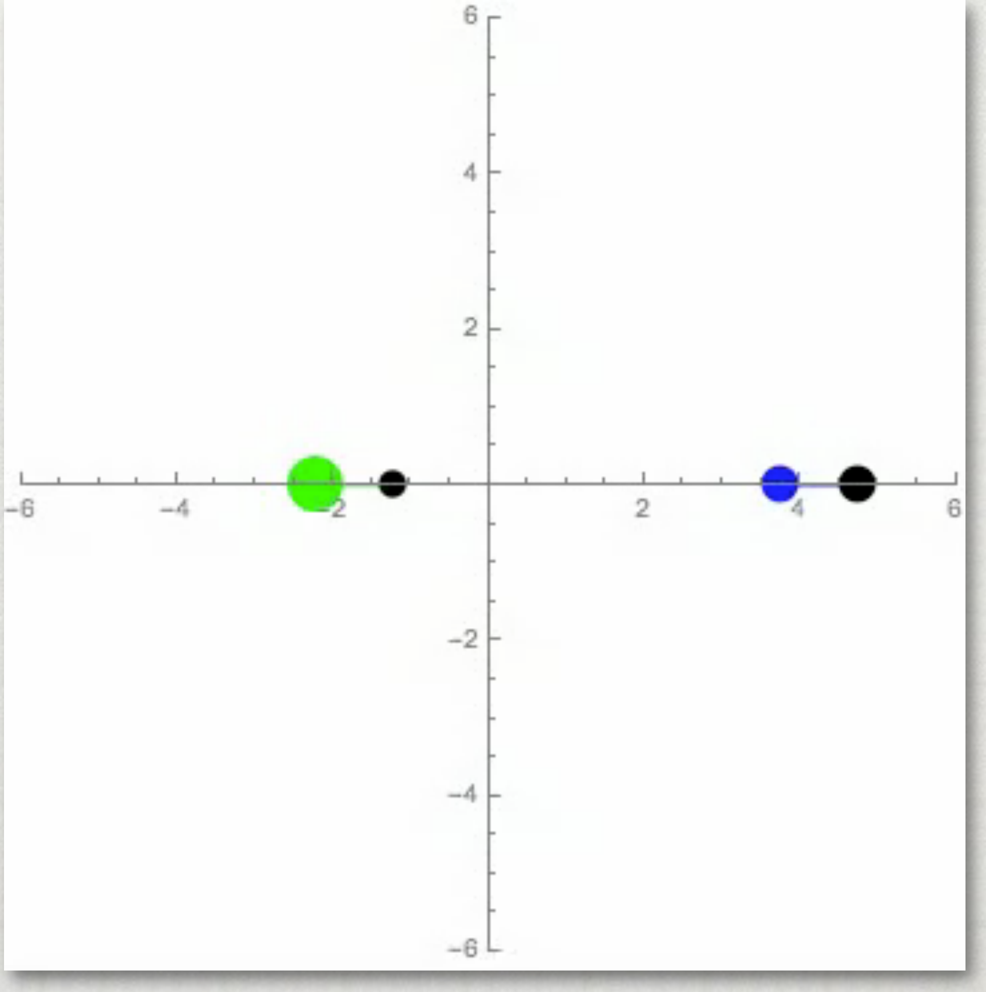
RELATIVE EQUILIBRIUM CONFIGURATIONS OF GRAVITATIONALLY INTERACTING RIGID BODIES



Lecture in Venice, June 2018

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Phase Locking of Gravitationally Interacting Rigid Bodies

Consider the familiar phenomemon of phase locking. In the moon-earth system, the moon is phase locked but the earth is not. We will consider the situation where all of the interacting bodies are phase locked..

Ideally, the whole system just rotates rigidly. It's a relative equilbrium (RE) motion.

"Real world" Examples

Pluto and Charon

Asteroid Pairs

For n = 2 bodies, some of these relative equilibrium states are local minima and can give an explanation for phase locking. What about $n \geq 3$?

Could these three “asteroids” evolve into a phase locked, RE motion where the whole group just rotates rigidly about the center of mass ?

Possible Mechanism for Phase Locking ?

Dissipative forces, such as tidal friction, act in addition to gravity. Assume that such forces cause the total energy of the system to decrease leaving the linear and angular momenta constant. This leads to the following mathematical problem:

Consider n gravitationally interacting rigid bodies in $\mathbb{R}^3$. Fix the center of mass at the origin and the total angular momentum vector $\lambda \in \mathbb{R}^3$. This defines a submanifold $\mathcal{M}_\lambda$ of the phase space. Then find the minima and local minima of the total energy function H restricted to $\mathcal{M}_\lambda$.

The study of phase locking goes back to Lagrange, Laplace, George Darwin, … But this formulation is due to Smale in "Topology and Mechanics I, II" which led to a lot of further work. In general

Critical Point of $H|_{\mathcal{M}_\lambda}$ $\longleftrightarrow$ Relative Equilibrium States

Main Results

Part I:

Theorem: For $n \geq 3$, relative equilibrium states in $\mathcal{M}_\lambda$ are never local minima of the energy.

- No restrictive assumptions about the shapes of the bodies

Implications:

- In a system of three or more masses, as energy decreases, either some of the bodies will collide (producing fewer bodies) or else some will escape
- In the system without dissipation, form the reduced mechanical system on

$$\mathcal{M}_\lambda / \mathbb{SO}(3)$$

Then the resulting equilibria are never extrema of the Hamiltonian (so nonlinear stability is unlikely)

This result was previously known for point masses where it follows from results about Morse indices of critical points (Palmore, Smale). My interest in the rigid body problem is due to a paper of D.Scheeres, "Minimum energy configurations …" where the theorem above was conjectured. Scheeres also has results about minima on the boundary, i.e., with some bodies in contact, at least when the bodies are spheres.

