Geometry of the Reduced and Regularized Three-Body Problem

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(joint work with R. Montgomery)

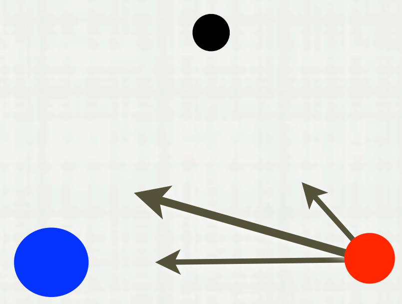
The Planar Three-Body Problem

Masses: $m_1, m_2, m_3 > 0$

Positions: $q_1, q_2, q_3 \in \mathbb{R}^2$

Momenta: $p_1, p_2, p_3 \in \mathbb{R}^{2*}$

$(q, p) \in T^*\mathbb{R}^6$



Hamiltonian system of 6 degrees of freedom.

$$H(q,p) = K(p) - U(q)$$

$$K(p) = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2} + \frac{|p_3|^2}{2m_3}$$

$$U(q) = \frac{m_1 m_2}{r_{12}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_3}{r_{23}}$$

Mutual distances: $r_{ij} = |q_i - q_j|$

Symmetries and Singularities

Symmetries: Translation and rotation of positions q_i in $\mathbb{R}^2$

=> Reduced Hamiltonian System of 3 degrees of freedom

Reduced phase space: T^*Q
Reduced configuration space: $Q = \mathbb{R}^+ \times \mathbf{S}^2$

Size of triangle ($\mathbb{R}^+$)

Shape of triangle
The "Shape Sphere" ($\mathbf{S}^2$)

Singularities: only due to collisions

Three binary collisions: $r_{12} = 0$ or $r_{31} = 0$ or $r_{23} = 0$

Triple collision: $r_{12} = r_{31} = r_{23} = 0$

Must restrict to $\mathcal{U} = \{r_{ij} \neq 0\} \subset Q$
Flow on $T^*\mathcal{U}$ is not complete.

Levi-Civita regularization extends orbits through binary collisions in a natural way. McGehee blow-up slows down triple collision solutions so they converge to equilibrium points in a boundary manifold.

Goal: Describe a global reduction, regularization and blow-up for the planar three-body problem. For each fixed energy and angular momentum, get a complete flow on a five-dimensional manifold. Emphasize the shape sphere point of view.

Some of the history this problem:

Levi-Civita: regularization of a single binary collision using the complex squaring map.
McGehee: blow-up of triple collision
Birkhoff, Thiele: regularization of two binary collisions in the R3BP using conformal maps
Bolotin: regularization of restricted n-body problem
Lemaitre: regularization of all three binary collisions, shape sphere point of view, local reduction using angle variables
Waldvogel: regularization of all three binary collisions, no shape sphere, no reduction
Waldvogel, Simo-Susin: reduction, regularization and blow-up for the zero angular momentum problem.
Heggie: Regularization using symmetrical variables, no blow-up, shape sphere or reduction

Some novel aspects of our approach:

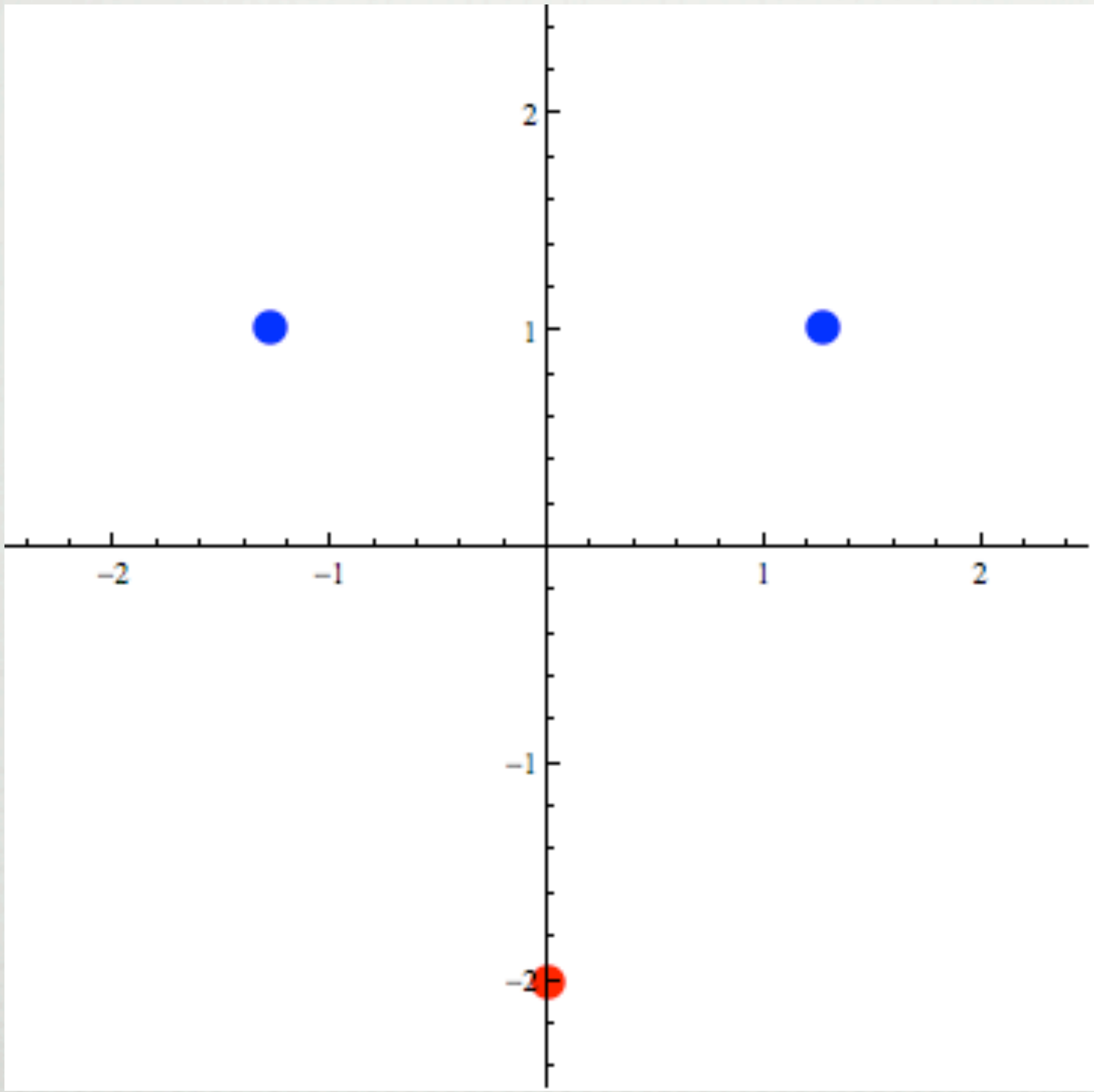
Combine global symplectic reduction with regularization of all three binary collisions and blow-up of triple collision.

Realization of the shape sphere and regularized shape sphere as complex projective lines, CP(1). Description of Lemaitre's beautiful conformal regularizing map based on simple projective formulas.

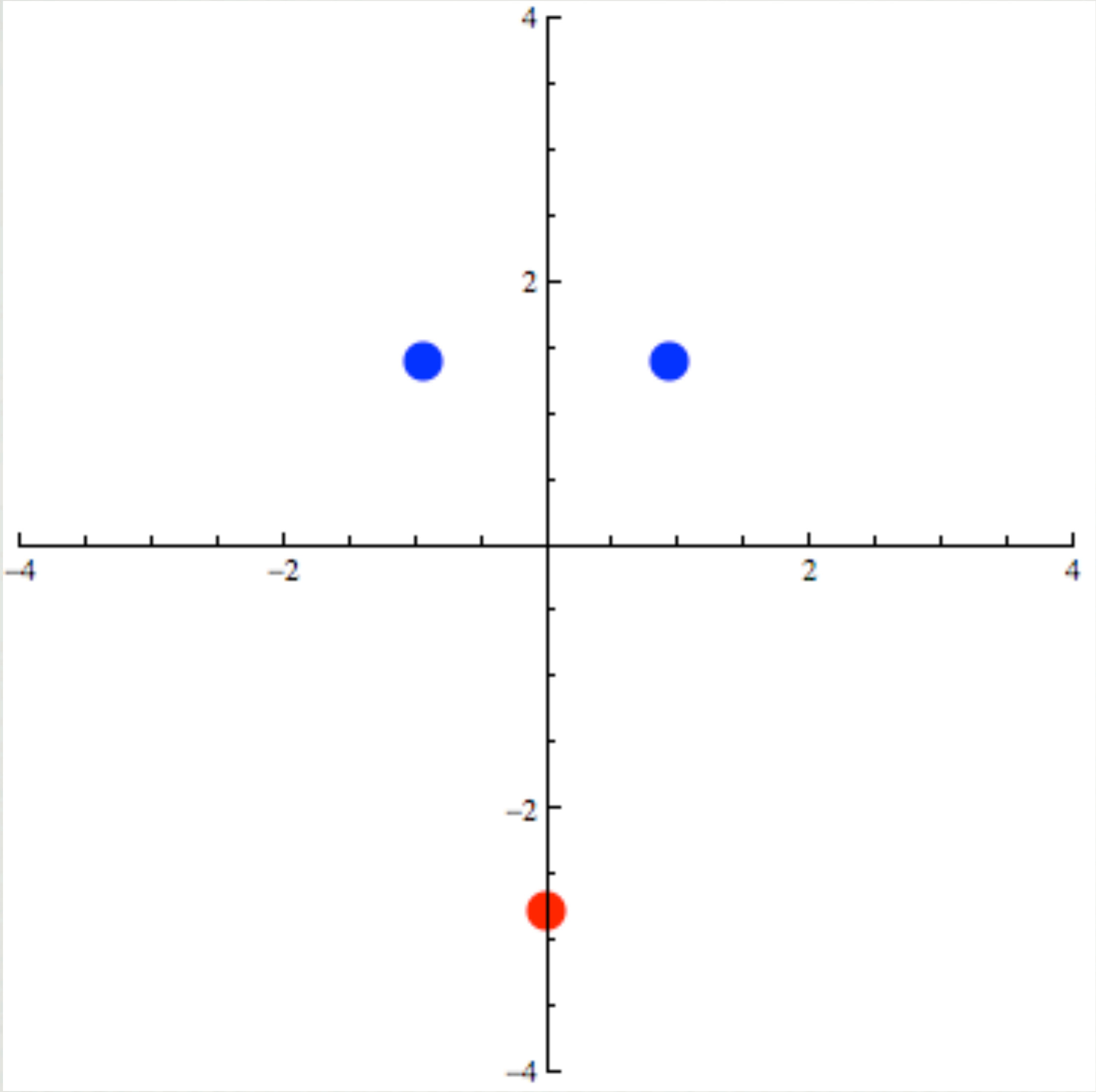
Global descriptions of the flows on reduced phase spaces using homogeneous coordinates.

Explicit formulas for the "curvature terms" in the reduced equations resulting from the twisted symplectic structures on the reduced phase spaces.

Some examples of orbits of the reduced and regularized flow.

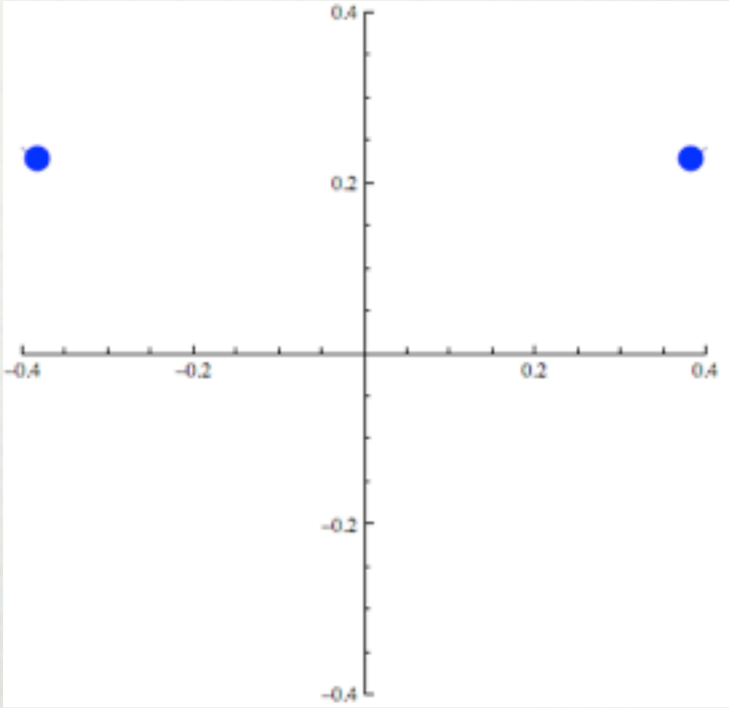


Simple periodic orbit of the isosceles 3BP. Binary collisions are unavoidable. Regularization is essential.

