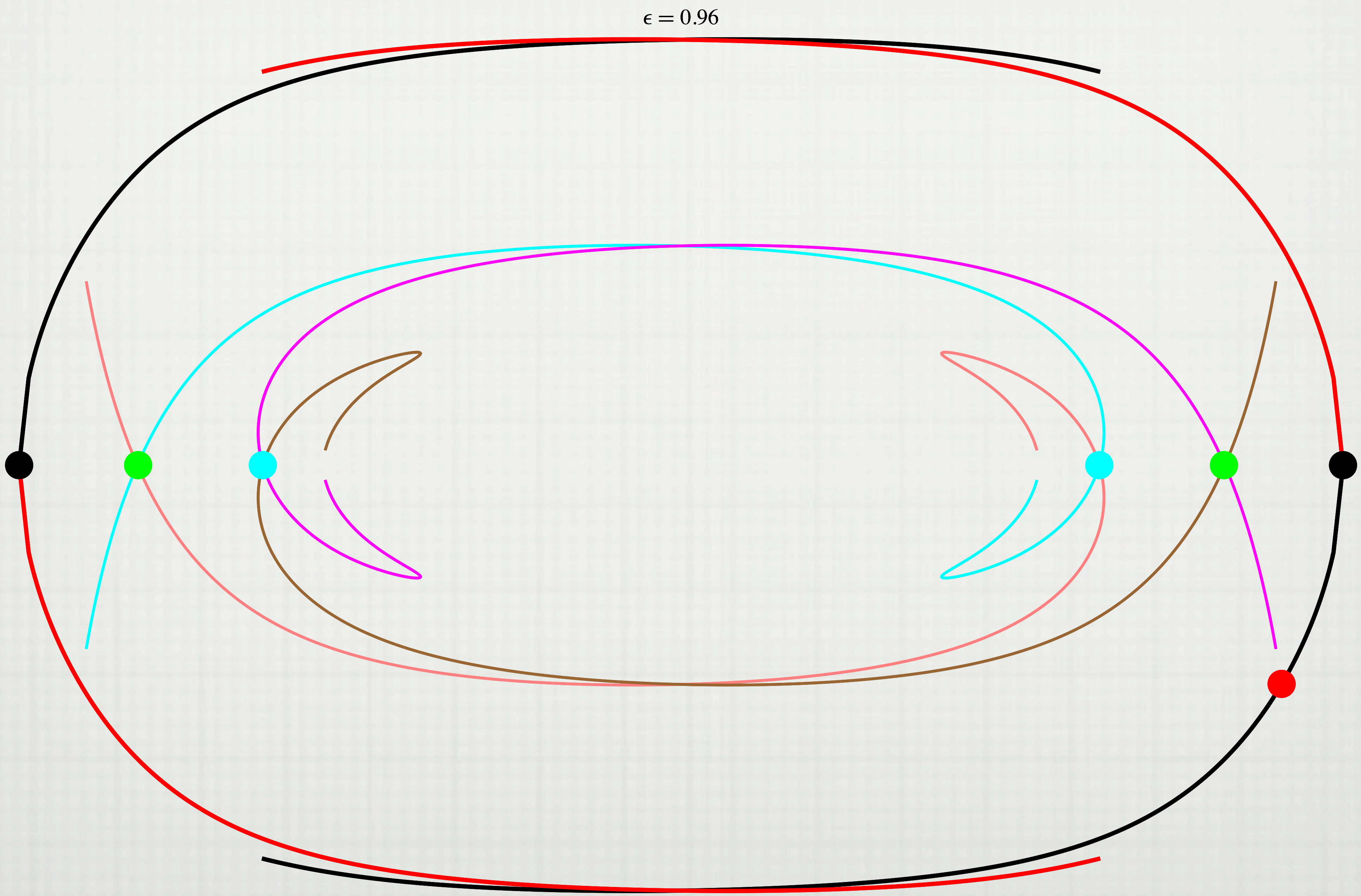