Main Results

Part II:

Study the case n = 2 in more detail. For general shapes the complexity of the gravitational potential makes it impossible to explicitly find the RE configurations. Instead try to give lower bounds for the number of critical points using Morse theory and Lusternik-Schnirelmann category

Theorem: For $|\lambda|$ sufficiently large there are at least 32 RE configurations (in a certain subset χ of configuration space, up to rotation about λ) provided the critical points of $H_{\mathcal{M}_\lambda}$ are nondegenerate. Without assuming nondegeneracy there must be at least 12 RE up to rotation. There are always at least two minima.

These estimates turn out to be rather weak. In the limit as $|\lambda| \to \infty$ one can get an exact count which continues to hold for $|\lambda|$ very large. Similar results were found by Maciejewski.

Theorem: For $|\lambda|$ sufficiently large there are exactly 576 RE configurations in χ, up to rotation about λ, all nondegenerate, including 16 minima.

Full n-Body Problem (n rigid bodies)

Consider a collection of n rigid, massive bodies in $\mathbb{R}^3$.

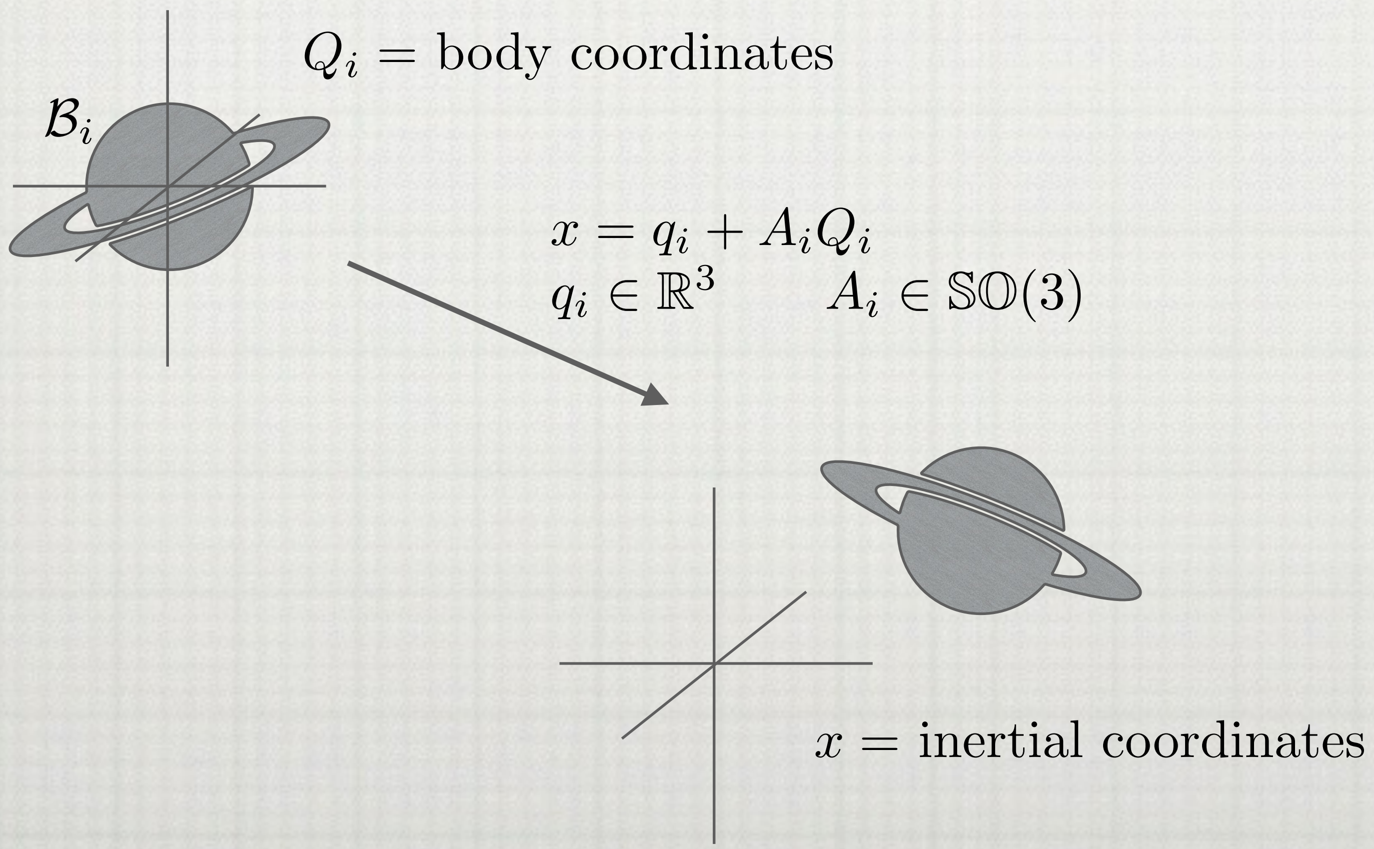
Body coordinate systems: $Q_i \in \mathbb{R}^3 \qquad \mathcal{B}_i \subset \mathbb{R}^3 \qquad dm_i, i = 1, \ldots, n.$

Total mass: $m_i = \int_{\mathcal{B}_i} dm_i > 0$

Center of mass: $\int_{\mathcal{B}_i} Q_i dm_i = 0.$

Symmetric 3×3 *inertia matrix* of $\mathcal{B}_i$:

$$I_i = \int_{\mathcal{B}_i} \left(|Q_i|^2 \,\mathrm{id} - Q_i Q_i^T \right) dm_i$$