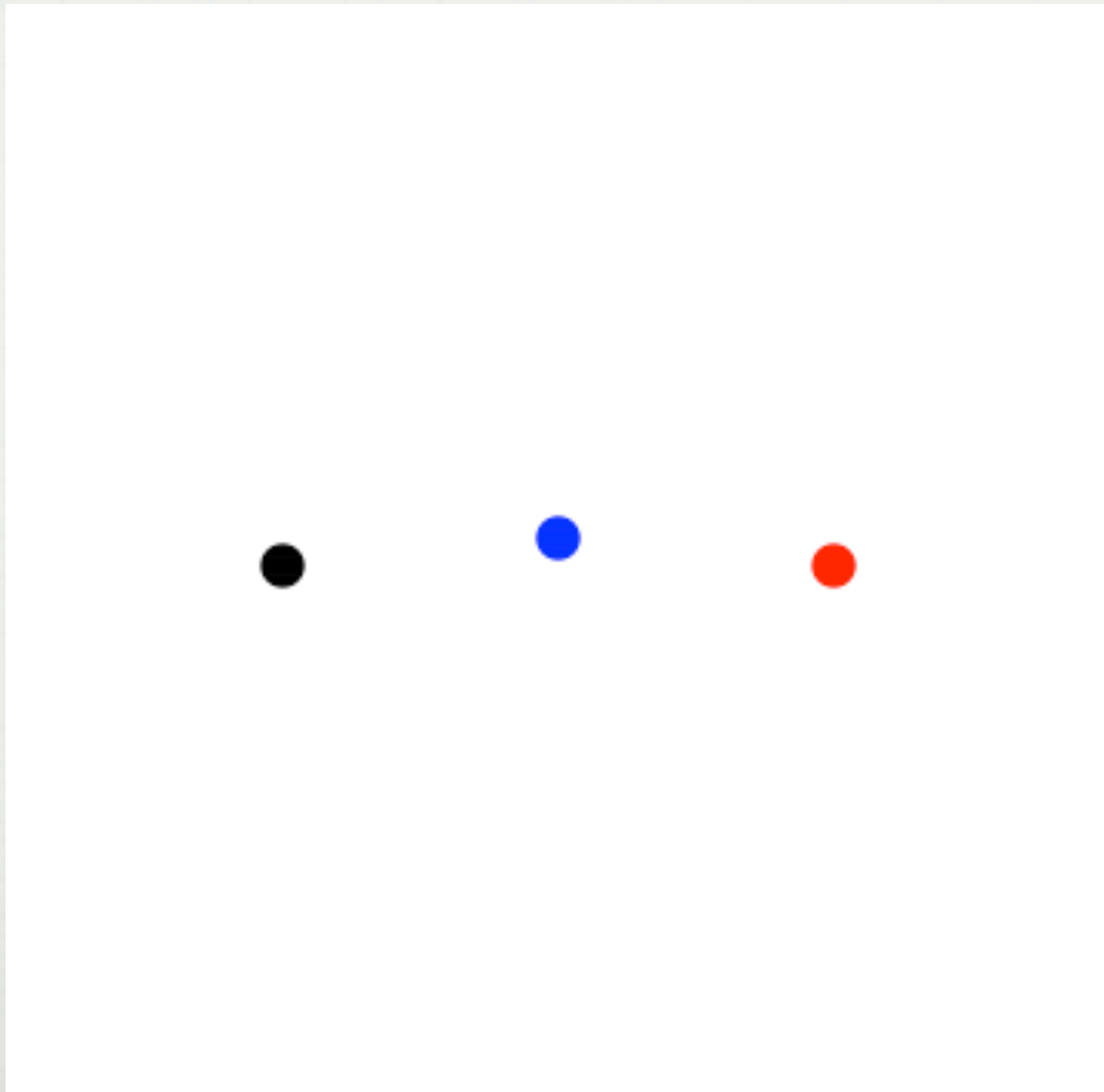


Isosceles periodic orbit with binary collisions and close approach to triple collision

Zoom showing behavior near triple collision



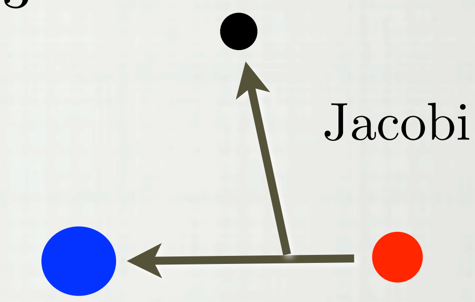
An orbit with many near collisions



Reduction by Translations

Usual method:

Set $p_{tot} = p_1 + p_2 + p_3 = 0 \in \mathbb{R}^{2*}$
Fix center of mass at origin: $m_1 q_1 + m_2 q_2 + m_3 q_3 = 0$.
Introduce coordinates on this invariant set.



Instead:

We start with

$$H(q,p) = K(p) - U(q) = \left(\frac{|p_1|^2}{2m_1} + \dots\right) - \left(\frac{m_1 m_2}{|q_1 - q_2|} + \dots\right)$$

and view $q_i \in \mathbb{C}, p_i \in \mathbb{C}^*$ so $(q,p) \in T^*\mathbb{C}^3$. Invariant under translations of positions, $q_i \mapsto q_i + c, c \in \mathbb{C}$.

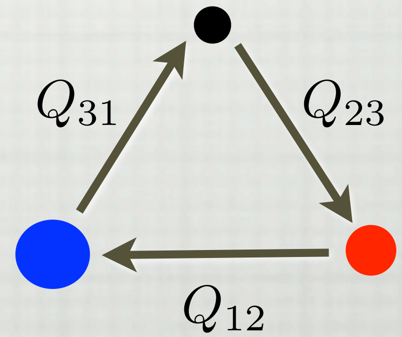
Define *relative coordinates*

$$Q_{12} = q_1 - q_2 \qquad Q_{31} = q_3 - q_1 \qquad Q_{23} = q_2 - q_3.$$

Note: linear map $L : \mathbb{C}^3 \to \mathbb{C}^3$, $Q = L(q)$ is not invertible.

Image of L is

$$\mathcal{W} = \{Q \in \mathbb{C}^3 : Q_{12} + Q_{31} + Q_{23} = 0\}.$$



Nevertheless, pull-back the momenta using the dual map L^*

$$p_1 = P_{12} - P_{31} \quad p_2 = P_{23} - P_{12} \quad p_3 = P_{31} - P_{23}$$

to get a new Hamiltonian on (Q, P)-space

$$\begin{aligned} H_{rel} &= K(P) - U(Q) \\ &= \frac{|P_{12} - P_{31}|^2}{2m_1} + \frac{|P_{23} - P_{12}|^2}{2m_2} + \frac{|P_{31} - P_{23}|^2}{2m_3} - \frac{m_1 m_2}{|Q_{12}|} - \frac{m_3 m_1}{|Q_{31}|} - \frac{m_2 m_3}{|Q_{23}|}. \end{aligned}$$

Strange duality: the new Hamiltonian is invariant under translations of *momenta*, $P_{ij} \mapsto P_{ij} + c, c \in \mathbb{C}^*$ and the momentum map for this symplectic group action is

$$Q_{tot} = Q_{12} + Q_{31} + Q_{23}.$$

In spite of the non-invertibility, it turns out that this new Hamiltonian represents the three-body problem in the following sense: The reduced Hamiltonian flow induced by $H(q, p)$ on the reduced space

$$\{(q, p) : p_{tot} = 0\}/\mathbb{C}$$

is equivalent to the reduced Hamiltonian flow induced by $H_{rel}(Q, P)$ on

$$\{(Q, P) : Q_{tot} = 0\}/\mathbb{C}^* \simeq T^*\mathcal{W}.$$

Reducing to 4 Degrees of Freedom

Now we have a different Hamiltonian system with Hamiltonian $H_{rel}(Q, P)$ on the (Q, P) space, which is still the twelve-dimensional space $T^*\mathbb{C}^3$. To get the reduction in dimension we need to restrict Q to the (complex) two-dimensional subspace

$$\mathcal{W} = \{Q : Q_{tot} = Q_{12} + Q_{31} + Q_{23} = 0\}$$

and quotient by momentum translation symmetry $P_{ij} \mapsto P_{ij} + c$. The result will be a Hamiltonian system with 4 degrees of freedom. Reduced phase space is the (real) eight-dimensional $T^*\mathcal{W}$.

Parametrizing $\mathcal{W}$:

Choosing any complex basis for the subspace $\mathcal{W}$ gives a diffeomorphism

$$T^*\mathcal{W} \simeq T^*\mathbb{C}^2$$

which carries out the reduction.

However, working with the variables Q_{ij} avoids arbitrary decisions about coordinates and makes regularizing binary collisions easier later on.

Homogeneous Spherical Coordinates

Replace $Q \in \mathcal{W} \subset \mathbb{C}^3$ by

$$r = |Q| \qquad \text{Size of triangle}$$

$$X = (X_{12}, X_{31}, X_{23}) \in \mathbf{S}(\mathcal{W}) \simeq \mathbf{S}^3 \subset \mathbf{S}^5 \qquad \text{Normalized configuration}$$

Hermitian mass metric on $\mathbb{C}^3$:

$$\langle V, W \rangle = \frac{1}{m}(m_1 m_2 \bar{V}_{12} W_{12} + m_3 m_1 \bar{V}_{31} W_{31} + m_2 m_3 \bar{V}_{23} W_{23})$$

$$r^2 = \langle Q, Q \rangle = \frac{1}{m}(m_1 m_2 r_{12}^2 + m_3 m_1 r_{31}^2 + m_2 m_3 r_{23}^2)$$

where $m = m_1 + m_2 + m_3$.

Note: $r = 0$ corresponds to triple collision.

View spheres as quotients under scaling, so

$$\mathbf{S}^5 = (\mathbb{C}^3 \setminus 0)/\mathbb{R}^+$$

$X \in \mathbb{C}^3 \setminus 0$ is a **homogeneous spherical coordinate** for a point in the sphere where

$$X' \sim X \text{ iff } X' = kX \qquad k > 0$$

Cotangent bundle $T^*\mathbf{S}^5$ can be viewed as symplectic reduction of action of $G = \mathbb{R}^+$ on $T^*(\mathbb{C}^3 \setminus 0)$

$$k \cdot (X, Y) = (kX, Y/k) \qquad k > 0.$$

The momentum map turns out to be

$$\mathrm{re}(\bar{Y}_{12} X_{12} + \ldots) = 0$$

and the quotient of the zero-momentum level is $T^*\mathbf{S}^5$.

In the end we get a Hamiltonian system on $T^*\mathbb{R}^+ \times T^*(\mathbb{C}^3 \setminus 0)$ with

$$H_{sph}(r, p_r, X, Y) = \frac{1}{2}p_r^2 + \frac{|X|^2}{r^2}K(Y) - \frac{1}{r}V(X)$$

where

$$K(Y) = \frac{|Y_{12} - Y_{31}|^2}{2m_1} + \ldots$$

$$V(X) = |X|U(X) = |X|\left(\frac{m_1 m_2}{\rho_{12}} + \ldots\right)$$

$$\rho_{ij} = |X_{ij}| \qquad \text{homogeneous mutual "distances"}$$

At this point we have actually **increased** the dimension to 14 ! But after remembering the constraint above, restricting X to $\mathcal{W}$ and quotienting by momentum translation and scaling symmetries we get back to an eight-dimensional reduced phase space

$$T^*\mathbb{R}^+ \times T^*\mathbf{S}(\mathcal{W}) \simeq T^*\mathbb{R}^+ \times T^*\mathbf{S}^3.$$

Reduction by Rotations -- Shape Sphere as CP(1)

Rotation group $SO(2)$ acts on normalized configurations $X \in \mathbb{C}^3$

$$e^{i\theta} \cdot X = (e^{i\theta} X_{12}, e^{i\theta} X_{31}, e^{i\theta} X_{23}).$$

X is already a homogeneous coordinate and the combined action of scaling and rotation amounts to an action of $G = \mathbb{C} \setminus 0$

$$k \cdot X = (kX_{12}, kX_{31}, kX_{23}) \qquad k \in \mathbb{C} \setminus 0.$$

The quotient space is $\mathbb{CP}(2)$. For $X \in \mathcal{W}$ we get an equivalence class

$$[X] \in P(\mathcal{W}) \simeq \mathbb{CP}(1) \qquad \text{Shape of Triangle.}$$

We call $P(\mathcal{W}) \simeq \mathbf{S}^2$ the **shape sphere**.

Momentum map for the action of $\mathbb{C} \setminus 0$ on phase space is

$$\bar{Y}_{12} X_{12} + \bar{Y}_{31} X_{31} + \bar{Y}_{23} X_{23} = 0 + i\,\mu$$

where μ is the **angular momentum**.

Reduced System for Size and Shape

All manifold of constant angular momentum are diffeomorphic to one another. Use a **momentum shift map** to translate the zero-angular momentum level onto the μ-level

$$(X, Z) \mapsto (X, Y)$$

where

$$\bar{Z}_{12}X_{12} + \bar{Z}_{31}X_{31} + \bar{Z}_{23}X_{23} = 0 + 0\,i$$

and Y is a certain μ-dependent translate of Z.

Hamiltonian becomes

$$H_\mu(r, p_r, X, Z) = \frac{1}{2}(p_r^2 + \frac{\mu^2}{r^2}) + \frac{|X|^2}{r^2}K(Z) - \frac{1}{r}V([X])$$

with K, V as before.

Remembering all the constraints and quotienting by all the symmetries gives a reduced system on the **reduced phase space**

$$T^*\mathbb{R}^+ \times T^*P(\mathcal{W}) \simeq T^*\mathbb{R}^+ \times T^*\mathbb{CP}(1).$$

However, there is a μ-dependent symplectic structure arising from the pullback of the standard structure under the momentum shift map.

Parametrizing the Shape Sphere--3 d.o.f.

Working with the X_{ij} variables will be advantageous later, but one can get an explicit reduction to 3 degrees of freedom as follows.

Choose any complex basis $\{e_1, e_2\}$ for $\mathcal{W}$ to get

$$X = \xi_1 e_1 + \xi_2 e_2.$$

This map $\mathbb{C}^2 \mapsto \mathcal{W}$ induces a parametrization of the shape sphere:

$$\mathbb{CP}(1) \mapsto P(\mathcal{W}).$$

Then choose your favorite way to handle $[\xi_1, \xi_2] \in \mathbb{CP}(1)$.