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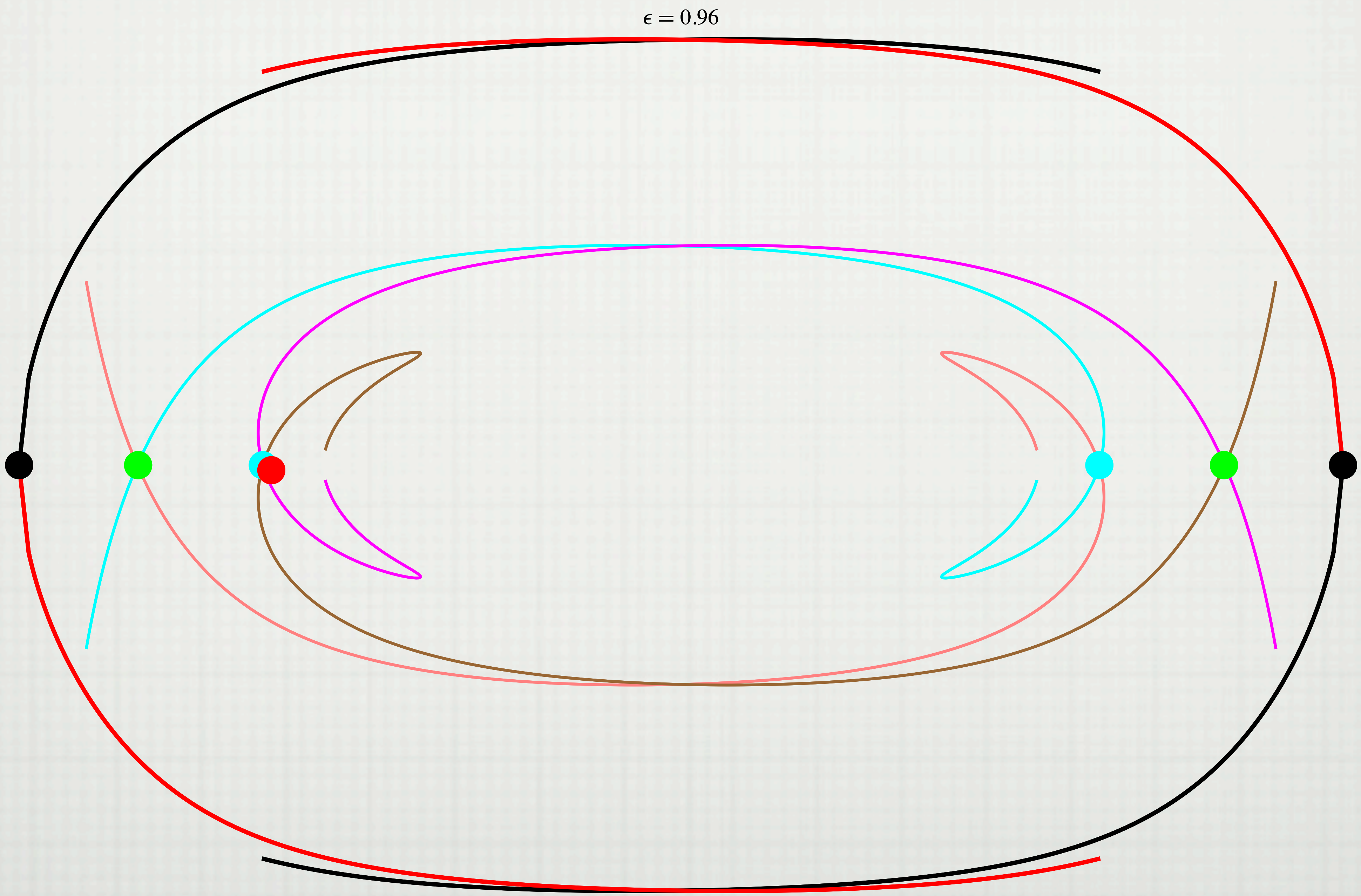