Configuration: $Z = (q_1, \ldots, q_n, A_1, \ldots, A_n) \in \mathcal{U} \subset R^{3n} \times \mathbb{SO}(3)^n$

$$\mathcal{U} = \{Z : \text{ center of mass} = 0, \text{ bodies disjoint } \}$$

Phase space: $T\mathcal{U}$

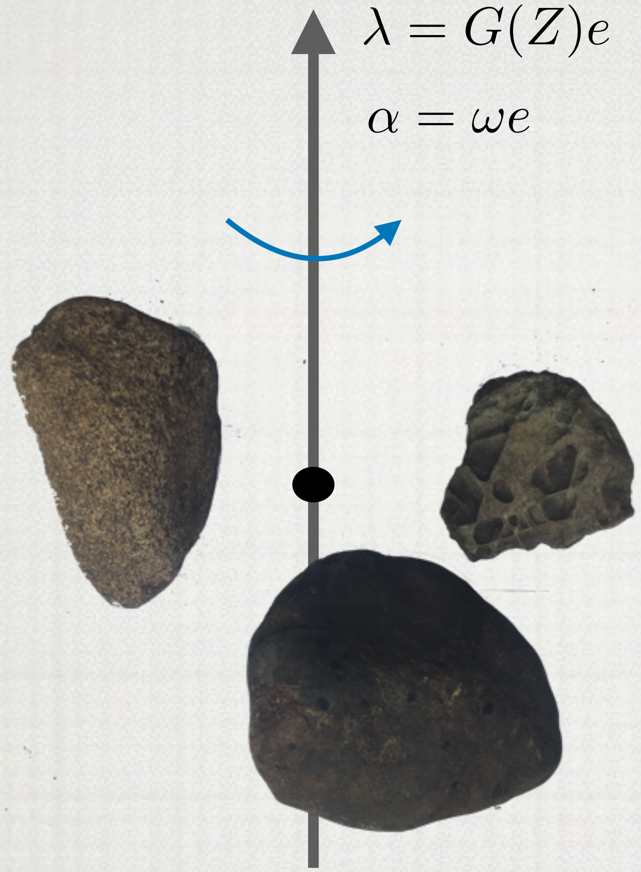
Velocities: $v_i = \dot{q}_i \qquad \Omega_i =$ angular velocity in body coordinates

Lagrangian system with $L(q_i, A_i, v_i, \Omega_i) = K - U$

The kinetic energy K is relatively straightforward, but the potential energy can be very complicated. $U = \sum U_{ij}$ where the relative potential between two bodies is given by an integral

$$U_{ij}(q_i, q_j, A_i, A_j) = \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \frac{dm_i(Q_i)\, dm_j(Q_j)}{|q_i - q_j + A_i Q_i - A_j Q_j|}$$

Relative Equilibria



Constant angular velocity vector $\alpha = \omega e$ $\quad e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Angular momentum vector $\lambda = I(Z)\alpha$ where

$$I(Z) = \sum_i m_i(|q_i|^2 \,\mathrm{id} - q_i q_i^T) + \sum_i A_i I_i A_i^T$$

is the 3×3 *total inertia matrix* of the whole configuration.

For RE, α is an eigenvector of $I(Z)$ so $\lambda = G(Z)e$

$$G(Z) = e^T I(Z) e = \textstyle\sum_i m_i(x_i^2 + y_i^2) + \sum_i e^T A_i I_i A_i^T e$$

$$q_i = (x_i, y_i, z_i)$$

For given λ, RE configurations, Z, can be characterized as critical points in 2 ways

Amended Potential (Smale, ...): $W_\lambda(Z) = \frac{1}{2}\lambda^T I(Z)\lambda - U(Z)$

Critical Energy Function (Scheeres): $H_\lambda(Z) = \dfrac{|\lambda|^2}{2G(Z)} - U(Z)$

- same critical points (RE)
- equal at critical points

Energy Minimizers

Recall that we are interested in local minima of the energy H on $\mathcal{M}_\lambda$ in phase space. Now the amended potential is obtained by minimizing over velocities (with given λ) so we have

$$\text{Local Minima of } H|_{\mathcal{M}_\lambda} \iff \text{Local Minima of } W_\lambda$$

Lemma: Local minima of $W_\lambda \iff$ local minima of H_λ (at least if the maximal eigenvalue of $I(Z)$ is not repeated).

Proof sketch for main theorem: Two steps. For $n \geq 3$, show that the simpler function H_λ has no local minima. Then deal with the case of repeated eigenvalues by considering W_λ directly.

Consider the partial Hessian of H_λ at a critical point with respect to the q_i variables (leaving the rotation matrices A_i fixed). Get an estimate for the trace of this matrix

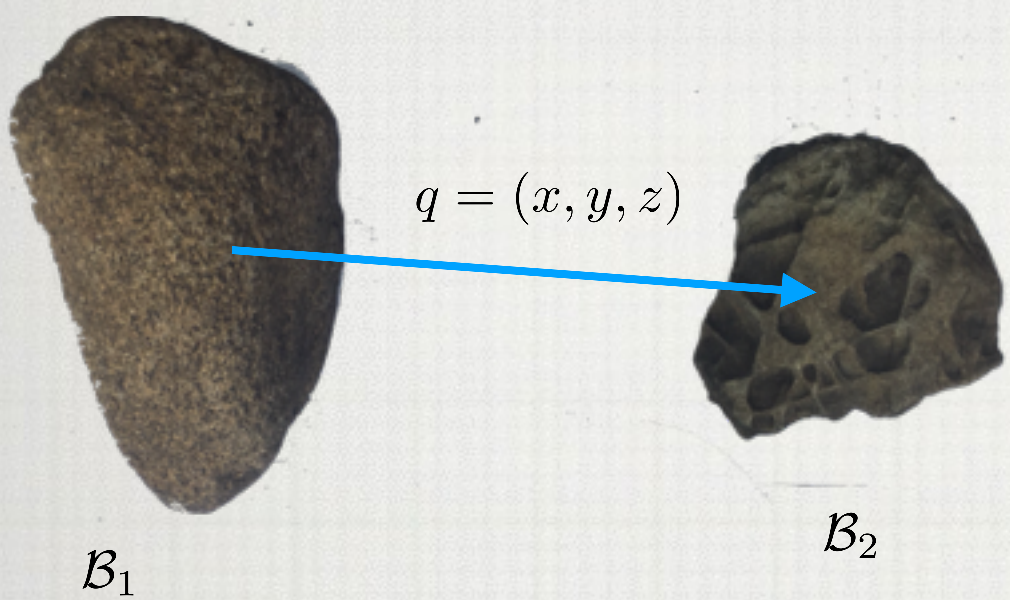
$$trace < \frac{(6-2n)|\lambda|^2}{G(Z)^2}$$

So $n \geq 3 \Longrightarrow trace < 0 \Longrightarrow$ not local min

In light of the complexity of $U(Z)$ how can one estimate the trace ? It turns out that the contribution to the Hessian from the potential $U(Z)$ is zero, essentially due to the fact that the Newtonian potential is a harmonic function on $\mathbb{R}^3$.