Affine local coordinates to Riemann sphere:

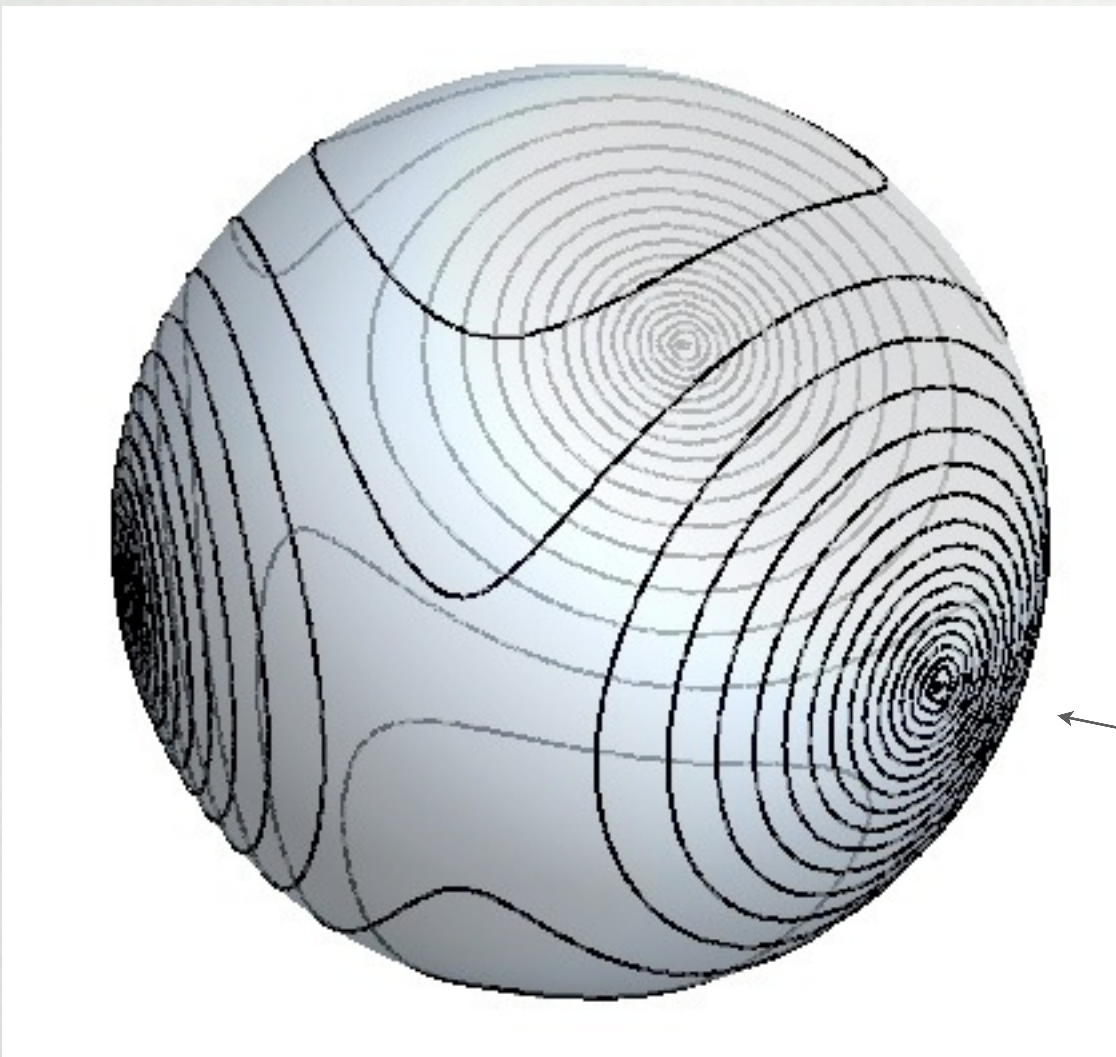
$$u = \frac{\xi_2}{\xi_1} \in \mathbb{C} \cup \infty.$$

Stereographic projection to round $\mathbf{S}^2$ in $\mathbb{R}^3$;

$$w = (w_1, w_2, w_3) = stereo(\xi_1, \xi_2).$$

Visualizing the Shape Sphere

Plot level curves of the shape potential V(X).
Round sphere model (with equal masses).



Poles: Lagrange's equilateral CC's

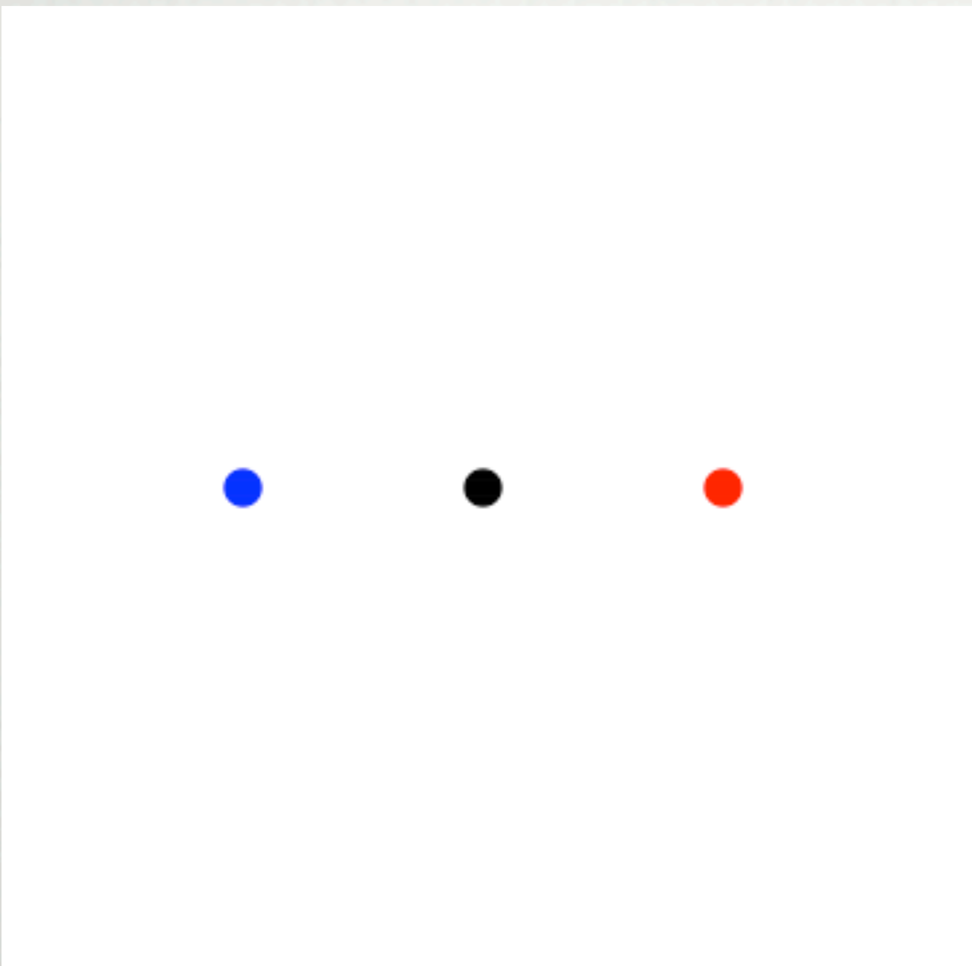
Equator: collinear shapes including Euler CC's and binary collisions

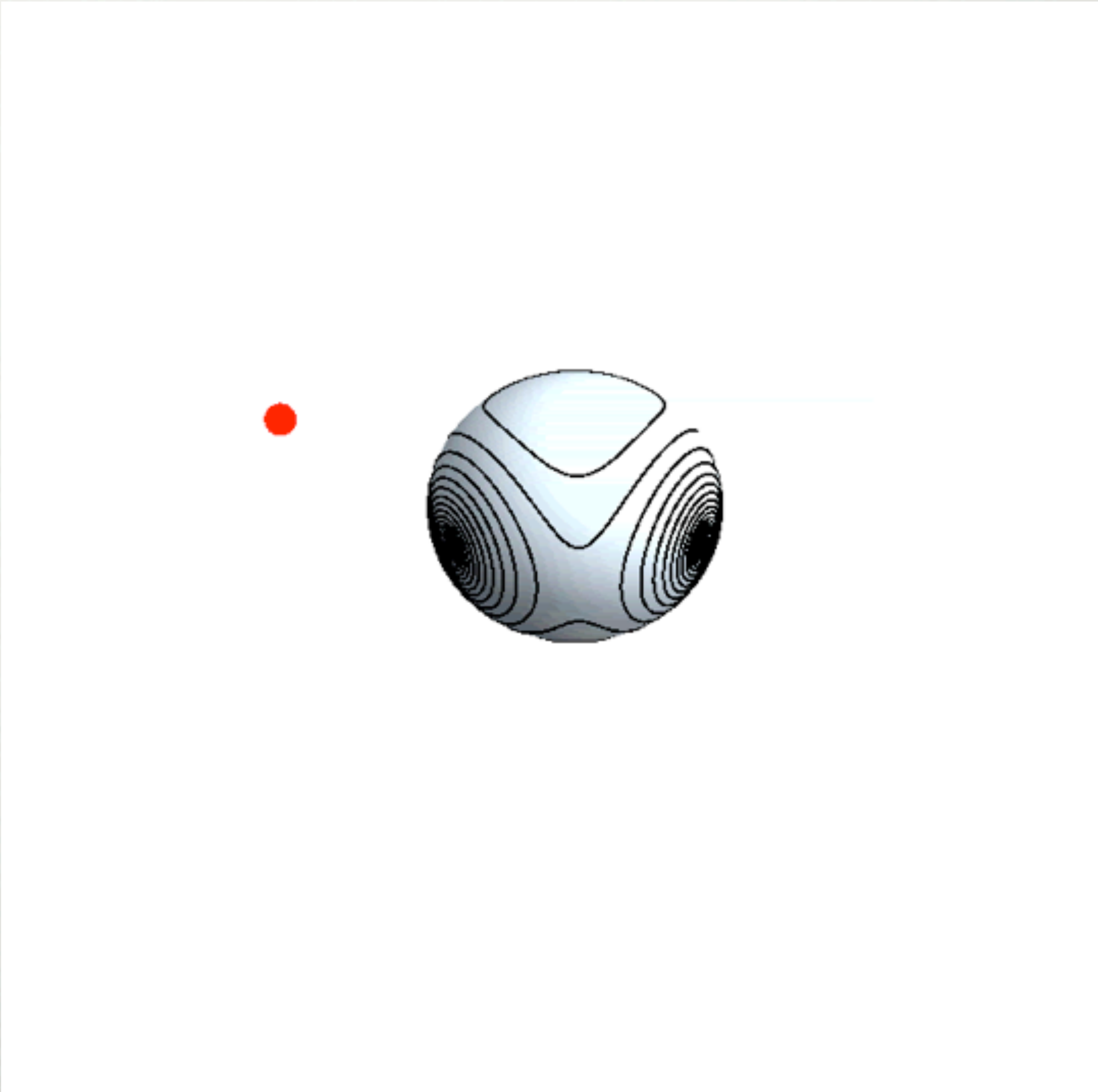
Binary collision

Reduced configuration space $\mathbb{R}^+ \times P(\mathcal{W}) \simeq \mathbb{R}^+ \times \mathbf{S}^2$ could be viewed as the solid region exterior of to this sphere.