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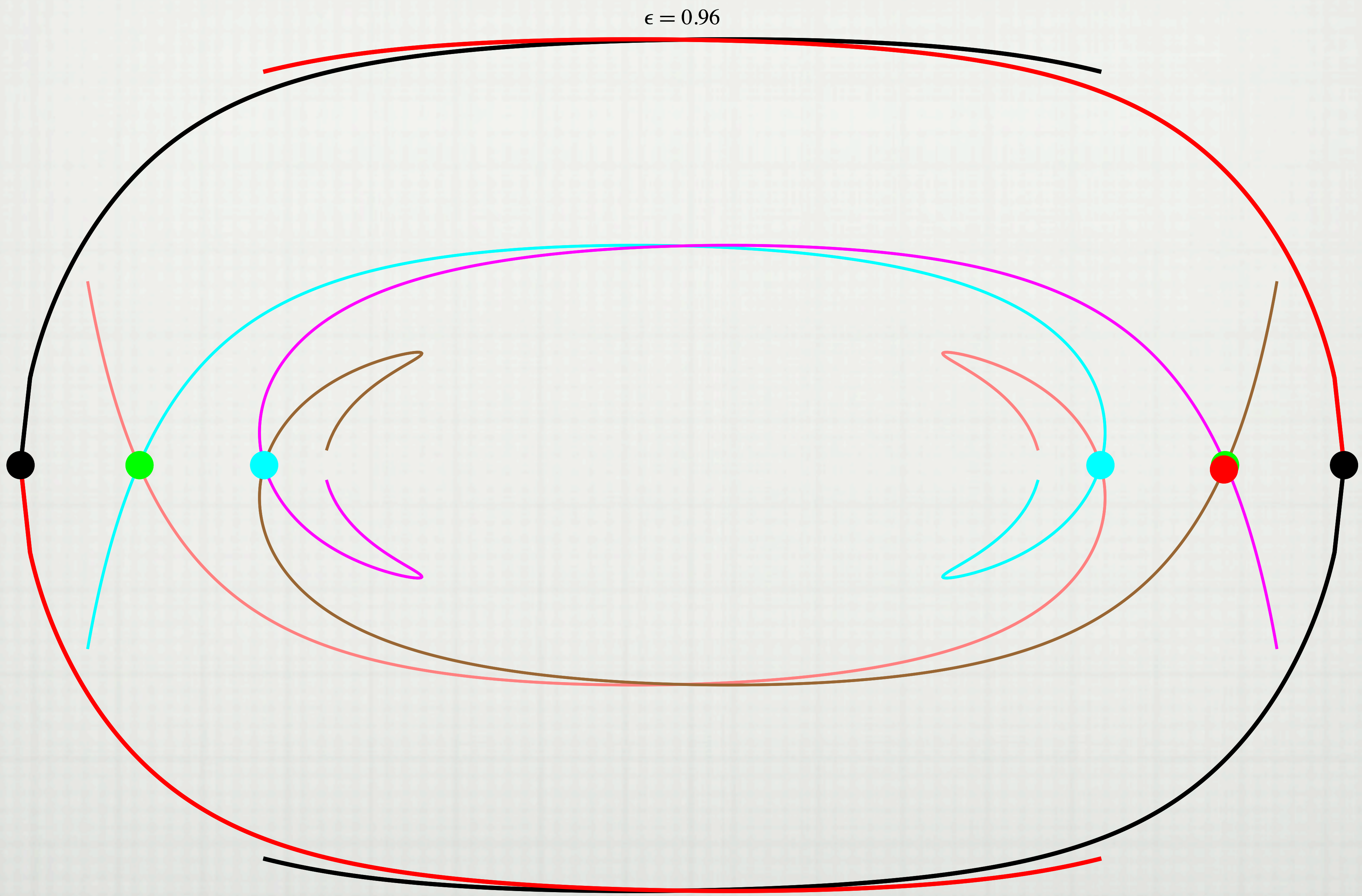

