As always, the review sheets in workshop are a great way to review for the exam. They're written by the same person who writes the exam, so you can get a good feel for what will be emphasized, and what the problems will look like. (Not surprisingly, they're similar to the problems in the lecture notes and homework.)

However, some of you won't get those review sheets until Thursday, mere hours before the exam. We'll also review for the exam on Wednesday in class, but that's not enough time to go over everything. So for those of you who would like to start studying early, here are some guidelines.

Instead of section-by-section, I'm organizing this by topic.

Update (Wednesday, 3/22): I've corrected a few minor typos. Also, just to make sure you understand, reading through this guide will give you a good idea of what to expect, but you won't be prepared until you actually *do* some problems like those described here. It's a long test, so the more practice you can get, the better. If you've done lots of problems you're more likely to do the ones on the exam confidentaly and quickly.

Eigenvalues and Eigenvectors

First, you need to know the definition of eigenvalue and eigenvector. If you can find a number λ and a vector $\mathbf{x}$ such that

$$B\mathbf{x} = \lambda\mathbf{x}$$

then λ is an *eigenvalue* of B, and $\mathbf{x}$ is an *eigenvector* of B associated to λ.

Given a 2×2 or 3×3 matrix B, you need to be able to find its eigenvalues. The lecture notes show you one way to set up the equations; in class I showed you another way to get the same thing. In either case, you need to evaluate a determinant:

$$\det(B - \lambda I)$$

The result of this determinant is a polynomial. Its roots are the eigenvalues of B. In the 2×2 case, you'll either have two real eigenvalues, or a pair of complex eigenvalues which are conjugates of each other. In the 3×3 case, you'll either have three real eigenvalues, or one real eigenvalue and a pair of complex ones. On the exam it should always be the case that the eigenvalues are either integers or, in the complex case, expressible using integers. (In other words, $3 + 4i$ is a possible eigenvalue for a matrix on the exam; $\frac{1}{2} + \frac{3}{7}i$ is not.)

If you're given an eigenvalue, or if you've already found one, you need to be able to find its associated eigenvectors. First write out the matrix equation:

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{or} \qquad B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Once you've written out the matrix B, this gives you a system of linear equations. By design this system is dependent – in other words, it should have infinitely many solutions. (This is because every eigenvalue has infinitely many corresponding eigenvectors.) That means there are certain tricks you can use to speed up the solution of the system; see your lecture notes for some of these.

In the 3×3 case, we'll make two concessions to make your life easier.

(1) You'll always have a zero somewhere in the matrix. Use it wisely to speed up the computation of the eigenvalues and eigenvectors. (Having a zero makes it faster to compute a determinant, as long as you expand along the row or column which contains the zero; there's also a trick we discussed in class to speed up solving the equations to find an eigenvector. You can find that trick in the "Three Complex Eigenvalues" section.)

(2) If you're asked to find the eigenvalues, either $\lambda = 0$ will be an eigenvalue, or you'll be told one of them in advance. (This may not seem like a concession, but it makes life easier, because otherwise you'd have to find all three roots of a cubic polynomial from scratch.)

Remember that, if you have a complex eigenvalue, you should expect the eigenvectors to have complex numbers. (And the "parameter" you use to solve the system of equations can now be any *complex* number.) The process of solving a system of equations with complex numbers is no different than solving a system with real numbers, but it takes some time to get comfortable working with complex numbers. Make sure you've done enough of these problems so that you don't panic at the sight of complex numbers.

Although it's awkward to work with complex numbers, it can save us some work in the end: complex eigenvalues come in conjugate pairs, and so do their corresponding eigenvectors. In other words, if $2 + 3i$ is an eigenvalue with eigenvectors

$$s \begin{pmatrix} 10 - 5i \\ -1 + 6i \end{pmatrix}, \quad s \in \mathbb{C},$$

then $2 - 3i$ is also an eigenvalue, and its eigenvectors are

$$s \begin{pmatrix} 10 + 5i \\ -1 - 6i \end{pmatrix}, \quad s \in \mathbb{C}$$

Second Order Linear Differential Equations

The bulk of our time since the last exam has been spent on Second Order Linear Differential Equations, so there's a lot to keep track of here. For me it helps to think of this as a progression, from easier equations to harder ones.