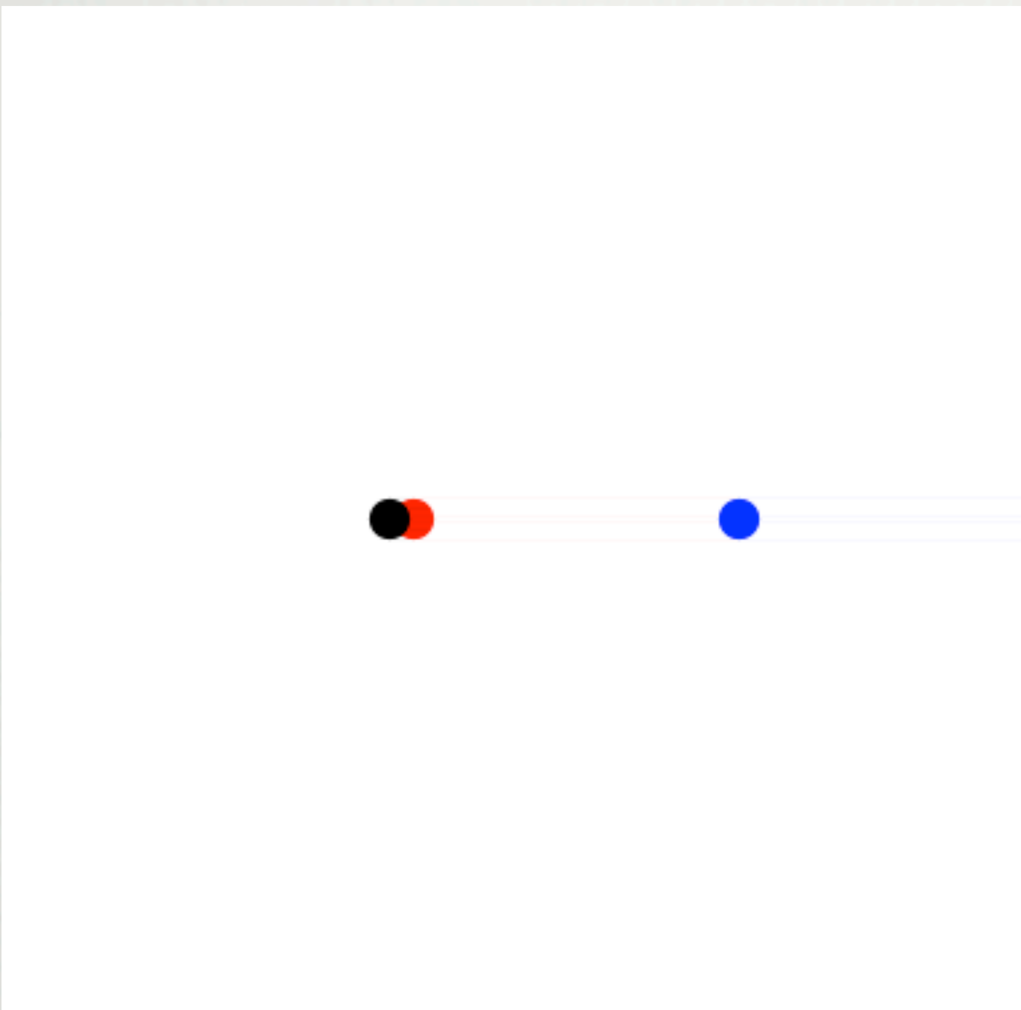
Some orbits plotted in reduced space

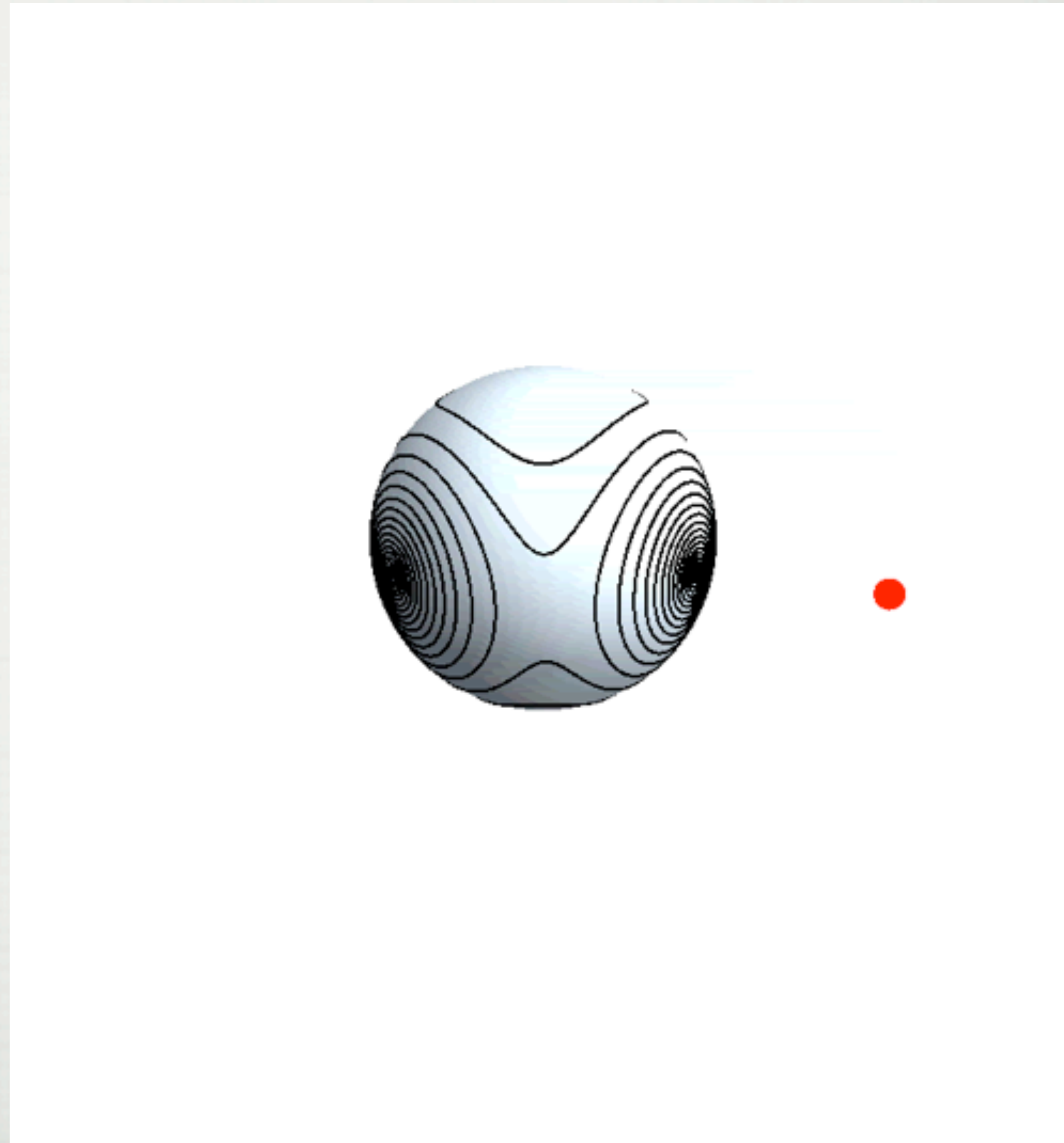
Figure eight orbit of Chenciner and Montgomery



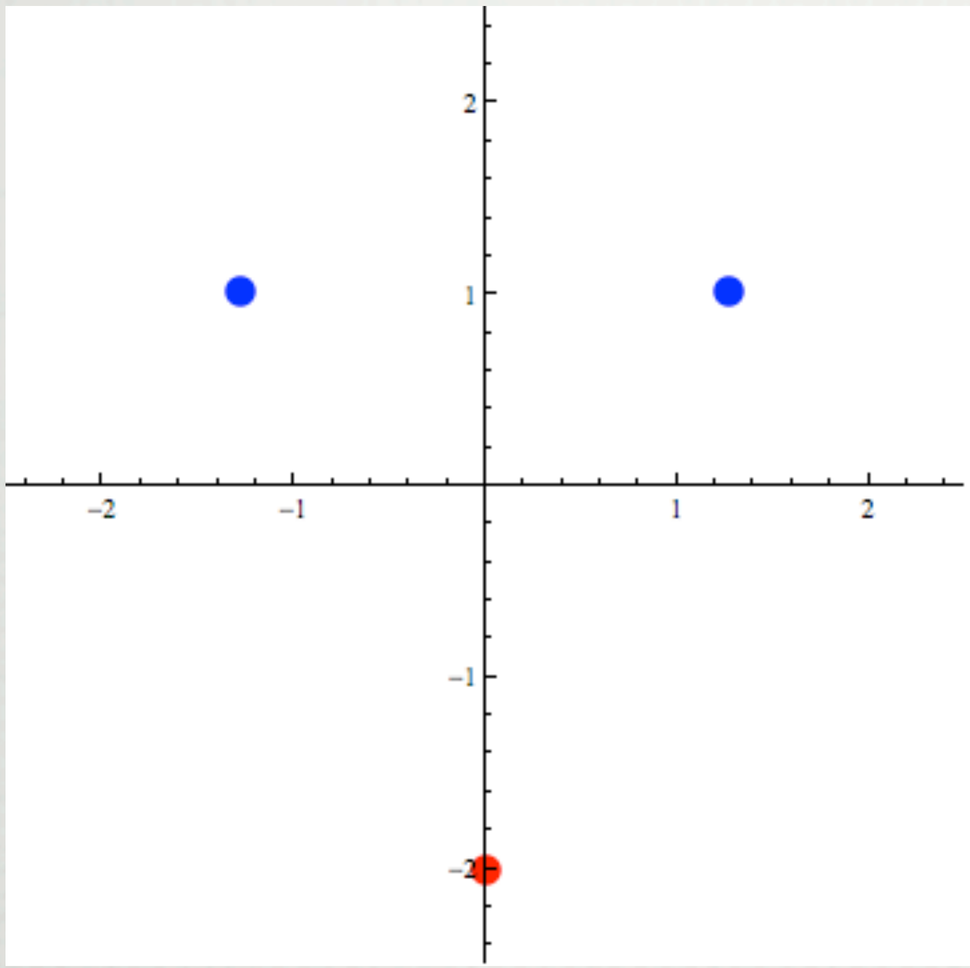


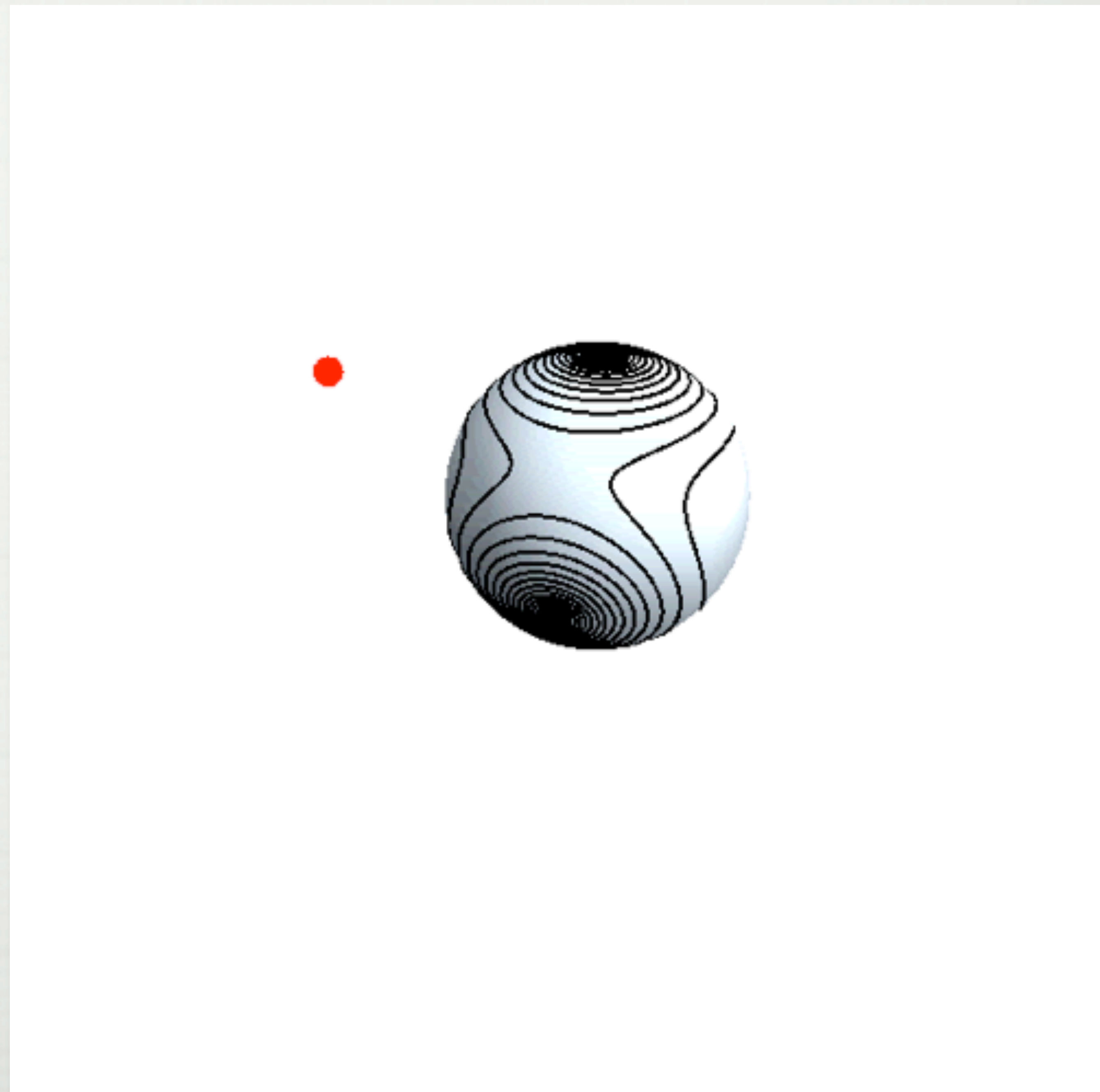
Broucke-Henon Orbit





Isosceles periodic brake orbit





Visualizing the Shape Sphere -- Affine Model, Equilateral Basis

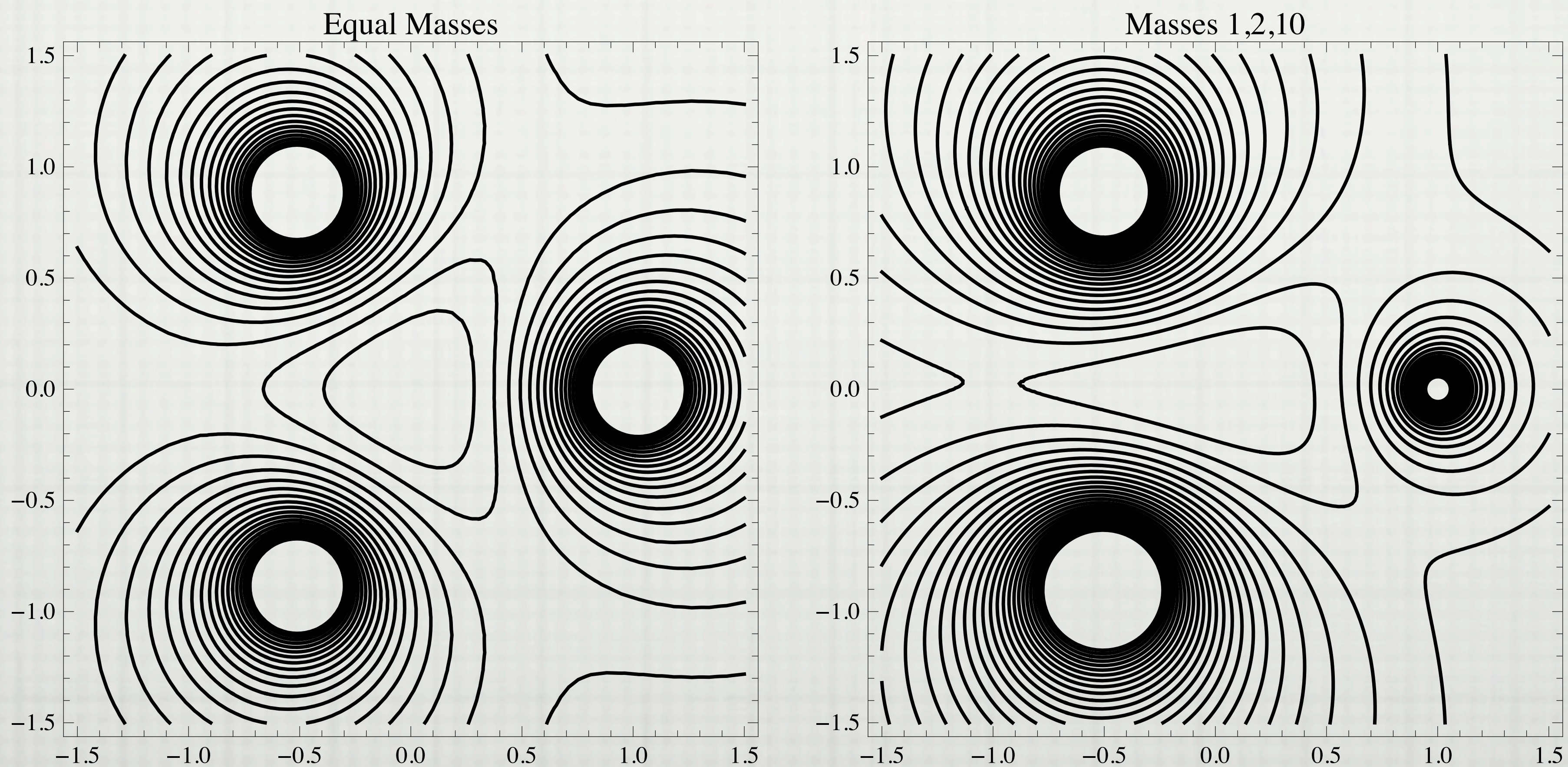
Choosing the basis for

$$e_1 = (1, \omega, \bar{\omega}) \qquad e_2 = (-1, -\bar{\omega}, -\omega) \qquad \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and using affine coordinates

$$u = \frac{\xi_2}{\xi_1}$$

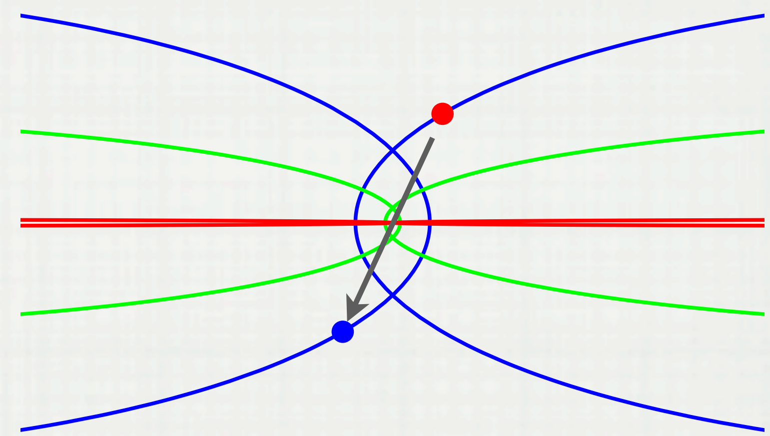
puts the collinear configurations on the unit circle, the binary collisions at the third roots of unity and the equilateral shapes at $u = 0, \infty$.