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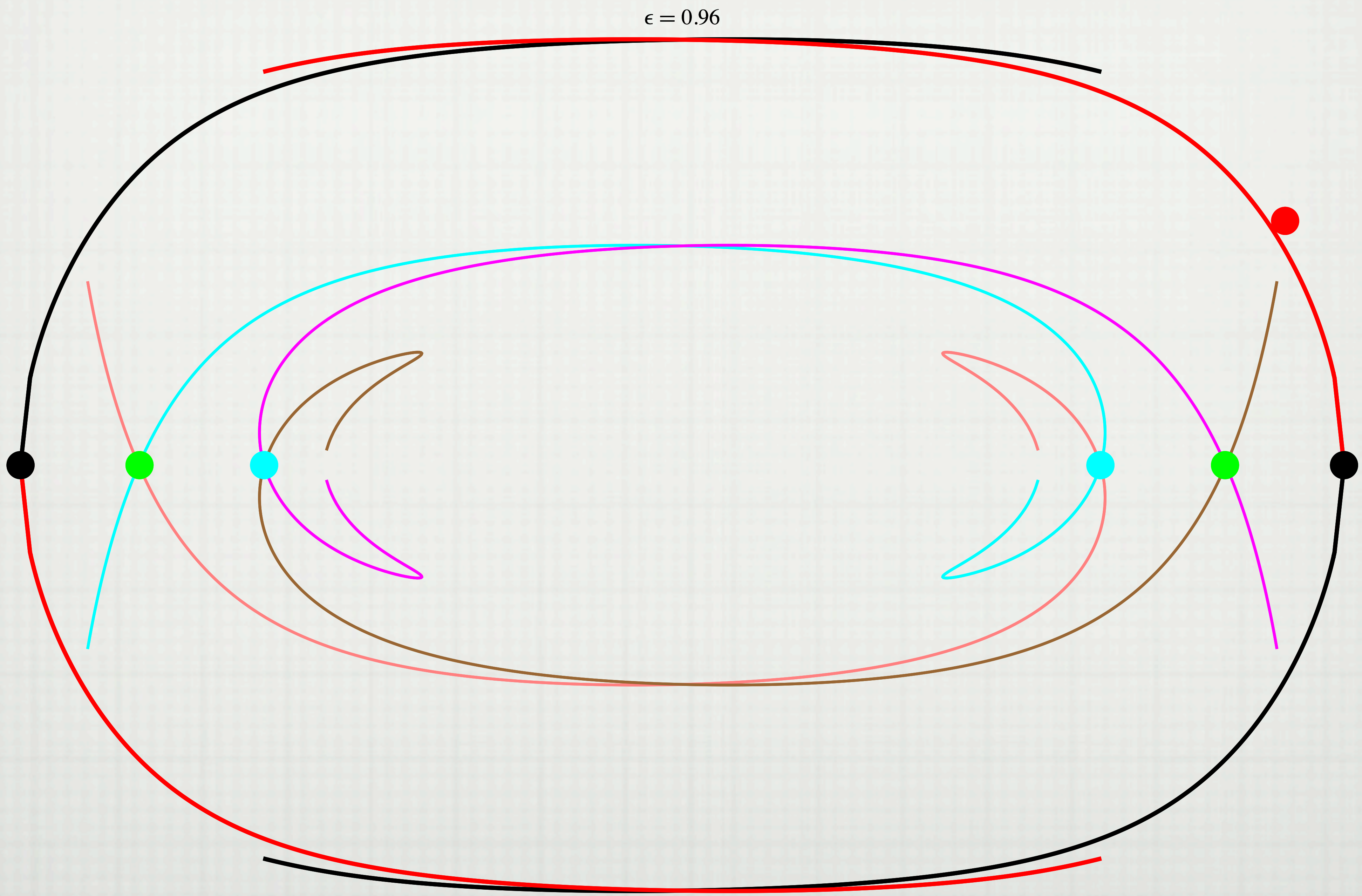