Constant Coefficients, Homogeneous. These are the equations

$$ay'' + by' + cy = 0$$

We always start by assuming that $y = e^{mt}$ is a solution; plugging that into the equation gives

$$e^{mt}(am^2 + bm + c) = 0$$

In other words, $y = e^{mt}$ is a solution as long as $am^2+bm+c = 0$. This is called the *auxiliary equation*, and because it's quadratic, there are generally two values of m that work.[1] To find the general solution to this equation we take all combinations of the resulting solutions:

Two distinct real roots ($b^2 - 4ac > 0$): For example, suppose $m = 2, 3$. Then $y = e^{2t}$ and $y = e^{3t}$ are solutions, and the general solution is

$$y = C_1e^{2t} + C_2e^{3t}$$

Two conjugate complex roots ($b^2 - 4ac < 0$): For example, suppose $m = 2 \pm 3i$. Then $y = e^{(2+3it}$ and $y = e^{(2-3i)t}$ are solutions, but we showed in class that we can transform these to the "nicer" solutions $y = e^{2t} \cos 3t$ and $y = e^{2t} \sin 3t$. Normally we skip straight to those nicer solutions. The general solution is

$$\begin{aligned} y &= C_1e^{2t} \cos 3t + C_2e^{2t} \sin 3t \\ &= e^{2t}(C_1 \cos 3t + C_2 \sin 3t) \end{aligned}$$

Special terms. When an equation like this describes the motion of a spring (or voltage in an LRC circuit), then the case with two complex roots is called *underdamped*. If you have two distinct real roots, the system is *overdamped*. The special case in between, where $b^2 - 4ac = 0$, is called *critically damped*.

Constant Coefficients, Nonhomogenous. These are the equations

$$ay'' + by' + cy = f(t)$$

The equation $ay''+by'+cy = 0$ is called the *associated homogenous equation*. If the general solution to the associated equation is $y_m(t)$, then the general solution to the nonhomogenous equation is

$$y(t) = y_m(t) + y_p(t)$$

where $y_p(t)$ is some particular solution to the nonhomogeneous equation. We generally find $y_p(t)$ by guessing what it looks like, based on $f(t)$. For example,

$f(t) = e^{2t}$: We'd guess $y_p(t) = Ae^{2t}$.
$f(t) = t^2$: We'd guess $y_p(t) = At^2 + Bt + C$.
$f(t) = 49 \sin 4t$: We'd guess $y_p(t) = A \cos 4t + B \sin 4t$.
$f(t) = e^{5t}5 \sin 4t$: We'd guess $y_p(t) = e^{5t}(A \cos 4t + B \sin 4t)$.

This isn't a comprehensive list of possible $f(t)$'s and corresponding guesses. See your lecture notes for more. Once you've made your guess, you need to take its first and second derivatives, plug them into the nonhomogeneous differential equation, collect like terms (this can be a lot of work) and solve for A, B, or whatever other constants you have.

[1] The case where $b^2 - 4ac = 0$, so we have one repeated real root, is special; see the lecture notes for how to deal with that.

Special terms. When an equation like this describes the motion of a spring (or voltage in an LRC circuit), then the general solution of the associated homogenous equation is called the *transient* solution. (It's transient because there will be a term like e^{-t} out front, which approaches zero as t grows larger, so this whole term approaches zero.) The particular solution $y_p(t)$ is called the *steady state* solution, because it does not degrade over time.

Non-constant Coefficients, Nonhomogenous. These are the equations

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

These are the toughest and longest problems we've had to deal with, and *you should be prepared to do one on the exam.* They're so hard that you can't even start unless you're already given a solution $y_1(t)$ to the associated homogeneous equation $a(t)y'' + b(t)y' + c(t)y = 0$. Then you guess that the general solution to the nonhomogeneous equation will look like $y = y_1 u$, where u is some function which will have two constants, C_1 and C_2.

First you have to calculate $y' = (y_1 u)'$ and $y'' = (y_1 u)''$ and plug them into the equation above. If all goes well. you should have a new equation which looks like this:

$$d(t)u'' + e(t)u' = g(t)$$

Aha! If we make the substitution $w = u'$,

$$d(t)w' + e(t)w = g(t)$$

then this is a *First* Order Linear Differential Equation, which we learned to solve at the beginning of the year using an integrating factor. So go ahead and solve for w. This will give you some solution involving a constant, which you can call C_1.