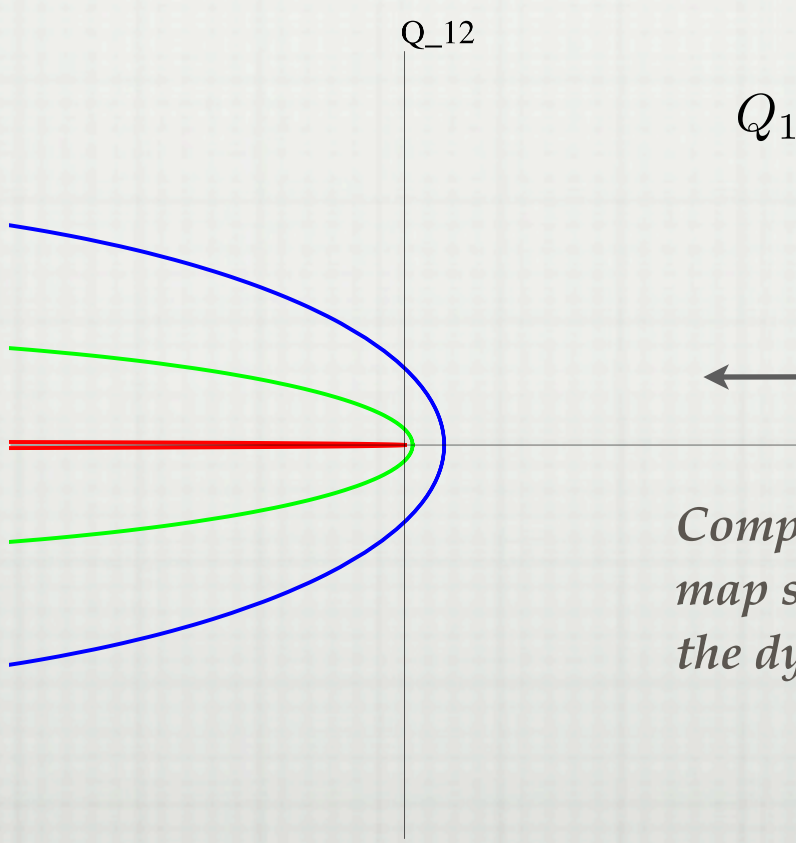


Regularization of Binary Collisions

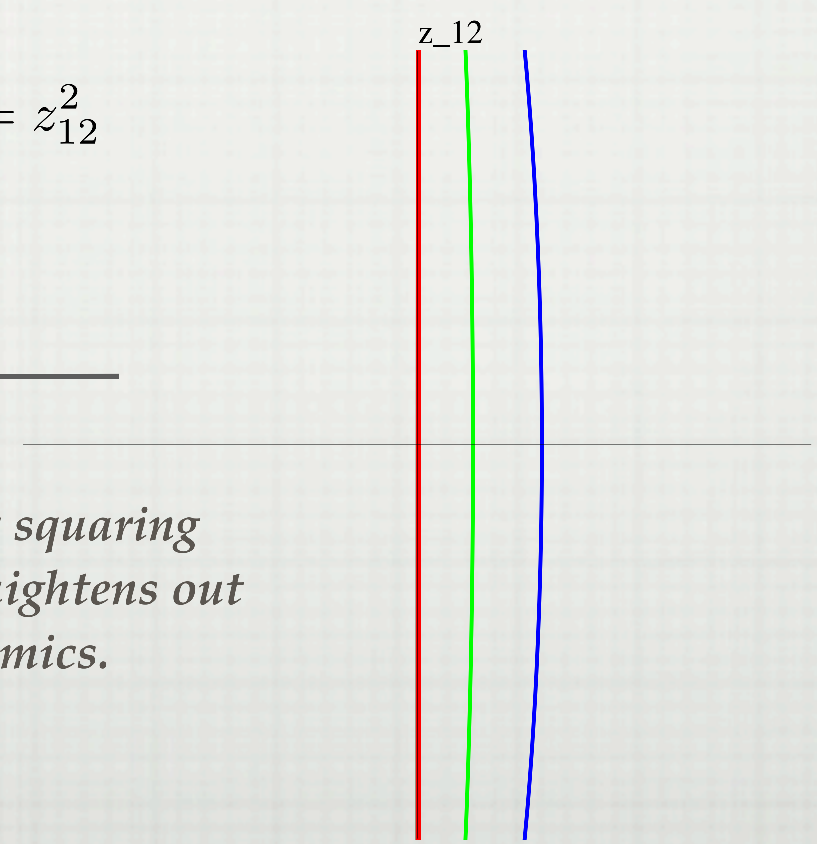
Levi-Civita regularization of a binary collision

Limiting behavior of near collision orbits is to have the masses bounce.





$$Q_{12} = z_{12}^2$$



Complex squaring map straightens out the dynamics.

Idea for regularizing all three binary collisions:

Regularize the *shape variables* for the planar three-body problem using a branched covering map of the shape sphere.

George Lemaitre: 50's and 60's via intricate trigonometric calculation.

We will pursue a simpler method based on the homogeneous variables X_{ij}

Three Levi-Civita maps $X_{ij} = z_{ij}^2$ where z_{ij} are three new complex variables.

Regularizing Map: $X = \phi(z)$ where $\phi : \mathbb{C}^3 \to \mathbb{C}^3$

$$\phi(z_{12}, z_{31}, z_{23}) = (z_{12}^2, z_{31}^2, z_{23}^2)$$

If

$$X \in \mathcal{W} : \qquad X_{12} + X_{31} + X_{23} = 0$$

then

$$z \in \mathcal{C} : \qquad z_{12}^2 + z_{31}^2 + z_{23}^2 = 0.$$

View ϕ as a map $\phi : \mathcal{C} \to \mathcal{W}$ of the complex algebraic surface $\mathcal{C}$ to the complex two-plane $\mathcal{W}$. Since everything is homogeneous we have induced maps of projective algebraic curves

$$\phi_{pr} : P(\mathcal{C}) \to P(\mathcal{W}) = \text{shape sphere}.$$

We will see that $P(\mathcal{C})$ is also a two-sphere (called the **regularized shape sphere**) and ϕ_{pr} is a branched covering map.

Geometry of $\mathcal{C}$ and $P(\mathcal{C})$

$$\mathcal{C} = \{z \in \mathbb{C}^3 : z_{12}^2 + z_{31}^2 + z_{23}^2 = 0\}$$

an algebraic surface (complex dimension 2 or real dimension 4). It turns out that topologically

$$\mathcal{C} \simeq \text{cone over } RP(3).$$

To see this note that if $z_{12} = a_{12} + i\, b_{12}$, etc., then

$$\begin{aligned} a_{12}^2 - b_{12}^2 + a_{31}^2 - b_{31}^2 + a_{23}^2 - b_{23}^2 &= 0 \\ 2(a_{12}b_{12} + a_{31}b_{31} + a_{23}b_{23}) &= 0 \end{aligned}$$

which shows that the vectors in $\mathbb{R}^3$

$$a = \operatorname{re} z = (a_{12}, a_{31}, a_{23}) \qquad b = \operatorname{im} z = (b_{12}, b_{31}, b_{23})$$

are orthogonal and have equal length:

$$|a|^2 = |b|^2 \qquad a \cdot b = 0.$$

After rescaling scaling, we get an orthonormal frame

$$[a, b, a \times b] \in SO(3) \simeq RP(3).$$

$$P(\mathcal{C}) = \{[z] \in \mathbb{CP}(2) : z_{12}^2 + z_{31}^2 + z_{23}^2 = 0\}$$

a projective curve (complex dimension 1 or real dimension 2). It is a non-degenerate conic and so, as is well-known,

$$P(\mathcal{C}) \simeq \mathbb{CP}(1) \simeq \mathbf{S}^2.$$

The map taking a conic to the projective line is essentially stereographic projection. A homogeneous version of this is given by the quadratic map

$$f : \mathbb{C}^2 \to \mathcal{C} \qquad z_{12} = 2ix_1x_2 \qquad z_{31} = x_1^2 + x_2^2 \qquad z_{23} = i(x_1^2 - x_2^2).$$

This is a two-to-one map. Normalizing the sizes gives an induced two-to-one map $f_{sph} : \mathbf{S}^3 \to \mathbf{RP}(3)$. Since f is homogeneous it also induces

$$f_{pr} : \mathbb{CP}(1) \to P(\mathcal{C}) \qquad [z] = f_{pr}([x_1, x_2])$$

which turns out to be a diffeomorphism.

Visualizing the Regularized Shape Sphere

The parametrization $f_{pr} : \mathbb{CP}(1) \to P(\mathcal{C}) = \{z_{12}^2 + z_{31}^2 + z_{23}^2 = 0\}$

$$z_{12} = 2ix_1x_2 \qquad z_{31} = x_1^2 + x_2^2 \qquad z_{23} = i(x_1^2 - x_2^2)$$

can be combined with

$$\text{Affine coordinates:} \qquad v = \frac{x_2}{x_1} \in \mathbb{C} \cup \infty$$

$$\text{Stereographic projection:} \qquad c = stereo(x_1, x_2) \in \mathbf{S}^2 \subset \mathbb{R}^3$$

to visualize $P(\mathcal{C})$ as a plane or round sphere.

Use six binary collision as landmarks. In homogeneous coordinates

$$0 = \rho_{12} = |X_{12}| = |z_{12}^2| = |2x_1x_2|^2 \implies [x_1, x_2] = [1, 0] \text{ or } [0, 1]$$

$$0 = \rho_{31} = |X_{31}| = |z_{31}^2| = |x_1^2 + x_2^2|^2 \implies [x_1, x_2] = [1, i] \text{ or } [0, -i]$$

$$0 = \rho_{23} = |X_{23}| = |z_{23}^2| = |x_1^2 - x_2^2|^2 \implies [x_1, x_2] = [1, 1] \text{ or } [0, -1]$$

Regularized Shape Sphere -- round version after stereographic projection

Coordinates $(c_1, c_2, c_3) \in \mathbb{R}^3$.
Can choose projection so

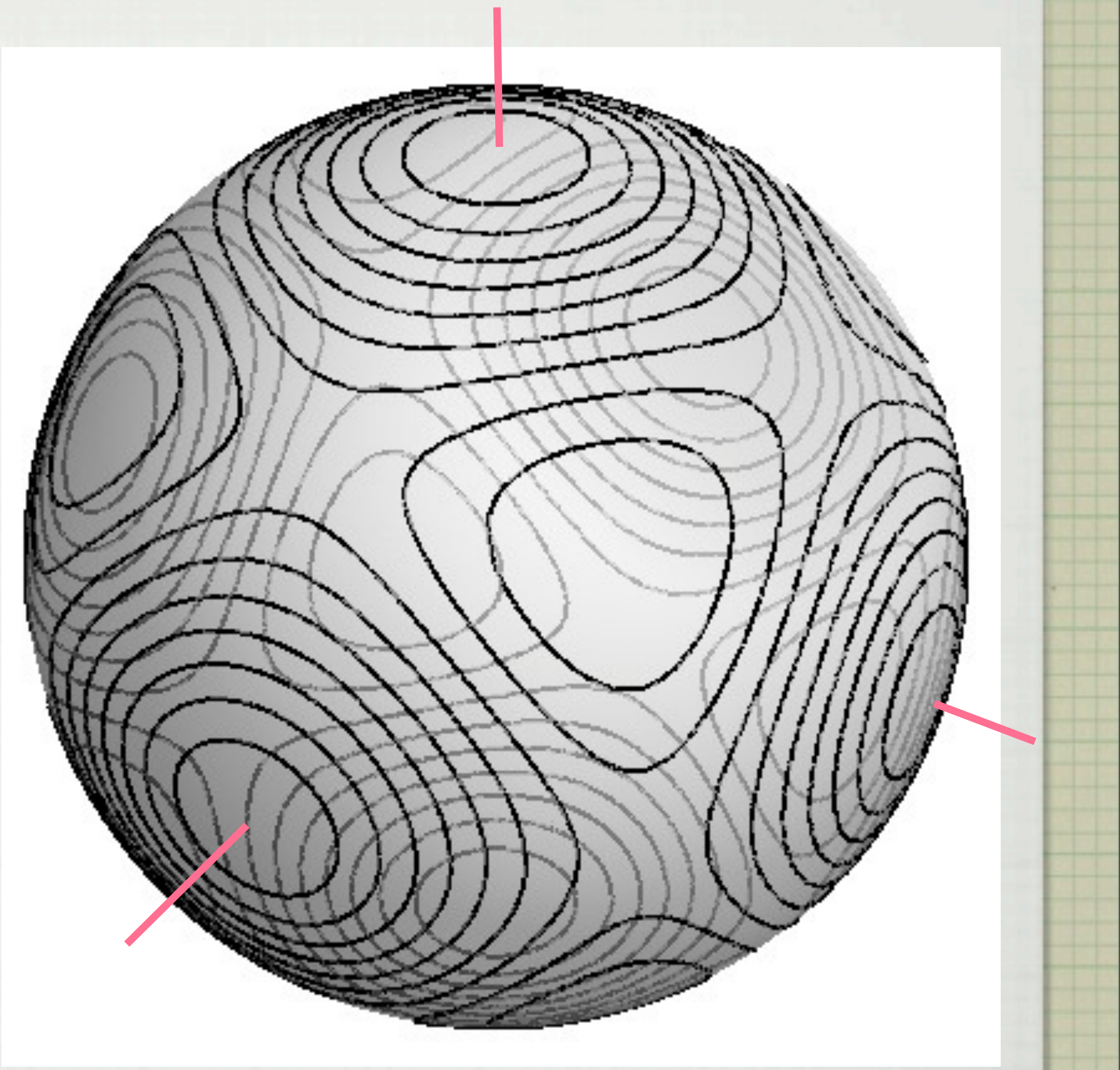
$$\rho_{12} = c_1^2 + c_2^2$$
$$\rho_{31} = c_3^2 + c_1^2$$
$$\rho_{23} = c_2^2 + c_3^2$$

Binary collisions are on coordinate axes.

$$\rho_{12} = 0 \implies c_1 = c_2 = 0.$$

Collinear shapes on coordinate planes.

$$\rho_{12} = \rho_{31} + \rho_{23} \implies c_3 = 0.$$



Octahedral symmetry -- imagine an octahedron inflated to become round.