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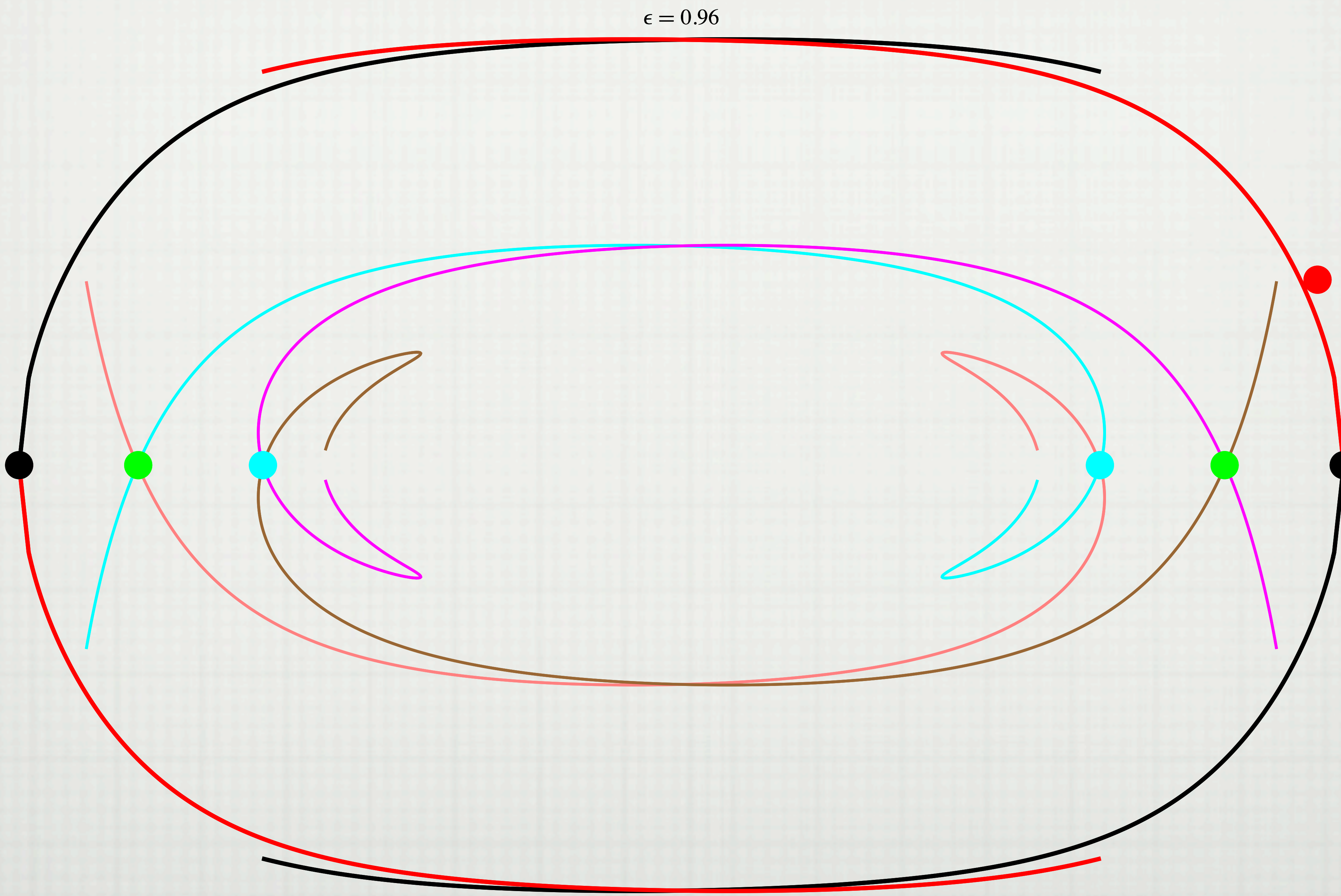

