Comments about Chapter 1 of the Math 5335 (Geometry I) text

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September 8, 2003; last revised September 9, 2004

Contents**:**

1. ***Chapter introduction.*** Our basic model for the Euclidean plane will be $\mathbb{R}^2$, the set of all ordered pairs of real numbers. Then we'll be able to use linear algebra to study the properties of lines in the plane. Actually, we'll use linear algebra for quite a more than that. Namely, we'll use the inner product (another name for the familiar dot product) and the closely related concept of the norm (or length) of a vector to study distance, angular measure, circles, and so forth.

   Of course, this was not how Euclid and other ancient Greeks studied geometry. And it's probably not how most of us learned plane geometry in high school. Indeed, things probably were the other way around: many of us used our knowledge of geometry as an aid to learning linear algebra concepts. But our approach does have a certain advantage from a purely logical viewpoint. Specifically, we just have to accept the existence of the real number system, and then almost everything in linear algebra follows in a completely logical way. Well yes, we do need some additional definitions, and some kind of understanding of why those definitions are significant. Once we have that body of knowledge, however, we have a coherent frame of reference for understanding geometry and how it relates to more advanced mathematics.

   So, let's just say this once more for emphasis. Our approach ***probably is not*** a recommended way for beginners to learn geometry. Instead it will provide us, now that we know some linear algebra, with a means to develop a higher level of understanding of some topics in geometry.

2. ***§1.1: A few basic properties of the inner product.***
   Here some properties that are very useful for calculations and also in proofs:
   (*i*) If X and Y are vectors, then $\langle X,Y\rangle = \langle Y,X\rangle$.
   (*iia*) If X, X' and Y are vectors, then $\langle X+X',Y\rangle = \langle X,Y\rangle + \langle X',Y\rangle$.
   (*iib*) If $X, Y,$ and Y' are vectors, then $\langle X,Y+Y'\rangle = \langle X,Y\rangle + \langle X,Y'\rangle$.
   (*iiia*) If X and Y are vectors and c is a real number, then $\langle cX,Y\rangle = c\langle X,Y\rangle$.
   (*iiib*) If X and Y are vectors and c is a real number, then $\langle X,cY\rangle = c\langle X,Y\rangle$.

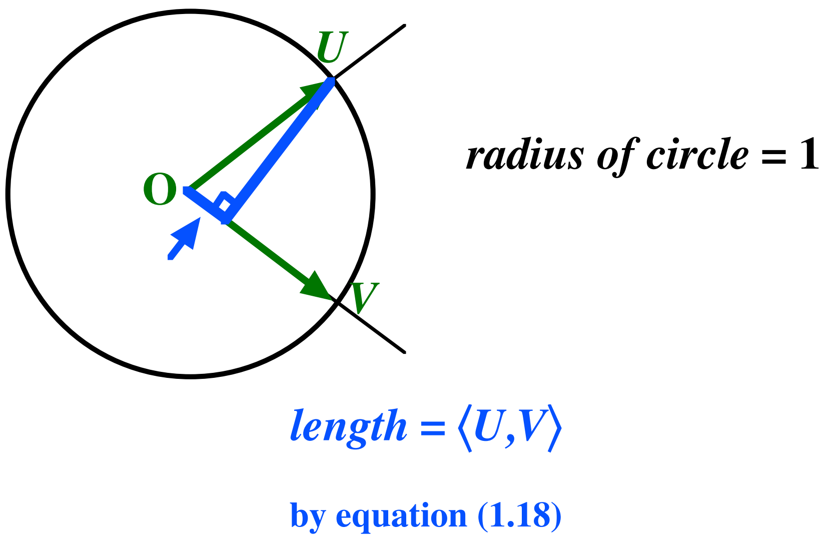
   In words: Property (*i*) says that the inner product is ***symmetric***; properties (*iia*) and (*iiia*) say that it is ***linear in the first variable***; while properties (*iib*) and (*iiib*) say that it also is ***linear in the second variable***.

   One uses these properties, together with the "norm identity" $||X|| = \langle X,X\rangle$ to prove **Lemma 1** on page 8 of the text. ***Suggestion:*** begin by writing $||X\pm Y|| = \langle X\pm Y,X\pm Y\rangle$, and then use the linearity and symmetry properties to expand and re-arrange the right side of the equation.

3. ***§1.4: Simple curves and their lengths.***
Examples 12 and 13 are very important since they're going to be the basis for our definition of angular measure in §1.5.

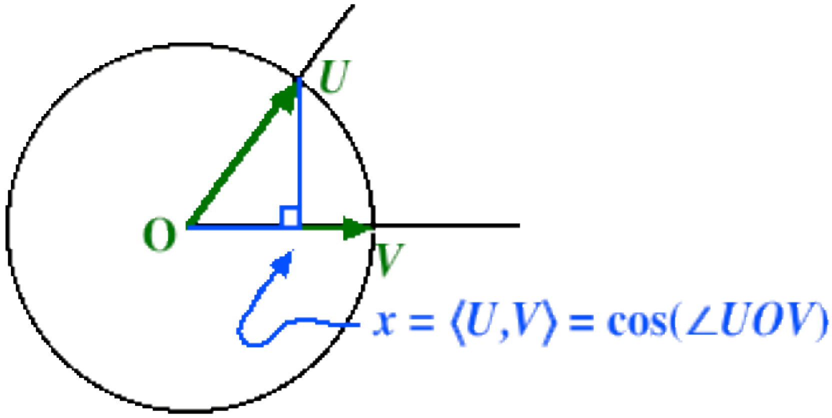
4. ***§§1.4 and 1.5: About the integral used in the definition of angular measure.*** If U and V are the unit direction indicators of the two rays that define the angle, then we have defined the angular measure to be $\int_{\langle U,V\rangle}^{1} \frac{ds}{\sqrt{1-s^2}}$. Now, the integrand $\frac{ds}{\sqrt{1-s^2}}$ doesn't seem too mysterious: it's what we encounter when we evaluate the length of an arc of a circle of radius = 1. But perhaps it's less clear why we would choose the particular lower bound that we chose for the definite integral, namely the inner product $\langle U,V\rangle$.