Once you have w, you need to go back and find u. Since $u' = w$, it follows that $u = \int w\,dt$. Evaluating that integral gives you another constant, which you can call C_2.

You're on the home stretch now. You want to solve for $y = y_1 u$. Now that you have u, you can calculate the general solution y, and you're finally done!

Other Topics

Boundary Value Problems. This is really just a variant of the Second Order Linear Differential Equations above. The difference is that, unlike an initial value problem where you're given values of $y(0)$ and $y'(0)$, in a boundary value problem you're given two values of the function at either end (i.e. the boundary) of an interval—say, $y(0)$ and $y(4)$. You aren't given any values of the derivative y'.

Boundary value problems might have no solution, one unique solution, or infinitely many solutions. (How can you have infinitely many? In workshop, you had a problem where the two boundary values implied that $C_1 = 0$, but there was no restriction on the value of C_2.)

In workshop, the biggest difficulty I noticed with these problems was sloppiness in plugging in values of t. I've been guilty of this myself. For example, if

$$y = C_1 \cos 2t + C_2 \sin 2t$$

and you want to use the boundary condition $y(\pi/2) = 0$, then you need to evaluate the sine and cosine functions at π, not $\pi/2$.

Autonomous Differential Equations and Phase Lines. These are the differential equations of the form $dy/dt = f(y)$, with no t's on the right hand side. In this class we're only concerned with autonomous differential equations of the type

$$\frac{dy}{dt} = A(y - B)(y - D)$$

The "equilibrium solutions" to this equation are $y = B$ and $y = D$. To see why, look at the equation; if $y = B$, for example, then dy/dt is always 0; in other words, y never changes. It's fixed at B, which is therefore an equilibrium value. The same reasoning works for $y = D$.

If $y \neq B, D$ then we can solve this differential equation by separating the variables and using partial fractions:

$$\frac{1}{(y - B)(y - D)}dy = Ady$$

This solution is generally a fraction where the top and bottom have a constant term and an exponential term.

We first covered equations like this back in the Logistic Growth section for Exam 1. This time around we were a little more careful about the solutions. In addition to the equilibrium solutions, you need to watch out for things like this: in the book the solution of $y' = -(y + 2)(y - 10)$, $y(0) = -8$ is found to be

$$y = \frac{10e^{12t} + 6}{e^{12t} - 3}$$

Because we're given the initial value of y when $t = 0$, we can assume that's the starting point for t. Notice that the solution becomes undefined once $e^{12t} - 3 = 0$, or when $t = (\ln 3)/12$ (which is about 0.179). So this solution is only true for values of t in the interval $[0, (\ln 3)/12]$.

On the other hand, if this were the solution resulting from an initial value for $y(10)$, then the solution would always be true as $t \to \infty$, because the denominator would never equal zero.

We also talked about *phase lines* with respect to autonomous equations. The basic idea is that you draw the y-axis, and make marks at the equilibrium solutions. Because these are the only places where $y' = 0$, they are the only places where y' can change from positive to negative. Now that you've split up the y-axis into three intervals, take a y value from each interval and plug it into your equation for y'. Make a mark in each interval to indicate if the derivative is positive or negative there. This is a quick way to see what happens for a given initial value of y, without figuring out the actual solution $y(t)$. For certain initial values, $y(t)$ increases, either to an equilibrium or to (positive) infinity. For other values, $y(t)$ decreases to an equilibrium or to (negative) infinity.

If arrows on either side of an equilibrium solution point towards that number, then that equilibrium is stable. If the arrows on either side point away from the equilibrium, its unstable.

Phase Angle Form. Given an expression such as

$$D \cos wt + E \sin wt$$

you need to be able to transform it into the form $A \sin(wt + d)$ or $A \cos(wt + d)$. There are examples of both kinds of transformations in the lectures notes. Double check this with me, but I believe you'll be given the formulas for $\sin(x + y)$, $\sin(x - y)$, $\cos(x + y)$ and $\cos(x - y)$. Once you've transformed the function, you should be able to find the amplitude (it's A) and the period (which is $2\pi/w$). To graph it you also need to be able to figure out the shift in the sine or cosine wave; it's d/w, and the direction of the shift depends on the sign of that quotient. Ask us if you need help remembering how these function transformations go.