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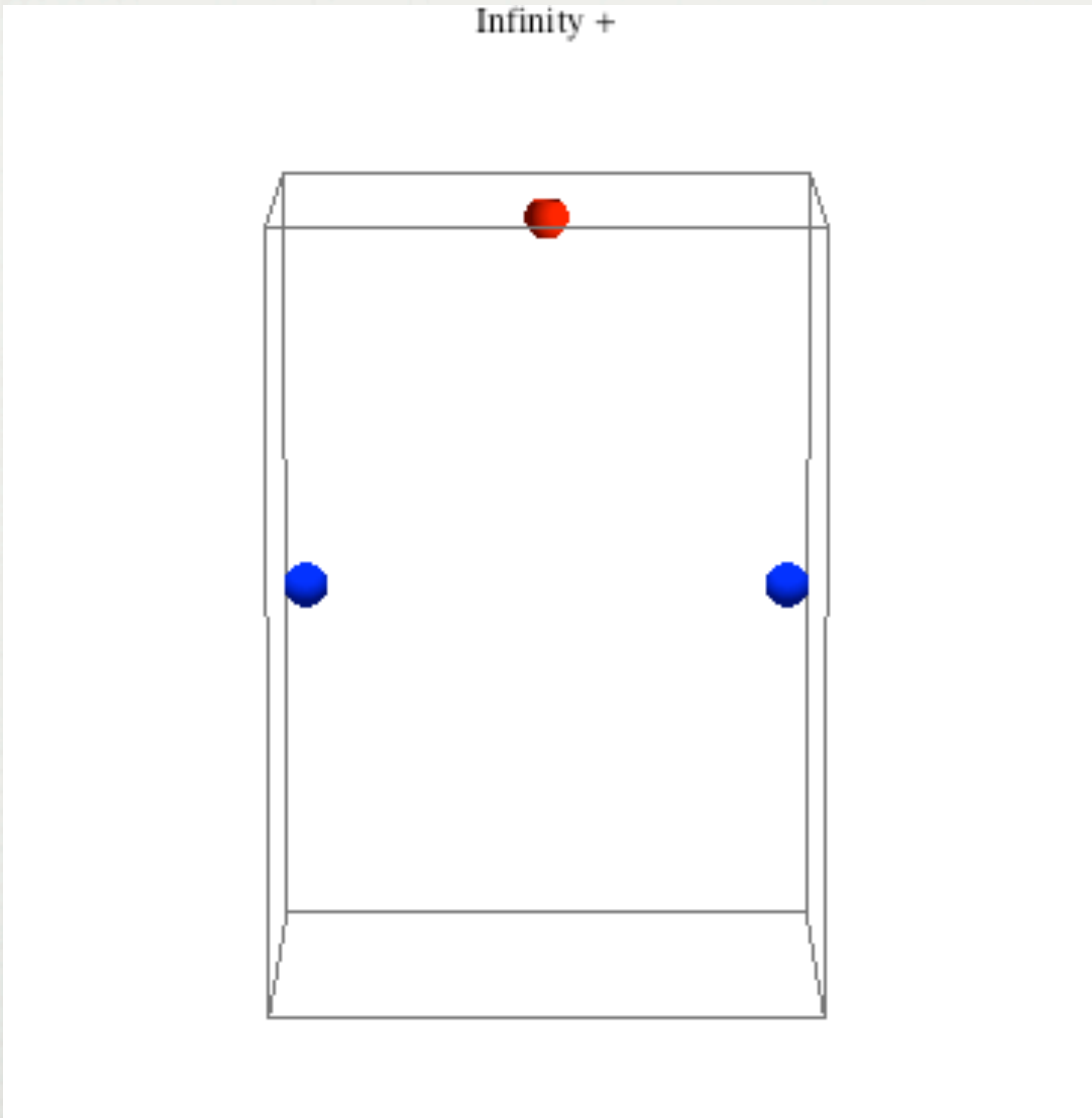
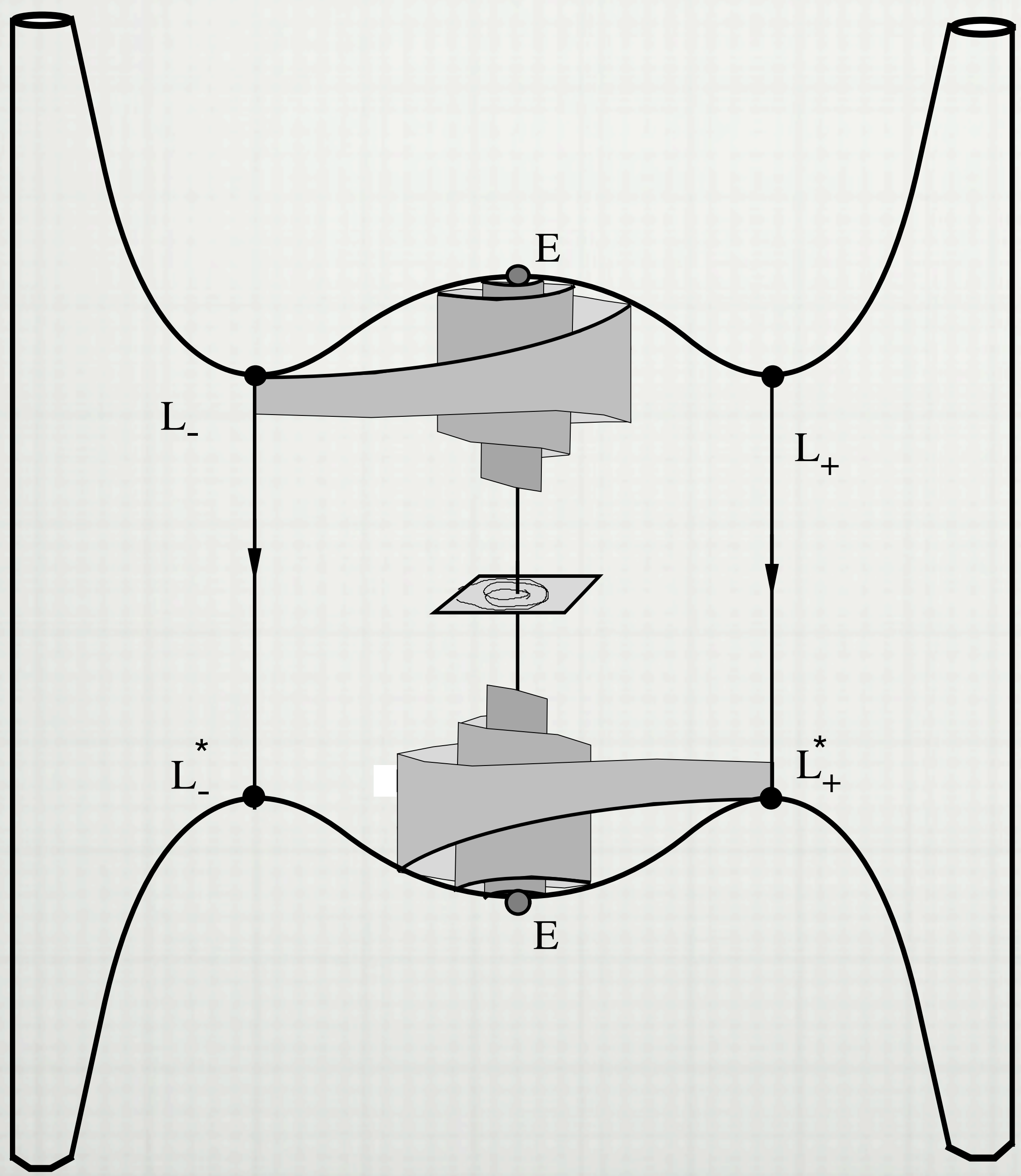