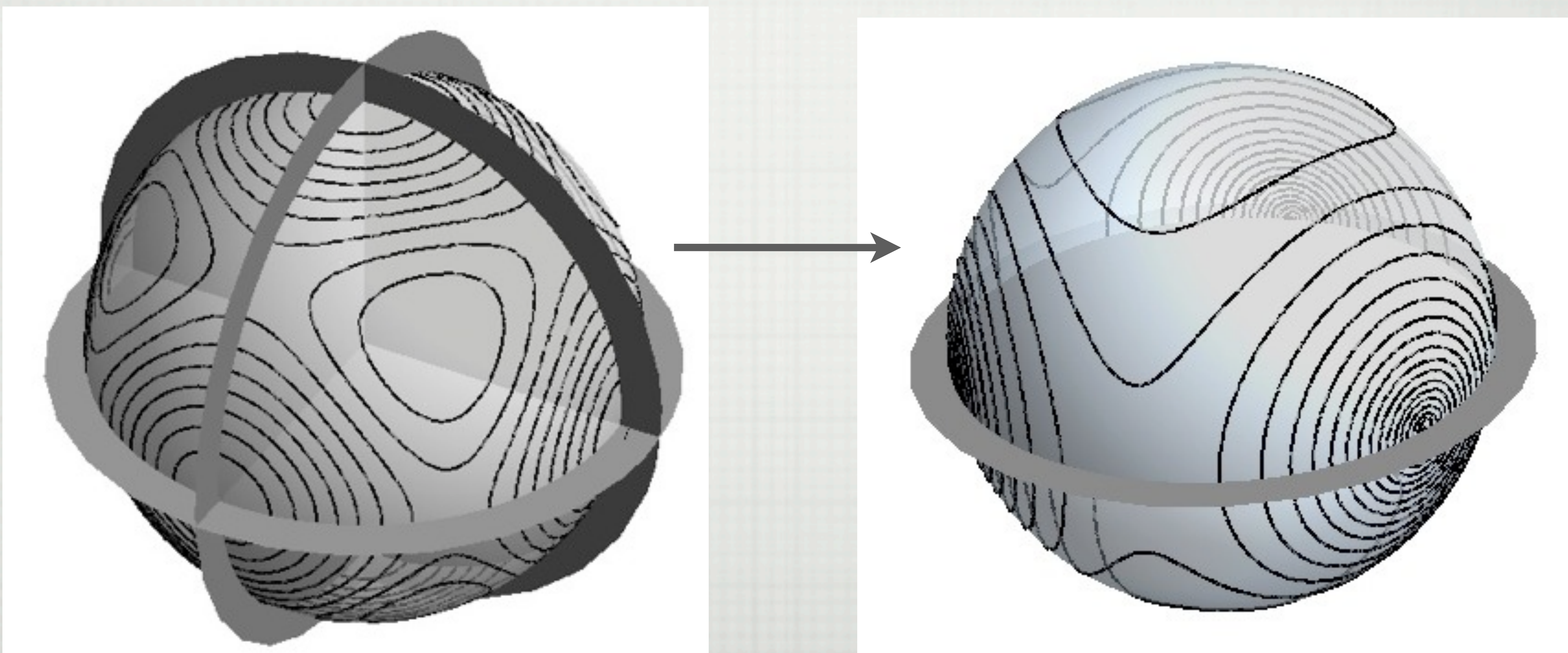
Lemaitre's Conformal Map: $\phi : \mathbb{C}^3 \to \mathbb{C}^3 \qquad X_{ij} = z_{ij}^2$

induces

$$\phi_{pr} : P(\mathcal{C}) \to P(\mathcal{W})$$

between regularized to unregularized shape spheres.

- **four-to-one cover branched over the binary collisions**
- **each octant of regularized sphere maps to a hemisphere**
- **behaves like the squaring map near the six regularized binary collision points**



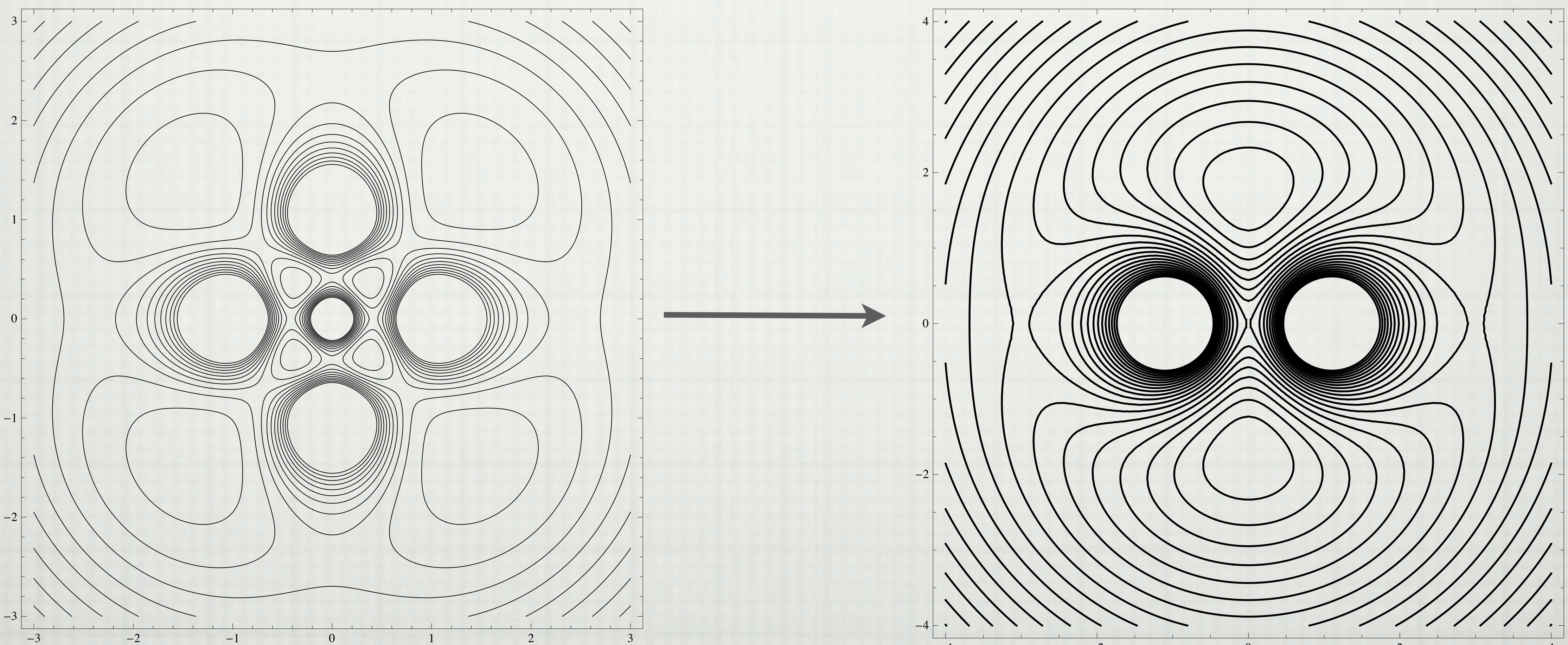
Affine version: Using affine local coordinates on both projective spaces

$$v = \frac{x_2}{x_1} \text{ on } P(\mathcal{C}) \qquad u = \frac{\xi_2}{\xi_1} \text{ on } P(\mathcal{W})$$

and sending one binary collision to infinity, the regularizing map takes the form

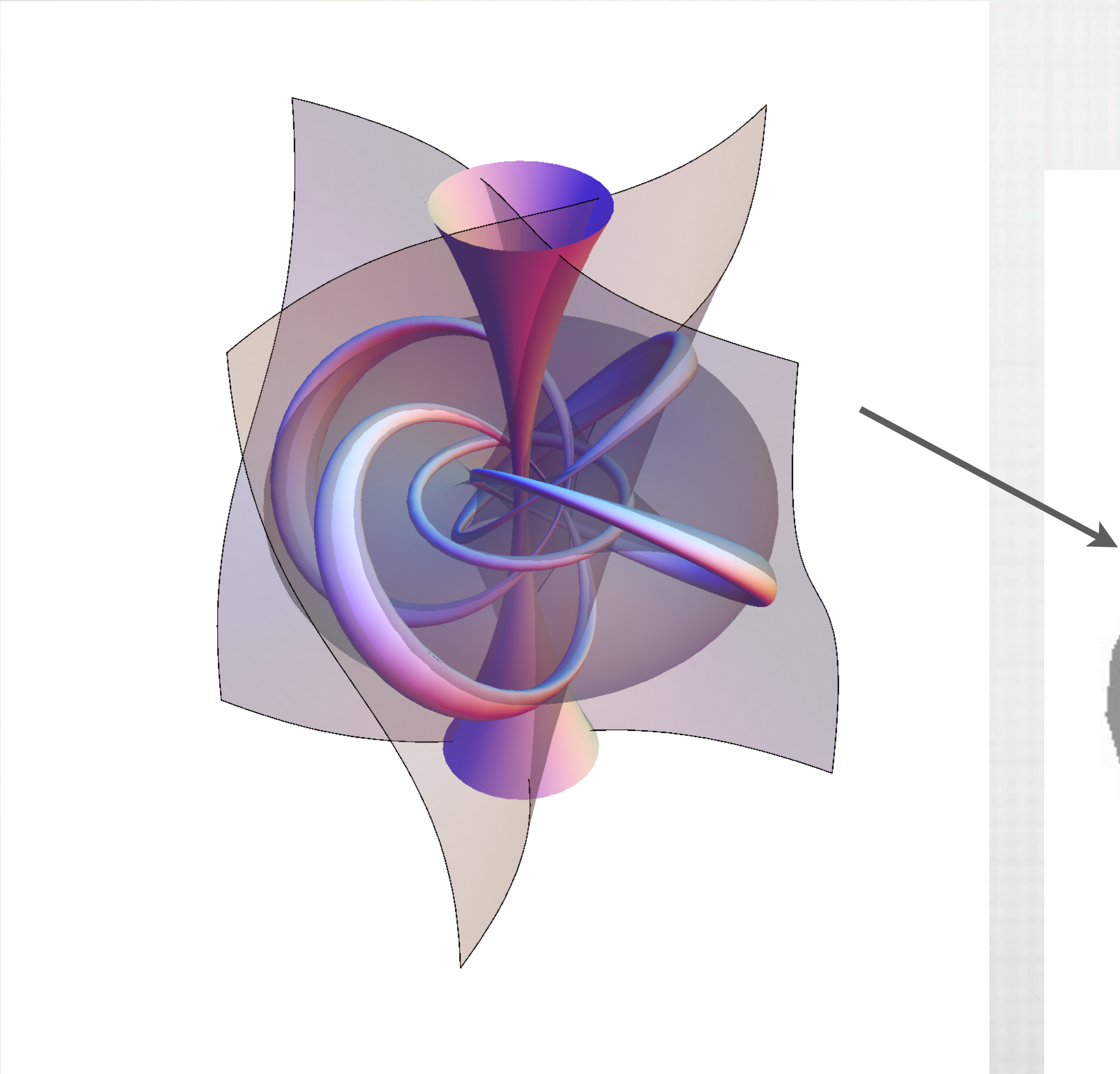
$$u = \phi_{pr}(v) = \frac{1}{2}(u^2 + u^{-2})$$