Part II: Morse theory for n=2

Configuration: $Z = (q, A_1, A_2) \in \mathcal{U} \subset \mathbb{R}^3 \times \mathbb{SO}(3) \times \mathbb{SO}(3)$



Goal: Count RE configurations $=$ Critical points of

$$H_\lambda(Z) = \frac{|\lambda|^2}{G(Z)} - U(Z)$$

$$G(Z) = \frac{m_1 m_2}{m_1 + m_2}(x^2 + y^2) + e^T A_1 I_1 A_1^T e + e^T A_2 I_2 A_2^T e$$

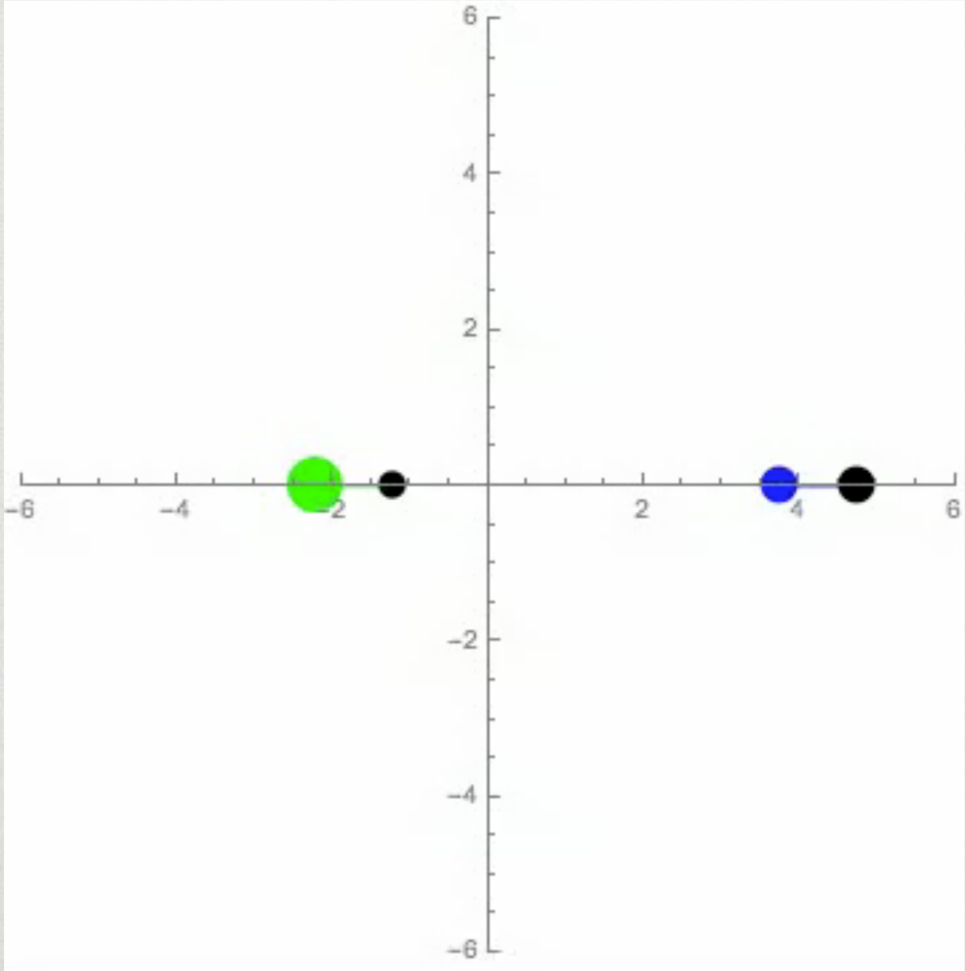
$$e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Rotation symmetry around **e** implies there will be circles of RE. May assume q=(x,0,z).

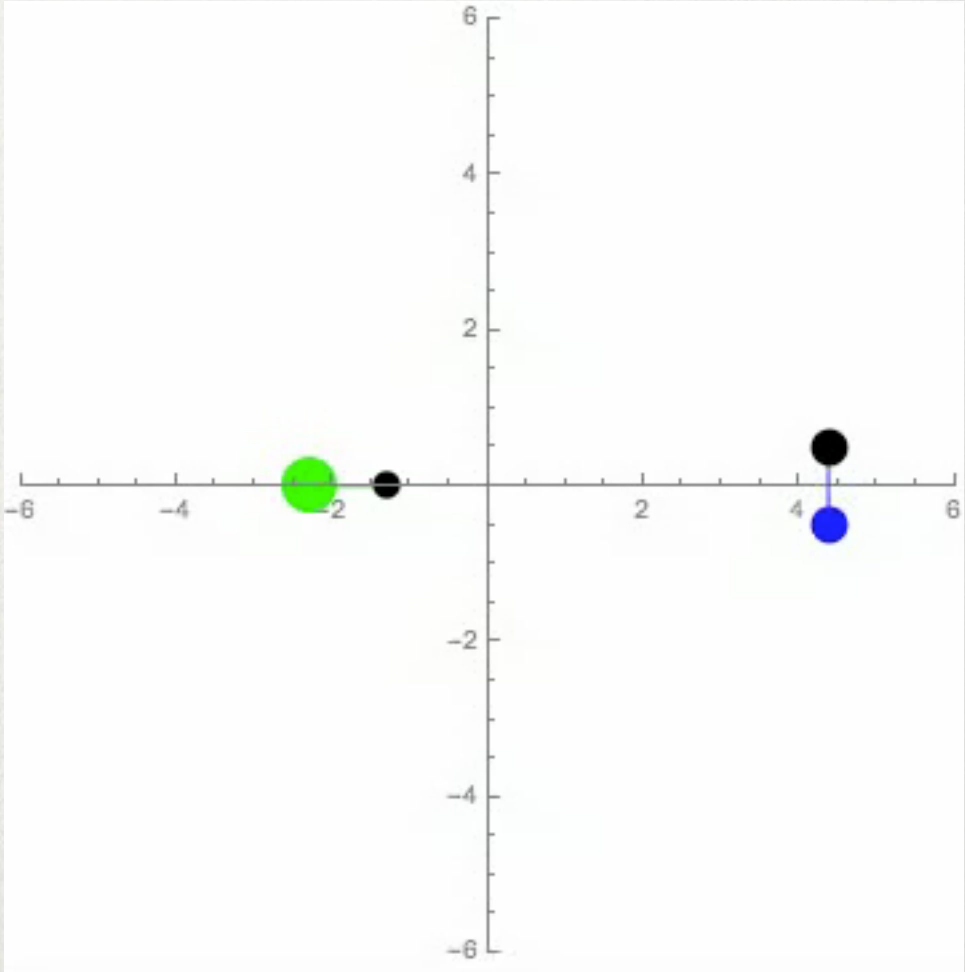
Because of the complexity of U(Z) this is still a difficult problem even when n=2. Previous authors have considered simpler special cases where one or both bodies is a sphere (the potential of a sphere reduces to that of a point mass). Or one can use only the first few terms in a Legendre expansion of the potential.