Mechanism for Chaos Near Collision

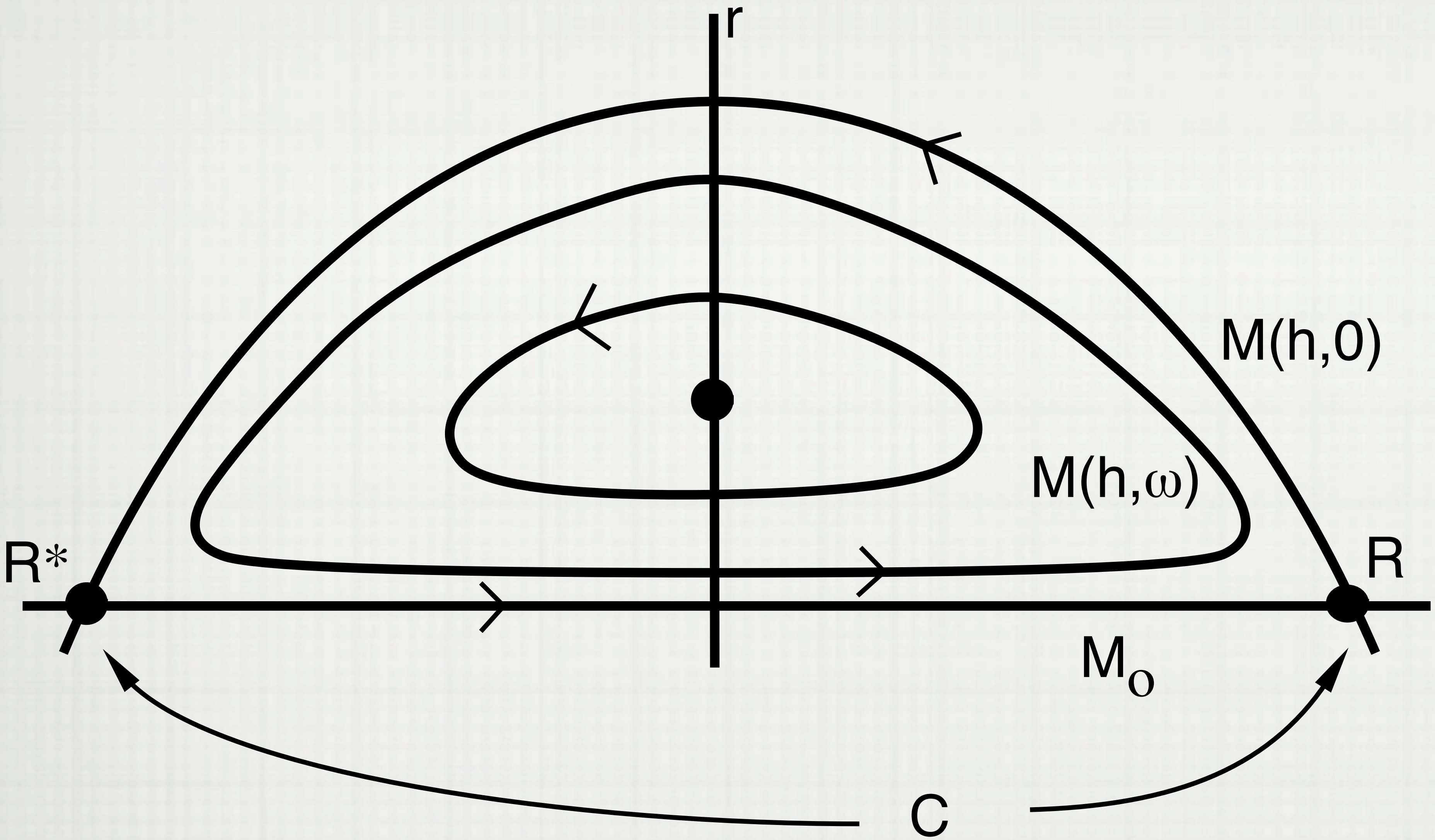
Why are there so many hyperbolic fixed points and heteroclinic points ?



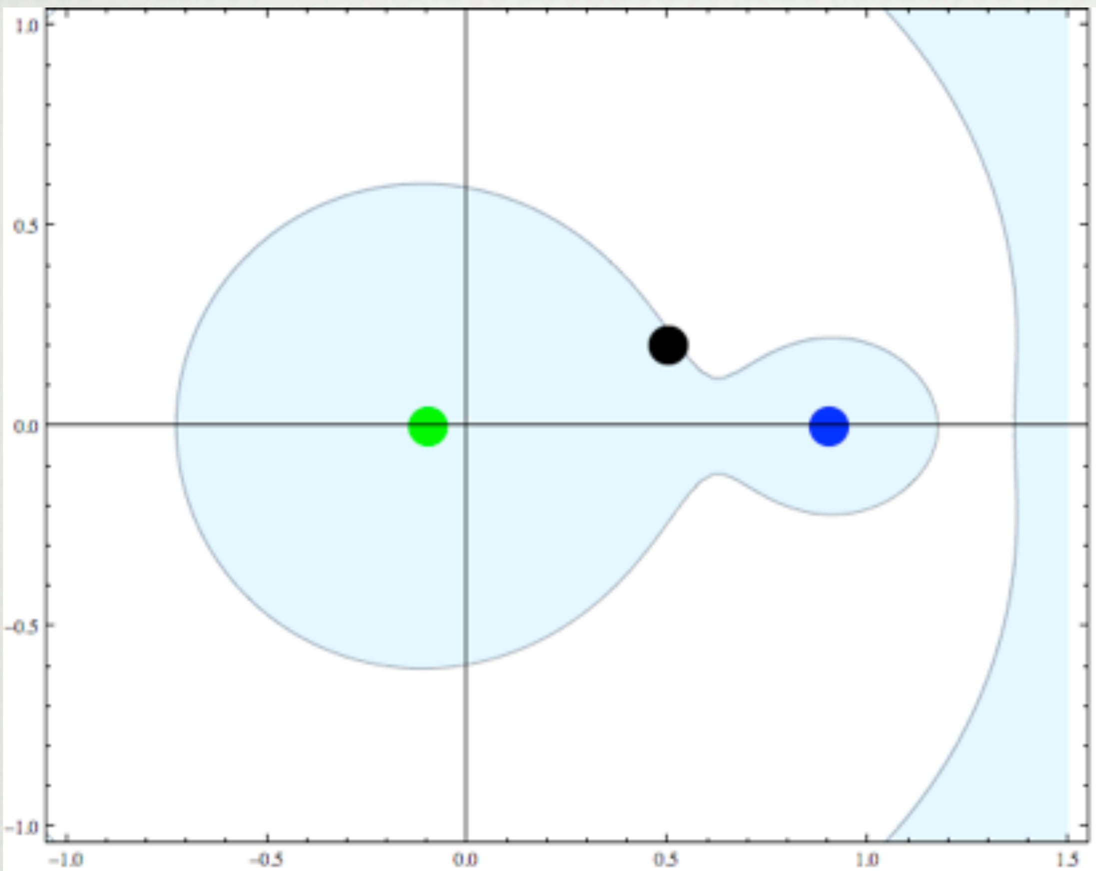
The limiting case ε = 1 is the planar isosceles problem where we have the Lagrange and Euler triple collision solutions. Using McGehee's blow-up method these become hyperbolic *equilibrium points* on an invariant triple collision manifold.