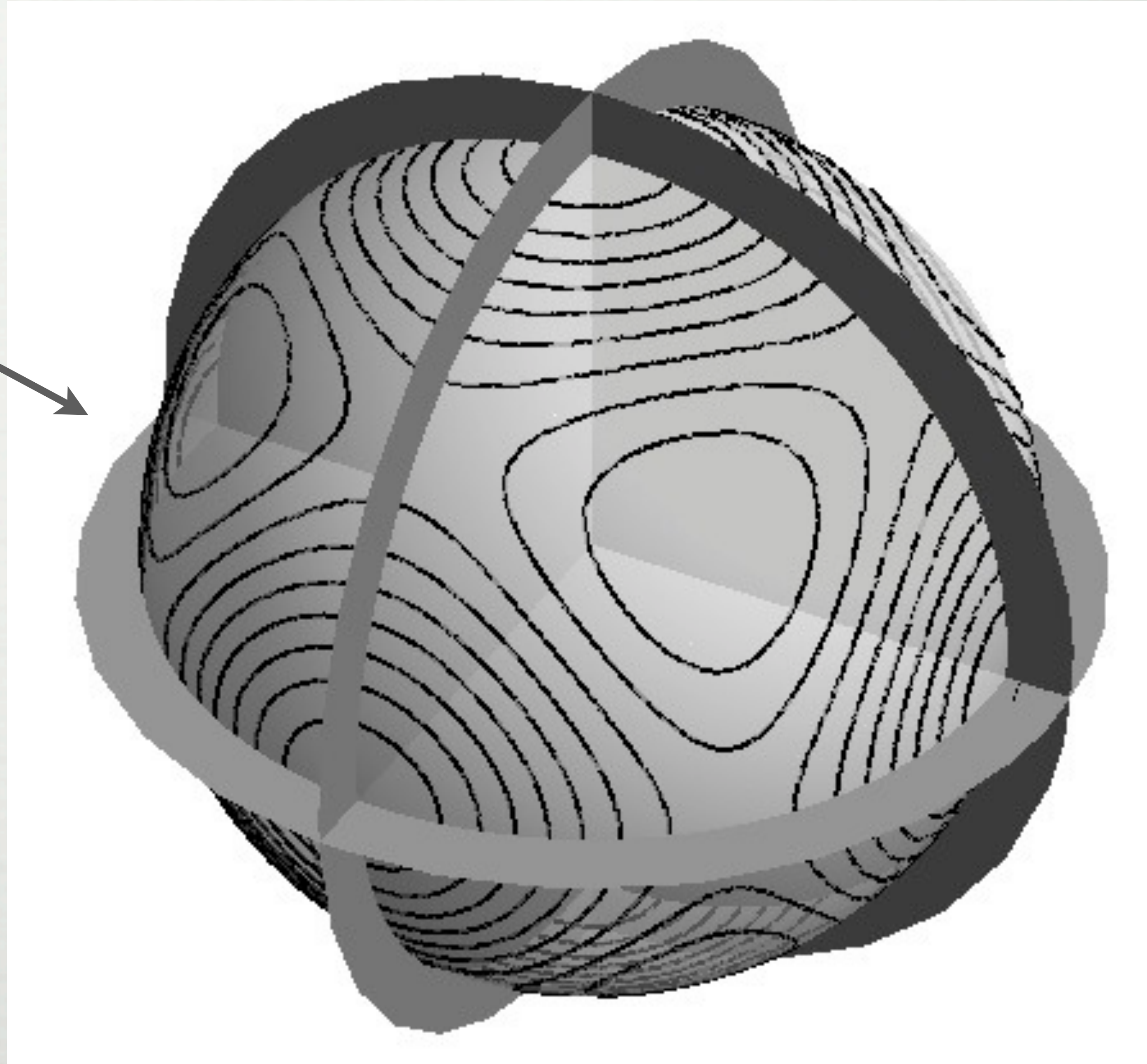
a degree-four rational map, reminiscent of the R3BP.



Regularized but unreduced ...

The preimage of the regularized shape sphere under the Hopf-like quotient map by SO(2). There are six binary collision circles (shown as tubes) and 3 collinear tori (transparent surfaces).





Regularized Hamiltonian

The symplectic extension of the Levi-Civita squaring map is

$$X_{ij} = z_{ij}^2 \qquad Y_{ij} = \frac{\eta_{ij}}{2\bar{z}_{ij}}$$

where η_{ij} are new momentum variables. Use these to transform the reduced Hamiltonian $H_\mu(r, p_r, X, Y)$.

Next rescale time using the Poincaré trick. Fix an energy level $H_\mu = h$ and set

$$\tilde{H}_\mu = \tau(H_\mu - h).$$

Solutions with $\tilde{H}_\mu = 0$ correspond to solutions with $H_\mu = h$ with time rescaled by the factor τ. We use

$$\tau = \frac{\rho_{12}\rho_{31}\rho_{23}}{(\rho_{12} + \rho_{31} + \rho_{23})^3} \qquad \rho_{ij} = |X_{ij}|$$

which vanishes at each of the binary collisions and is invariant under the scaling. The factors of ρ_{ij} in the numerator of τ will cancel out the corresponding factors in the denominator of the potential.

The form of the regularized Hamiltonian depends on the choice of coordinates. Here is one:

Using $x_1, x_2 \in \mathbb{C}$ where

$$z_{12} = 2i\, x_1 x_2 \qquad z_{31} = x_1^2 + x_2^2 \qquad z_{23} = i(x_1^2 - x_2^2)$$

and momenta y_1, y_2 gives

$$\tilde{H}_\mu = \frac{\tau\, p_r^2}{2} + \frac{\tau\, \mu^2}{2r^2} + \frac{\kappa}{4r^2}|y_1 x_2 - x_1 y_2|^2 - \frac{1}{r}W(x) - h\tau$$

Shape kinetic energy

where

$$W(x) = \frac{|X(x)|\,(m_1 m_2 \rho_{31}\rho_{23} + m_1 m_3 \rho_{12}\rho_{23} + m_2 m_3 \rho_{12}\rho_{31})}{(\rho_{12} + \rho_{31} + \rho_{23})^3}$$

$$\rho_{12} = |2x_1 x_2|^2 \quad \rho_{31} = |x_1^2 + x_2^2|^2 \quad \rho_{23} = |x_1^2 - x_2^2|^2$$

$$\tau = \frac{\rho_{12}\rho_{31}\rho_{23}}{(\rho_{12} + \rho_{31} + \rho_{23})^3} \qquad |X(x)|^2 = \frac{m_1 m_2 \rho_{12}^2 + m_1 m_3 \rho_{31}^2 + m_2 m_3 \rho_{23}^2}{m_1 + m_2 + m_3}$$

$$\kappa = \frac{m|X(x)|^4}{4m_1 m_2 m_3\,(\rho_{12} + \rho_{31} + \rho_{23})^4}$$

Regularization: The factors of ρ_{ij} in the denominator of the potential term are gone. Note that

$$\rho_{12} + \rho_{31} + \rho_{23} \neq 0$$

since the **homogeneous** mutual distances never vanish simultaneously.

The phase space is $T^*\mathbb{R}^+ \times T^*(\mathbb{C}^2 \setminus 0)$ but x, y are homogeneous coordinates and there is a regularized induced system on $T^*\mathbb{R}^+ \times T^*\mathbb{CP}(1)$. There is still a singularity at triple collision: $r = 0$.

Shape Kinetic Energy and the Fubini-Study Metric

The **shape kinetic energy** term

$$\frac{\kappa}{4r^2}|y_1x_2 - x_1y_2|^2$$

is conformal to the dual of the Fubini-Study metric on $\mathbb{CP}(1)$.

The standard Hermitian metric on $\mathbb{C}^3 = \{z = (z_{12}, z_{31}, z_{23})\}$ induces the **Fubini-Study metric** on $\mathbb{CP}(2)$ and hence also on the projective curve $P(\mathcal{C})$, the regularized shape sphere. There is a dual metric on the cotangent bundle $T^*P(\mathcal{C})$ which turn out to be given in homogeneous coordinates by:

$$\frac{1}{2}|y_1x_2 - x_1y_2|^2$$

Note: this is invariant under the complex scaling $(x, y) \mapsto (kx, y/\bar{k})$ defining the cotangent bundle.

After stereographic projection, the Fubini-Study metric is a constant times the usual round metric on $\mathbf{S}^2$.

Curvature Terms

The reduced phase space is diffeomorphic to

$$T^*\mathbb{R}^+ \times T^*S^2$$

but with a non-standard symplectic structure (due to the use of the momentum shift map). This adds certain **"curvature terms"** to the usual Hamilton's equations.

For example, using the reduced Hamiltonian $\tilde{H}_\mu(r, p_r, x, y)$ we get

$$\begin{aligned} \dot{r} &= (\tilde{H}_\mu)_{p_r} \\ \dot{p}_r &= -(\tilde{H}_\mu)_r \\ \dot{x} &= (\tilde{H}_\mu)_y \\ \dot{y} &= -(\tilde{H}_\mu)_x - \frac{2\mu\tau}{r^2}\, i\, y \end{aligned}$$

The curvature term is a computed from a **"curvature two-form"** on $\mathbb{CP}(1)$ which is a multiple of the imaginary part of the Fubini-Study Hermitian metric.

Blow-up of Triple Collision

The use of size and shape variables makes it easy to carry out McGehee's blow-up of triple collision. It just requires rescaling of the momentum variables and a further change of timescale to slow down the triple collision orbits.

McGehee used the timescale factor $r^{\frac{3}{2}}$ which gives the right behavior as $r \to 0$ but speeds up orbits near $r = \infty$. To get a complete flow we used

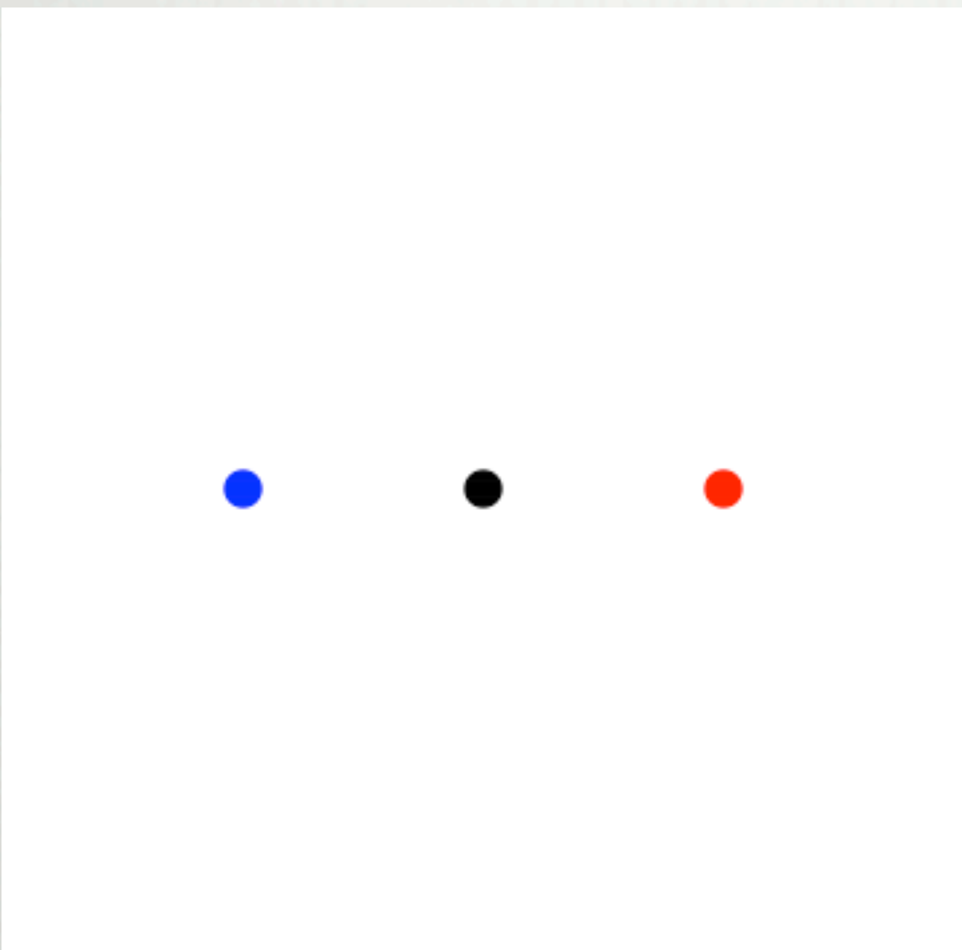
$$\left(\frac{r}{r+1}\right)^{\frac{3}{2}}.$$

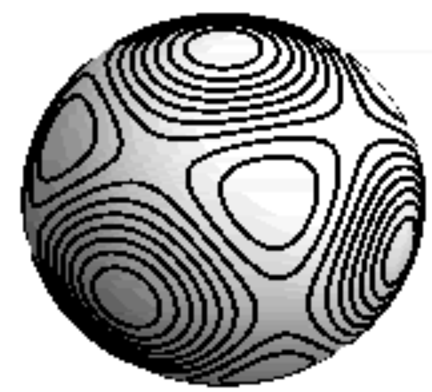
As in the usual blow-up, the resulting system of ODE's extends smoothly to $\{r = 0\}$ which becomes an invariant **collision manifold**. Triple collision orbits (which all have angular momentum $\mu = 0$), now converge to equilibrium points in the collision manifold.

Since collision singularities have been removed and since orbits cannot become unbounded in finite time, the resulting flow is complete.