Simulation with simple shapes — two "batons"

For shapes consisting of spheres joined by massless rods, it is possible to write down the potential energy, but still nontrivial to find all of the RE.



Minimum energy RE — stable



Saddle point RE — unstable

Scheeres' work on two spheres

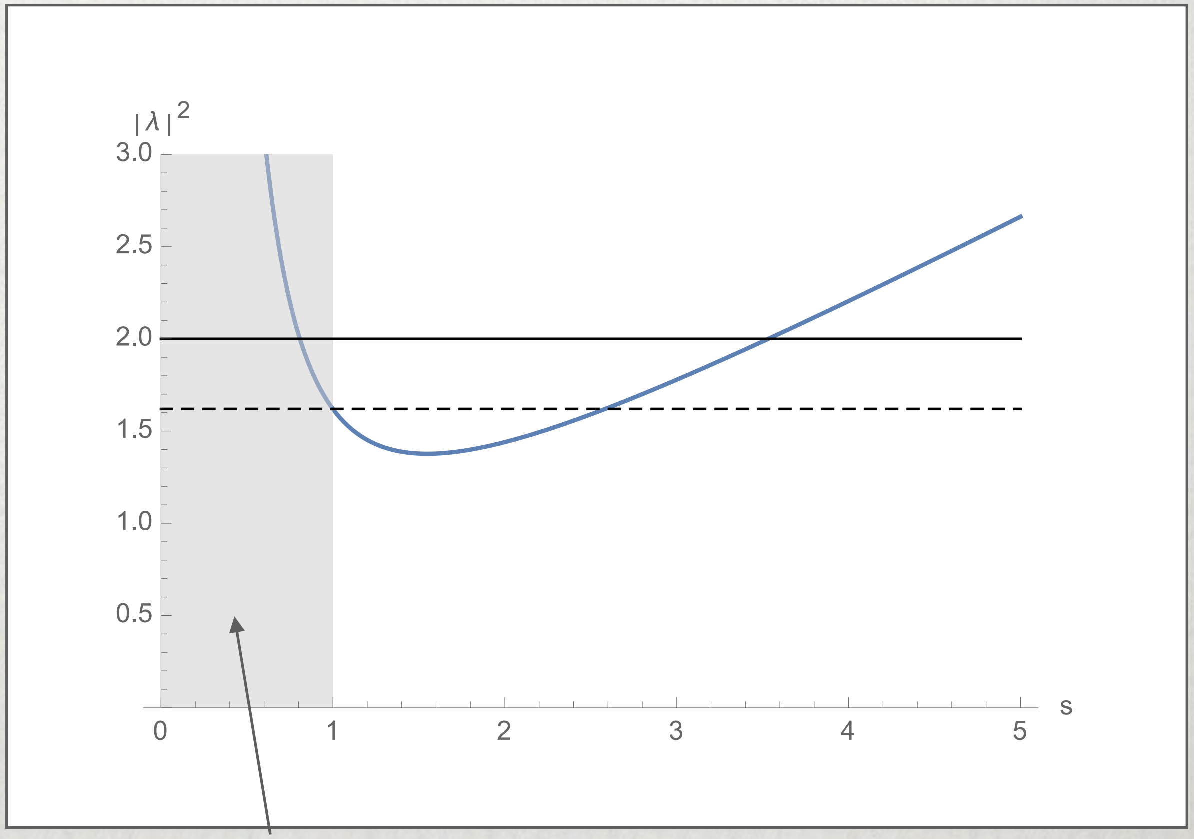
If each body is a uniform sphere, the mutual potential is the same as for two point masses. But the spheres problem is different because the solid spheres have moments of inertia which contribute to the energy when rotated. This problem was studied by Dan Scheeres, who has also considered problems with more than two spheres.

Result for two spheres of equal mass and equal radius $\frac{1}{2}$: $q = (s, 0, 0)$, rotations irrelevant

Blue curve shows positions of the RE (critical points of minimal energy function) for various angular momenta.

As angular momentum is increased

- low angular momentum: no RE
- two RE
- one of them reaches contact at the "fission parameter"
- only one RE for larger angular momentum



Bodies in contact

Fix $\lambda = (0, 0, |\lambda|)$.