Spiraling of the Lagrange stable and unstable manifolds produces infinitely many solutions bi-asymptotic to equilateral triple collision.

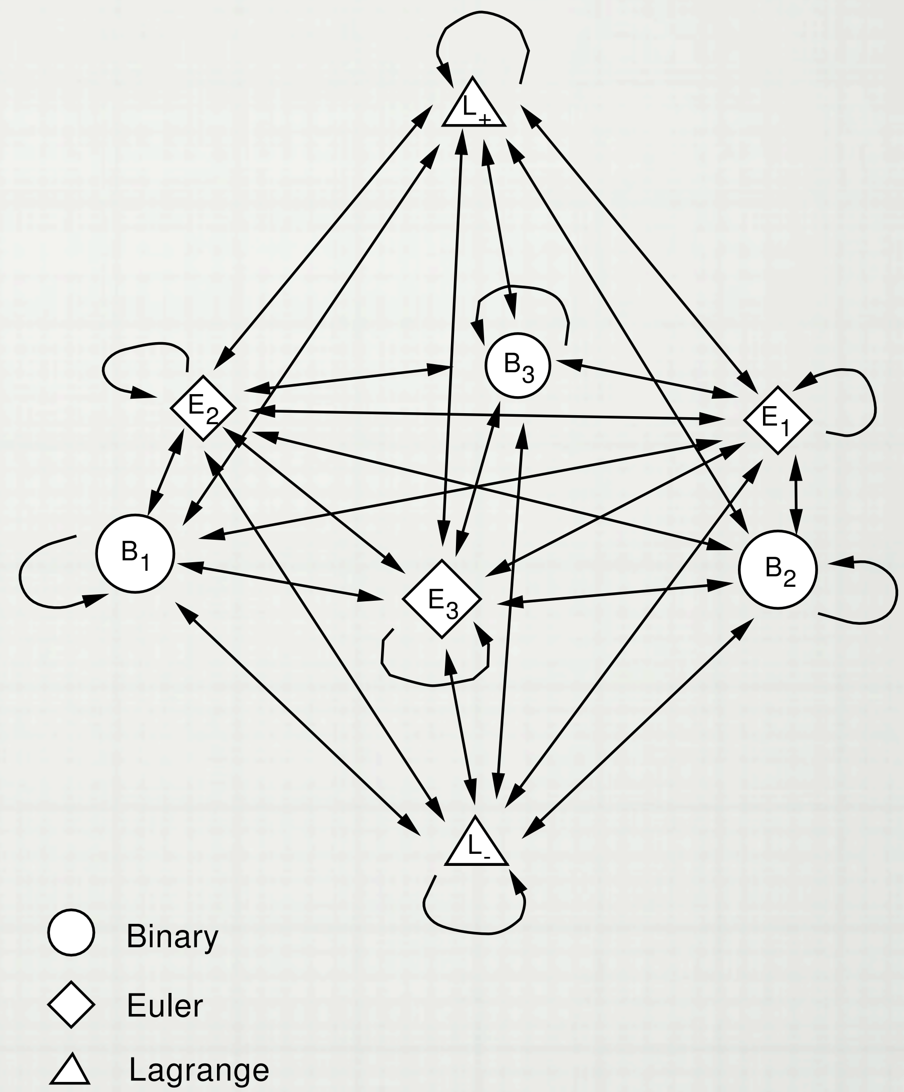
Heteroclinic Loops of Equilibria



Other kinds of chaos



Near collinear Lagrange points



Symbolic dynamics for the low angular momentum planar three-body problem. Use 4D "windows".

Mapping Poincaré ...

... with the Sitnikov problem Poincaré map with $\varepsilon = 0.1$

Fin

Mapping Poincaré ...

... with the Sitnikov problem Poincaré map with $\varepsilon = 0.1$

Fin