Math 5336
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Direction indicators and ideal points

Let's begin by stating two propositions from the text, namely Proposition 9 and Proposition 10 of §8.3. Please recall that a projective line is *ordinary* if it contains at least one ordinary point. Also, we have the usual bijective correspondence between $\mathbb{R}^n$ and the set of all ordinary points of $\mathbb{P}^n$, given (as explained in §8.2) by the map φ: $(x_1,\ldots,x_n) \rightsquigarrow (1, x_1,\ldots,x_n)^{\wp}$. A particular property of this correspondence is that if U and V are points of $\mathbb{R}^n$, then φ maps the line joining U and V to the set of ordinary points of the projective line joining $\varphi(U)$ and $\varphi(V)$. (This property is stated in Proposition 7 of §8.2 of the text, in the more general context of affine subspaces of $\mathbb{R}^n$.)

Proposition 9. Every ordinary projective line contains exactly one ideal point.

Proposition 10. Two ordinary projective lines in $\mathbb{P}^n$ meet at a single ideal point if and only if their corresponding lines in $\mathbb{R}^n$ are parallel

My main objective in this writeup is to provide another way to understand these facts, by relating the direction indicator of the affine line to the ideal point on the corresponding projective line. I won't try to give an alternative proof of Proposition 9, although the ideas used here can be adapted for that purpose. On the other hand, Proposition 10 of the text can be deduced as a fairly immediate corollary of the main result below, and this

Proposition. Let P and Q be points of $\mathbb{R}^n$, with $P \neq Q$, and let $X^{\wp} = \varphi(P)$ and $Y^{\wp} = \varphi(Q)$, where φ is the map described above. Then a vector $(c_1,\ldots,c_n)$ is a direction indicator for the (affine) line joining P and Q in $\mathbb{R}^n$ if and only if $(0,c_1,\ldots,c_n)^{\wp}$ is the unique ideal point on the projective line joining $X^{\wp}$ and $Y^{\wp}$ in $\mathbb{P}^n$.

Proof of the Proposition. We begin by introducing coordinates. Thus, we can assume that $P = (a_1,\ldots,a_n)$ and $Q = (b_1,\ldots,b_n)$ in $\mathbb{R}^n$. Therefore, $Q - P = (b_1\text{-}a_1,\ldots,b_n\text{-}a_n)$, is a direction indicator of the line joining P and Q. More generally, a vector $(c_1,\ldots,c_n)$ is a direction indicator of this line if and only if it is a scalar multiple of $(b_1\text{-}a_1,\ldots,b_n\text{-}a_n)$.

On the other hand, $X^{\wp} = (1,a_1,\ldots,a_n)$ and $Y^{\wp} = (1, b_1,\ldots,b_n)$, so that the points of the projective line that contains $X^{\wp}$ and $Y^{\wp}$ are of the form $Z^{\wp}$, where $Z^0 = rX^0 + sY^0$, for suitable real numbers r and s, not both equal to 0. {As usual, $X^0 = (1,a_1,\ldots,a_n)$ and $Y^0 = (1,b_1,\ldots,b_n)$.} We can obtain the ideal point on the line joining $X^{\wp}$ and $Y^{\wp}$ by making the 0^{th} coordinate = 0, or equivalently by taking $r + s = 0$. Up to a scalar multiple of the pair (r,s), we can do this by taking $r = -1$ and $s = 1$. This leads to the conclusion that $Z^{\wp} = (0,b_1\text{-}a_1,\ldots,b_n\text{-}a_n)^{\wp}$ is the unique ideal point on this line. In slightly different words, we can say that $(0,c_1,\ldots,c_n)^{\wp}$ is the unique ideal point on this line if and only if $(0,c_1,\ldots,c_n)$ is a scalar multiple of $((0,b_1\text{-}a_1,\ldots,b_n\text{-}a_n))$.

By combining the affine and projective parts of the discussion, we obtain the conclusion of the Proposition.

Corollary (= Proposition 10 of §8.3). Two ordinary projective lines in $\mathbb{P}^n$ meet at a single ideal point if and only if their corresponding lines in $\mathbb{R}^n$ are parallel.

Proof of the Corollary. Let $\mathcal{L}$ and $\mathcal{L}'$ be two affine lines in $\mathbb{R}^n$, and let $\mathcal{L}^\wp$ and $\mathcal{L}'^\wp$ be the corresponding ordinary projective lines. Let $(c_1,\ldots,c_n)$ and $(c'_1,\ldots,c'_n)$ be the direction indicators of $\mathcal{L}$ and $\mathcal{L}'$ respectively. The proposition tells us that $(0,c_1,\ldots,c_n)^\wp$ is the unique ideal point on $\mathcal{L}^\wp$, and that $(0,c'_1,\ldots,c'_n)^\wp$ is the unique ideal point on $\mathcal{L}'^\wp$.

If we assume that $\mathcal{L}$ and $\mathcal{L}'$ are parallel, then $(c'_1,\ldots,c'_n)$ is a scalar multiple of $(c_1,\ldots,c_n)$, say:

$$(c'_1,\ldots,c'_n) = \lambda(c_1,\ldots,c_n) = (\lambda c_1,\ldots, \lambda c_n) \text{ in } \mathbb{R}^n$$

for some $\lambda \in \mathbb{R}$ But then $(0,c'_1,\ldots,c'_n) = (0,\lambda c_1,\ldots, \lambda c_n)$ in $\mathbb{R}^{n+1}$, which implies that :

$$(0,c'_1,\ldots,c'_n)^\wp = (0,\lambda c_1,\ldots, \lambda c_n)^\wp = (0,c_1,\ldots, c_n)^\wp \text{ in } \mathbb{P}^n,$$

so that $\mathcal{L}^\wp$ and $\mathcal{L}'^\wp$ certainly have a single ideal point in common.

The converse is proved in a completely similar way -- essentially, all of the steps of the above discussion can be reversed. So, I'll indicate the steps more tersely. Thus, if we assume that $\mathcal{L}^\wp$ and $\mathcal{L}'^\wp$ have a single ideal point in common, then $(0,c'_1,\ldots,c'_n)^\wp = (0,c_1,\ldots, c_n)^\wp$ in $\mathbb{P}^n$, and it follows that $(0,c'_1,\ldots,c'_n)$ is a scalar multiple of $(0,c_1,\ldots, c_n)$ in $\mathbb{R}^{n+1}$. But this implies that $(c'_1,\ldots,c'_n)$ is a scalar multiple of $(c_1,\ldots, c_n)$ (in $\mathbb{R}^n$), so that $\mathcal{L}$ and $\mathcal{L}'$ are parallel.