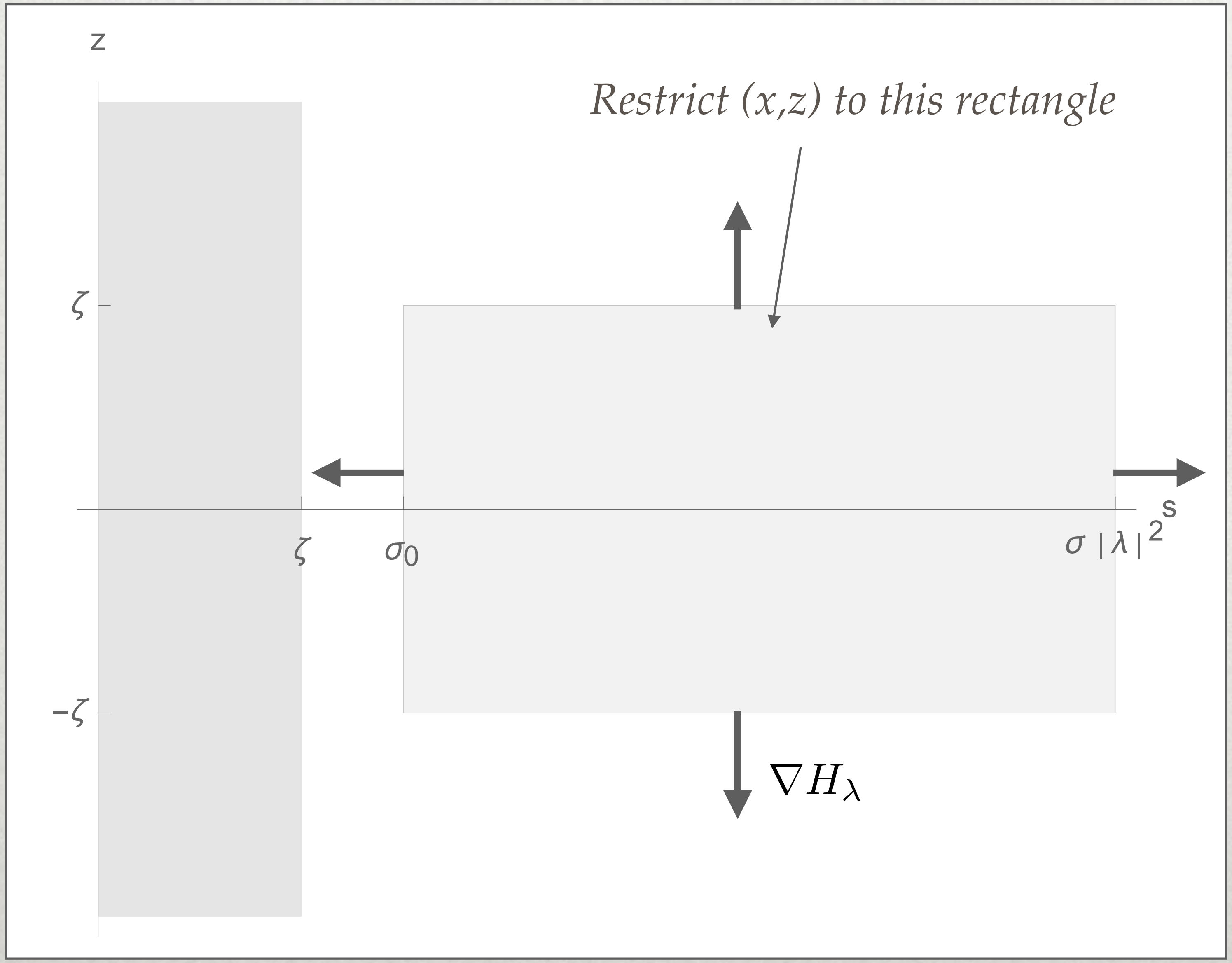
Count critical points of the minimal energy function $H_\lambda(Z)$ where

$$Z = (q, A_1, A_2) \in \mathbb{R}^3 \times \mathbb{SO}(3)^2$$

By rotation around λ, may assume $q = (x, 0, z), x > 0$

To avoid contact or interpenetration of nonconvex shapes, assume

$$x > \sigma_0 = \text{ separation constant}$$



$\times\ \mathbb{SO}(3)^2$

Lemma: For $|\lambda|$ moderately large ∇H_λ points out on boundary

$\mathbb{Z}_2$ Symmetry

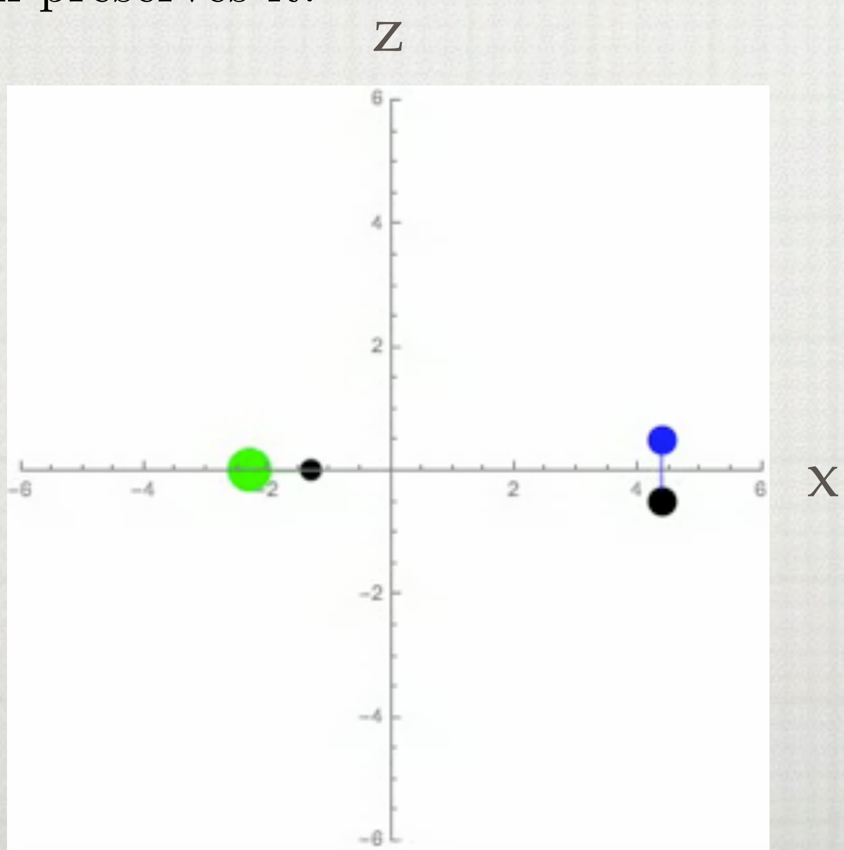
At this point one could use Morse theory to count critical points in the manifold with boundary above:

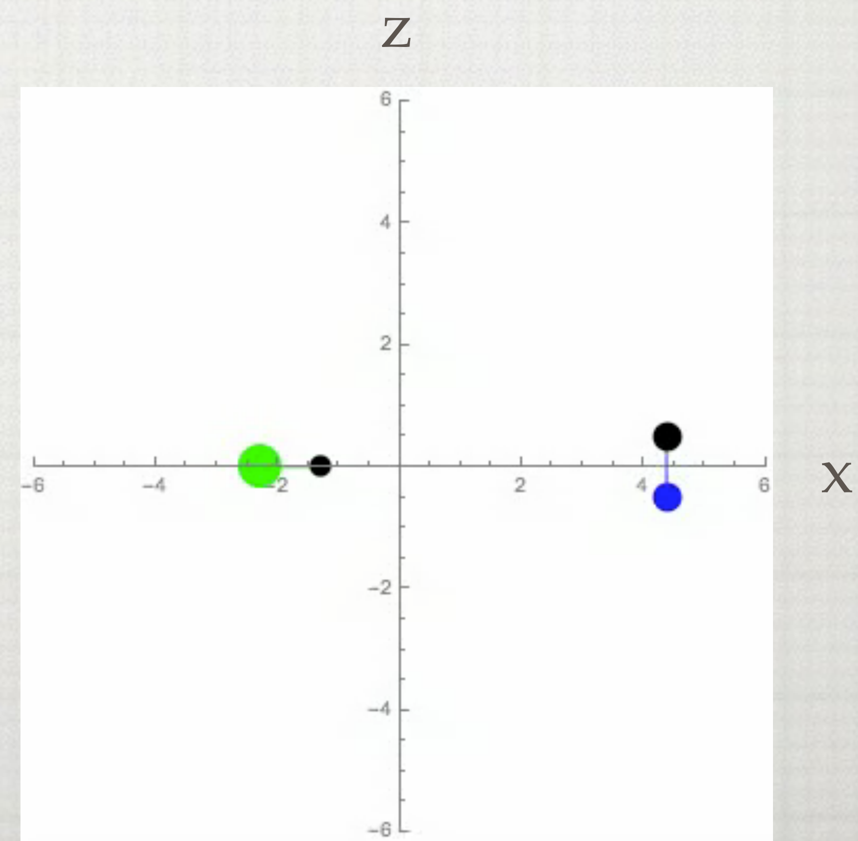
$$\chi = \text{Rectangle} \times \mathbb{SO}(3) \times \mathbb{SO}(3)$$

but to get a better estimate one should take account of a $\mathbb{Z}^2$ symmetry. The symmetry is the composition of two other symmetries:

- *Time reversal*
- *Rotation by π around x-axis*

Separately, each reverses the angular momentum vector, λ, but the composition preserves it.





Quotient Space Topology

The critical energy function determines a smooth function on the quotient $\chi/\mathbb{Z}_2$
Let

$$Z = (x, 0, z, A_1, A_2) \in \chi = \text{Rectangle} \times \mathbb{SO}(3) \times \mathbb{SO}(3)$$

and

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

be the matrix of rotation by π around the x-axis. The $\mathbb{Z}_2$ action is

$$RZ = (x, 0, -z, RA_1, RA_2).$$

$\chi/\mathbb{Z}_2$ is a disk bundle over

$$(\mathbb{SO}(3) \times \mathbb{SO}(3))/\mathbb{Z}_2 \simeq \mathbb{SO}(3) \times (\mathbb{SO}(3)/\mathbb{Z}_2)$$

where the action is moved to the second factor by using $(A_2^{-1}A_1, A_2)$ as coordinates.

Now it is well known that

$$\mathbb{SO}(3) \simeq \mathbb{RP}(3) \simeq \mathbf{S}^3/\mathbb{Z}_2$$

and this implies

$$\mathbb{SO}(3)/\mathbb{Z}_2 \simeq \mathbf{S}^3/\mathbb{Z}_4 = \mathbb{L}(1, 4)$$

a Lens Space !!!

Cohomology and Estimates

Using $\mathbb{Z}_2$ coefficients, the cohomology groups $H^i(\mathbb{SO}(3))$, $0 \leq i \leq 3$ all have rank 1. The Poincaré polynomial is

$$1 + t + t^2 + t^3.$$

It is a standard result that the $\mathbb{Z}_4$ cohomology of $\mathbb{L}$ is the same. Converting to $\mathbb{Z}_2$ coefficients the result is again that $H^i(\mathbb{L})$, $0 \leq i \leq 3$ all have rank 1. The Poincaré polynomial of the product space is then

$$P(t) = (1 + t + t^2 + t^3)^2 = 1 + 2t + 3t^2 + 4t^3 + 3t^4 + 2t^5 + t^6$$

Morse theory then gives at least $1+2+3+4+3+2+1 = 16$ critical points in $\chi/\mathbb{Z}_2$ and therefore at least 32 critical points in χ, assuming all critical points are nondegenerate.

Lusternik-Schnirelmann category gives a lower bound on the number of critical points without assuming nondegeneracy.

$$\text{Number critical points } \geq \mathrm{Cat}(\mathbb{SO}(3) \times \mathbb{L}) \geq 1 + \mathcal{CL}(\mathbb{SO}(3) \times \mathbb{L})$$

where $\mathcal{CL}$ is the cup length (maximal length of a nontrivial cup product in cohomology). Using $\mathbb{Z}_2$ coefficents

$$\mathcal{CL}\,\mathbb{SO}(3) = 3 \qquad \mathcal{CL}\,\mathbb{L} = 2 \qquad \mathcal{CL}\,(\mathbb{SO}(3) \times \mathbb{L}) = 5$$

So there must be at least 6 critical points in $\chi/\mathbb{Z}_2$ and at least 12 in χ

Limit as $|\lambda| \to \infty$

These estimates seem to be far from optimal. It is possible to show that in a limiting problem as $|\lambda| \to \infty$ there are exactly 576 relative equilibria, all nondegenerate, and this remains true for all $|\lambda|$ sufficiently large. This result is very close to those of Macjiewski. It makes the generic assumption that for each of the bodies $\mathcal{B}_1, \mathcal{B}_2$, the three principle moments of inertia are distinct. In the limit the RE configurations consist of all configurations with these axes aligned with the coordinate axes.