One way to explain this choice involves looking at Theorem 22 of chapter 1. Let's put the vertex of our angle at the origin, and use the Theorem to evaluate (*i*) the distance from the point U to the line $\overleftrightarrow{OV}$ and (*ii*) the point closest to U on that line. Well, actually, that closest point is the main item of interest, and it turns out to be $\langle U,V\rangle V$. Since the vector V is a unit vector, the segment $\overline{OV}$ has length = 1, and therefore the distance from O to the foot of the perpendicular is $= \langle U,V\rangle$, as shown in the figure.



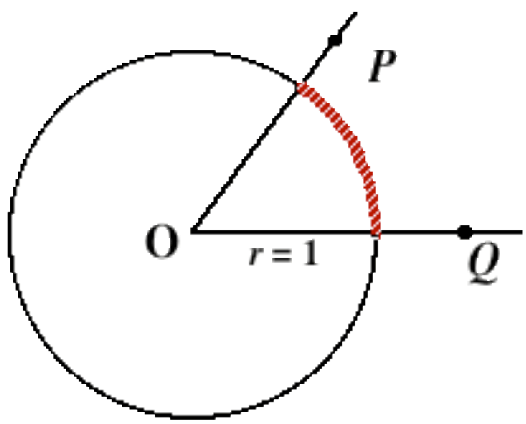
The length of this segment is what one ordinarily calls the ***cosine*** of the angle, namely the ratio (adjacent side)/(hypotenuse). {Note that the hypotenuse has length = 1, since we're working with the unit circle.} Now, in the special case where V is on the positive x-axis, ***i.e.,*** $V = (1,0)$, and the equation of the circle is $y = \sqrt{1-x^2}$, the arc-length calculation that we're doing is about finding the length of this curve between the endpoints $x = \langle \mathrm{U,V}\rangle = \cos(\angle UOV)$ and $x = 1$. In other words, it's just the length of the circular arc subtended by the angle.

(The figure below shows the relation between the inner product and the cosine.)



5. ***§1.5: A concrete interpretation of angular measure.*** To re-emphasize our "down-to-earth" interpretation of this definition, suppose that a circle of radius = 1 is drawn with its center at the vertex of the angle. The angular measure is then the length of the arc of this circle which is subtended by the angle, ***i.e.*** the length of the arc whose endpoints lie on the two sides of the angle. This turns out to coincide with the usual ***radian measure***.

   (In the figure below, the measure of $\angle POQ$ is the length of the circular arc that is covered by the red shading.)



6. **[optional]** ***§1.5: Angular measure: our definition compared with protractor measure.*** In the usual protractor, a circle is divided into 360 equal arcs -- or more literally, a half circle is divided into 180 equal arcs. If the arc subtended by an angle is covered by an integral number m of these arcs, then the protractor measure, or ***degree measure***, of the angle is = m. To extend this to include fractional degrees, we can pick a positive integer n (the denominator of the desired fraction) and divide the half circle into $180n$ equal parts. For instance, if $n = 7$, then we divide the circle into $180 \cdot 7 = 1260$ equal arcs. Each small arc should have degree measure = $1/n$. If the arc subtended by an angle is covered by an integral number m of these smaller arcs, then its degree measure should be = m/n.

   On the other hand, each of the small arcs has length = $\pi/180n$. Since the arc subtended by our angle is covered by m of these, its length must be $m(\pi/180n) = (m/n)\,(\pi/180)$. This proves that if the degree measure of an angle is a rational number, then we have the [expected] relationship:

(radian measure) = $(\pi/180)\cdot$(degree measure).

Conversely, if the radian measure [or subtended arc length measure] of an angle is of the form (rational number)$\cdot\pi$, then we have

(degree measure) = $(180/\pi)\cdot$(radian measure)

and accordingly the degree measure is a rational number.

Finally, let's try asking whether or not the degree measure of an angle could be an ***irrational*** number. Well, it's not really possible to divide our half circle into an irrational number of equal portions. On the other hand, it's completely possible that the arc subtended by an angle could be of the form (irrational number)$\cdot\pi$. Accordingly, we could just use the equation:

(degree measure) = $(180/\pi)\cdot$(radian measure)

as our ***definition of degree measure***. Our previous discussion shows that it agrees with the usual definition in the case of rational degree measure. Since every real number can be approximated with arbitrarily good accuracy by rational numbers, this seems to settle the issue at least in principle. Actually, the process of "dotting the i's and crossing the t's" in the verification of the irrational multiple case can be viewed as more of an advanced calculus question than a geometry question. Accordingly, we won't pursue it any further here.

Go back:

To the class homepage

To the homework page

To the course description