Some Three-Body Orbits in the Regularized Configuration Space

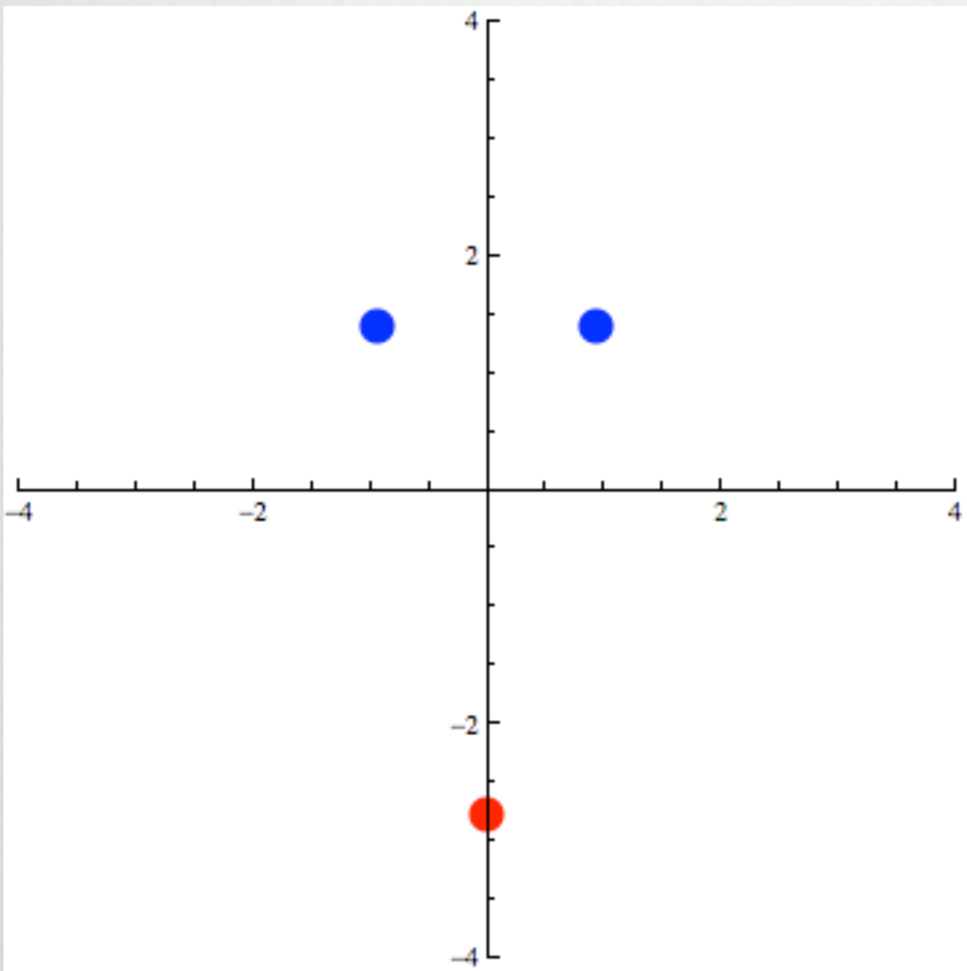
Figure-eight orbit

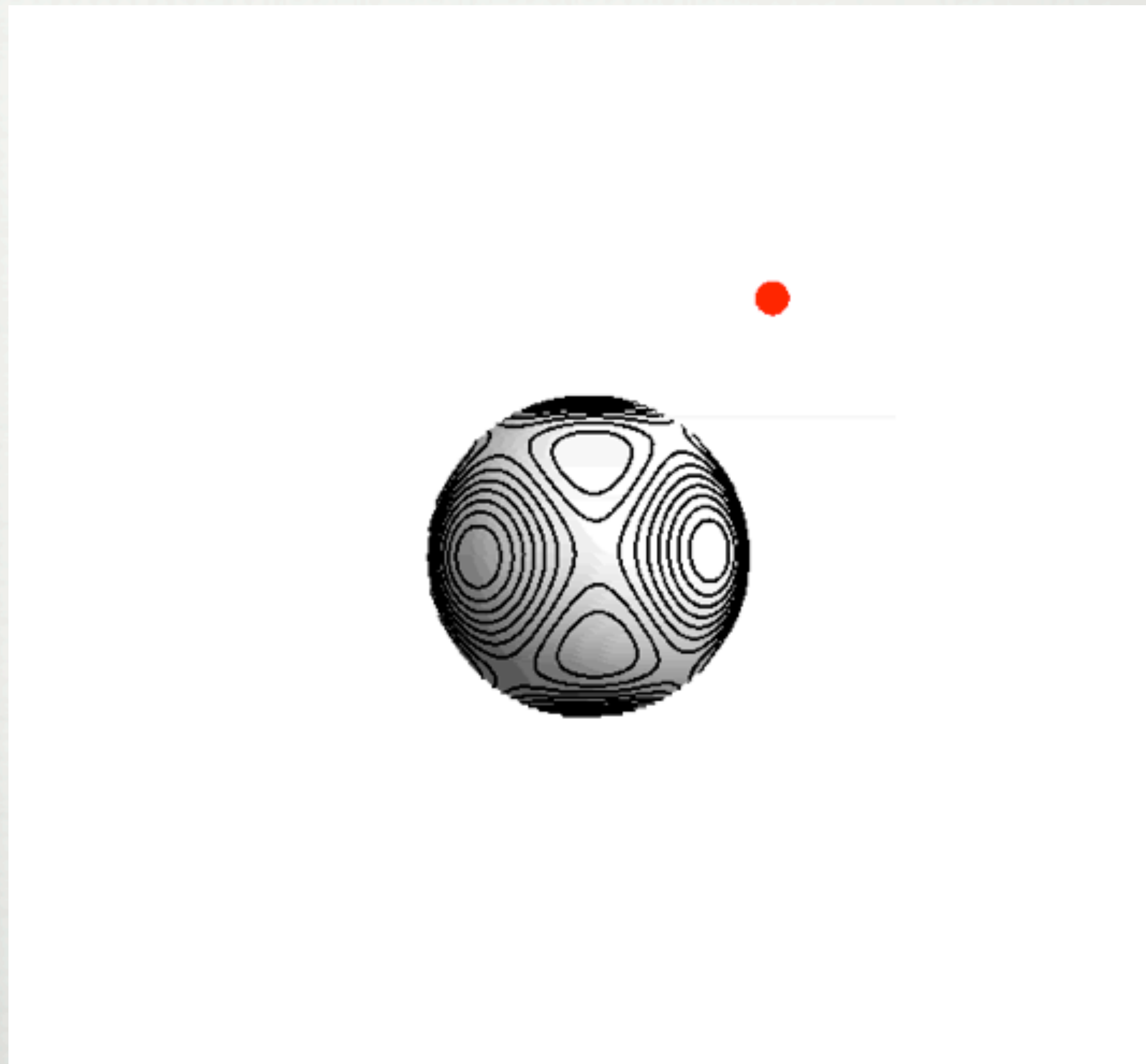




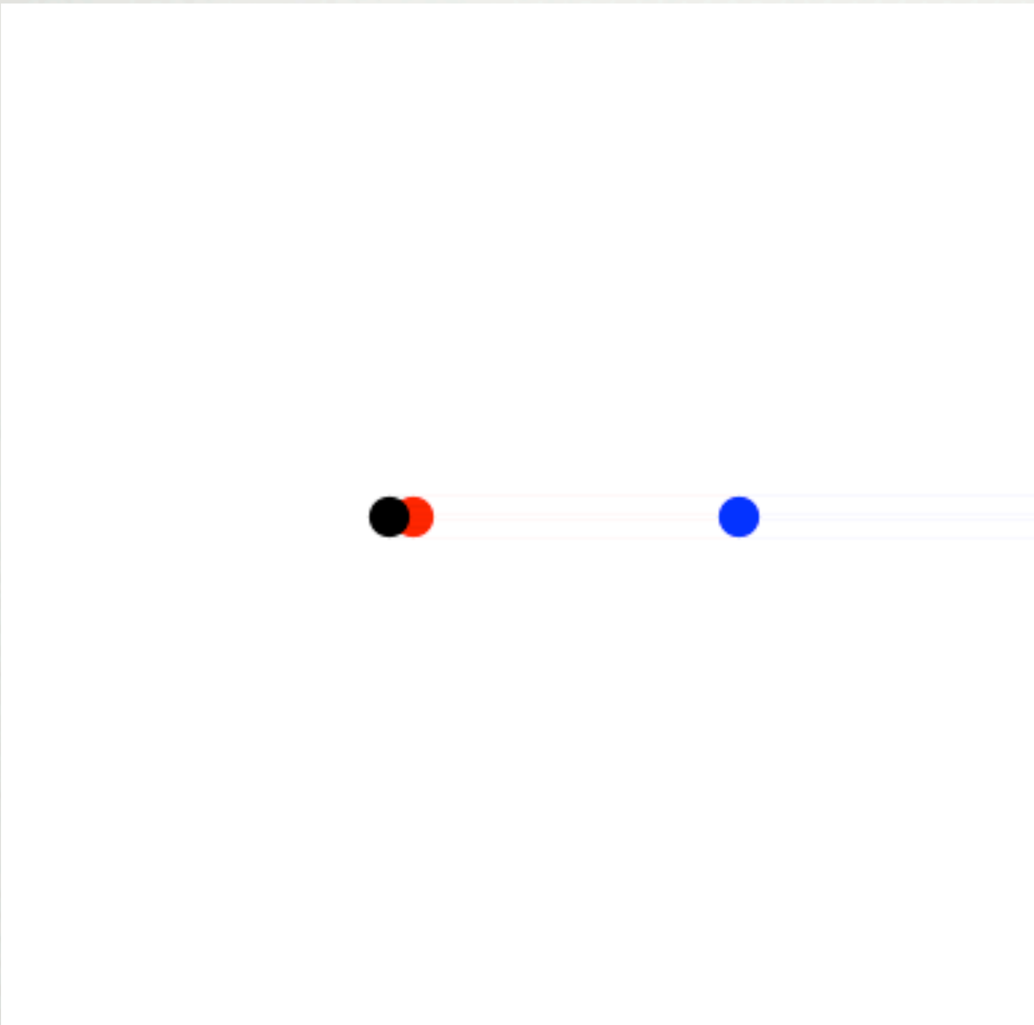
The orbit in regularized shape space is remarkably simple!

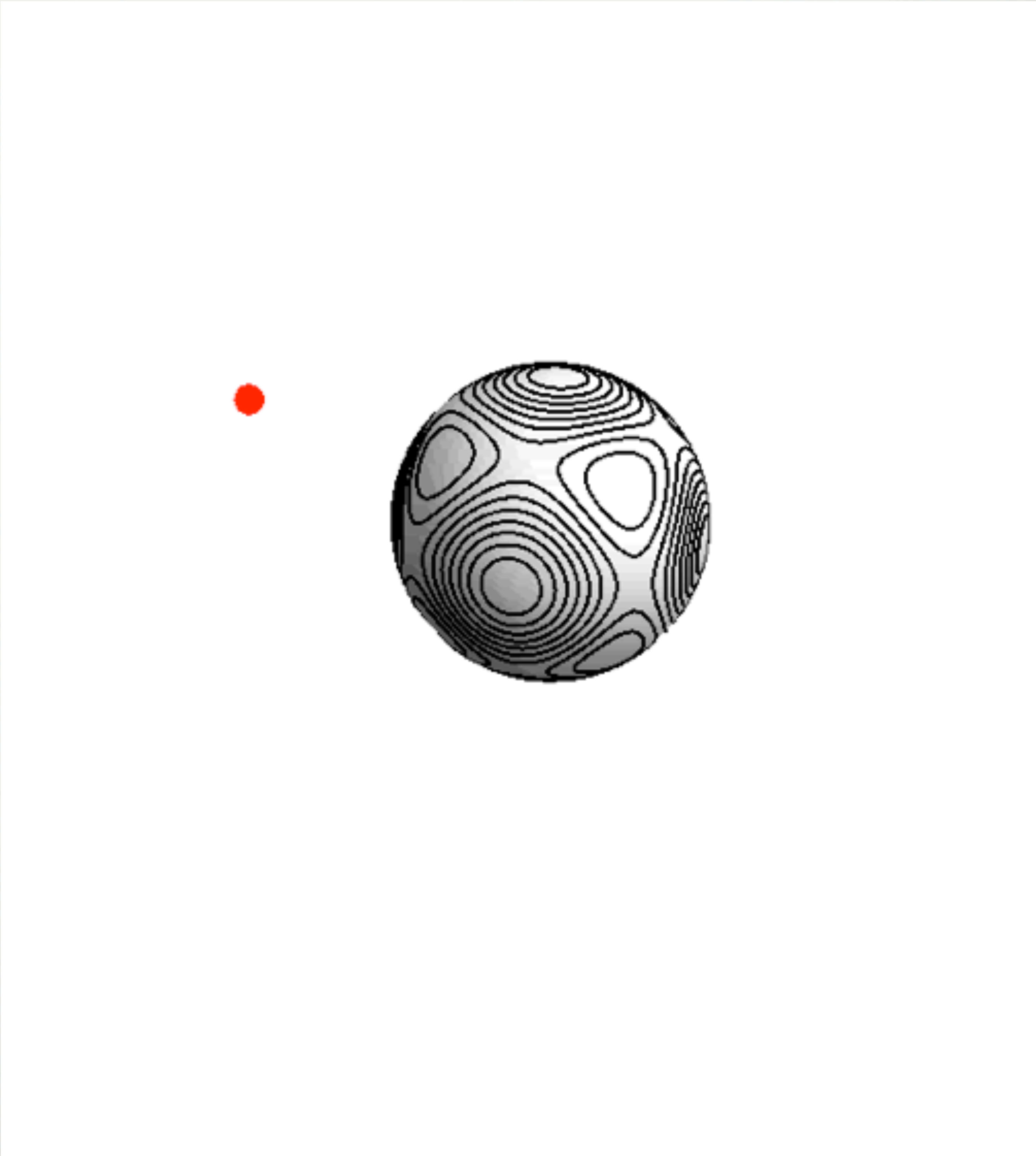
Isosceles orbit with a close approach to triple collision.





Broucke-Henon Orbit





Generalizations ....

Three-Body Problem in Space: Problems with reduction due to the non-free action of $SO(3)$. Get a shape disk (upper hemisphere of the shape sphere). It seems that there will be singularities of the reduced system on the boundary (collinear configurations).

Alternatively one can use Kustanheimo-Steifel regularization instead of Levi-Civita regularization:

$$X_{ij} = KS(z_{ij}) \qquad X_{ij} \in \mathbb{R}^3, z_{ij} \in \mathbb{R}^4.$$

This introduces additional T^3 symmetry. It still seems hard to carry out a complete reduction without introducing singularities on the collinear configurations.

Planar Four-Body Problem We have

$$X = (X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}) \in \mathbb{C}^6$$

subject to four linear constraints of the form $X_{ij} \pm X_{jk} \pm X_{ki} = 0$. Only three of the equations are independent and we get $X \in \mathcal{W}^3 \subset \mathbb{C}^6$. Projectively, we have $[X] \in P(\mathcal{W}) \simeq \mathbb{CP}(2)$. Each of the six binary collisions defines a projective line in this projective plane.

Setting $X_{ij} = z_{ij}^2$ gives quadratic equations $z_{ij}^2 \pm z_{jk}^2 \pm z_{ki}^2 = 0$ which gives a projective algebraic surface $P(\mathcal{C}) \subset \mathbb{CP}(5)$. The regularizing map

$$\phi_{pr} : P(\mathcal{C}) \to P(\mathcal{W}) \simeq \mathbb{CP}(2)$$

is branched over the six binary collision lines.

The surface $P(\mathcal{C})$ is singular over the points where three binary collision lines intersect. The "desingularization" of the surface is apparently a K3 surface.

Not clear how to work with such objects and obtain a useful reduced and regularized problem.

The result would still not be a complete flow since there are solutions tending to infinity in finite time, a result of Mather and McGehee.

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