There are 6×6 ways to choose which oriented axes are along the x direction and then 4×4 ways to rotate the other axes: $6 \times 6 \times 4 \times 4 = 576$

If one chooses an axis of minimal moment of inertia along x and maximal along z the resulting RE is a minimum. There are 16 minima. The other Morse indices are given by the *Morse polynomial*

$$M(t) = 16(1+t)^2(1+t+t^2)^2 = 16 + 64t + 128t^2 + 160t^3 + 128t^4 + 64t^5 + 16t^6$$

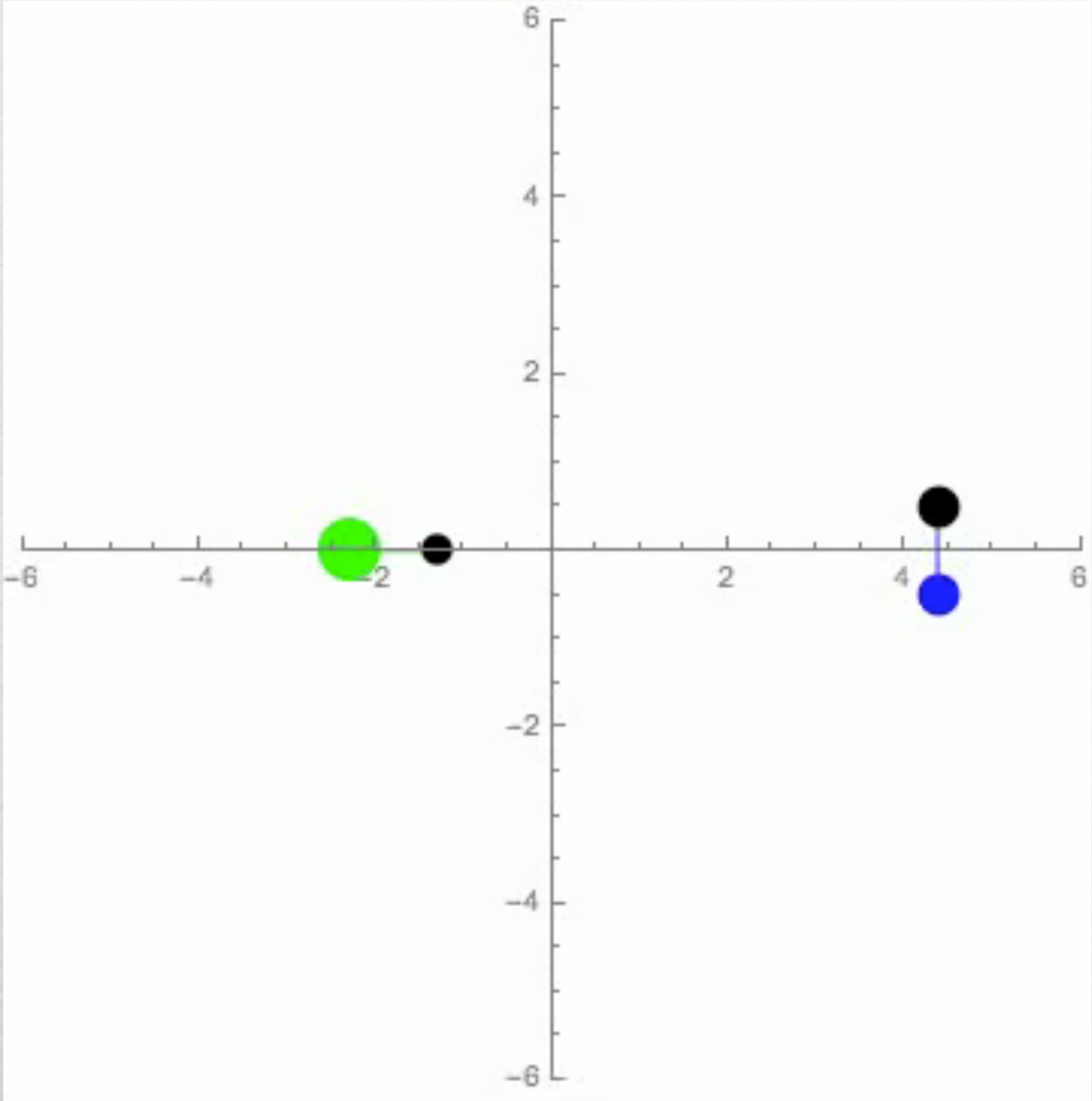
Some questions for further research

- Are there examples with fewer than 576 RE ? Irregular shapes with no discrete symmetries at lower angular momenta ?
- What kind of bifurcations occur as angular momentum is decreased (preliminary work by Jodin Morey) ?
- Are there examples with more than the expected number of minima ?
- For simple shapes, can one prove existence of homoclinic and heteroclinic orbits connecting the unstable RE ? Nearby chaotic motions ?
- Is there a reasonable theory for irregular bodies in contact (following Scheeres' work with spheres) ?
- What about RE with irregular, possible nonconvex bodies very close but not in contact ?

References to this work:

R. Moeckel, *Minimal energy configurations of gravitationally interacting rigid bodies*, Celestial Mechanics and Dynamical Astronomy, 128, 1, 3-18, (2017)

R. Moeckel, *Counting relative equilibrium configurations of the full two-body problem*, Celestial Mechanics and Dynamical Astronomy, 130, 2 (2018)



Not a RE !

Grazie Mille !!!!