1. (10 points) Use a truth table to determine whether the following statements are logically equivalent. If so, explain how this is demonstrated in the truth table; if not, give a situation in which their truth values are different.

Statement 1: $p \Rightarrow q$

Statement 2: $\sim (p \wedge \sim q)$

p	q	$p \Rightarrow q$	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$\Leftrightarrow$
T	T	T	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the columns $p \Rightarrow q$ and $\sim(p \wedge \sim q)$ are identical, the two statments are equivalent.

2. (10 points) Consider the following implication: *If* $x \in \mathbb{N}$ *and* $y = \pi$, *then* $\sin(xy) = 0$.

(a) (4 points) Write the converse of the implication.

If $\sin(xy) = 0$, then $x \in \mathbb{N}$ and $y = \pi$.

(b) (6 points) Write the negation of the implication.

$x \in \mathbb{N}$ and $y = \pi$, and $\sin(xy) \neq 0$.

3. (16 points) Write each statement using mathematical quantifiers and symbols where possible. Then write the negation of each statement, again using quantifiers and other symbols. You should not just put the negation symbol in front of the statement; rather, change quantifiers and other symbols as needed to express the negation as a new statement.

   (a) (8 points) For all positive real numbers x, there is a real number y such that $f'(xy) = 0$ or $f'(xy)$ is undefined.

   Statement: $\forall x \in \mathbb{R}, x > 0, \exists y \in \mathbb{R} \ni f'(xy) = 0 \vee f'(xy)$ is undefined.

   Negation: $\exists x \in \mathbb{R}, x > 0, \ni \forall y \in \mathbb{R}, f'(xy) \neq 0 \wedge f'(xy)$ is well defined.

   (b) (8 points) There exists an integer n such that $\sqrt{2} < n < \sqrt{2} + 1$.

   Statement: $\exists n \in \mathbb{Z} \ni \sqrt{2} < n < \sqrt{2} + 1$.

   Negation: $\forall n \in \mathbb{Z}, \quad n \leq \sqrt{2} \vee n \geq \sqrt{2} + 1$.

4. (4 points) Give a **BRIEF** (2-3 sentences at most) description of how to prove a mathematical statement p by contradiction.

   Assume the negation of p, $\sim p$, is true. Use this assumption to prove a contradiction. Then the assumption is false, so p is true.

Let A, B, and C be subsets of a universal set U. Prove: $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof: To show that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$, take some $x \in A \setminus (B \cap C)$. Then, by definition, $x \in A$ and $x \notin B \cap C$. $x \notin B \cap C$ is equivalent to $x \notin B$ or $x \notin C$ $[\neg(x \in B \wedge x \in C) \iff x \notin B \vee x \notin C]$. So $x \in A \setminus B$ or $x \in A \setminus C$. I.e.,$x \in (A \setminus B) \cup (A \setminus C)$, which shows that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Conversely, suppose $x \in (A \setminus B) \cup (A \setminus C)$. Without loss of generality, assume that $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. So $x \notin B \cap C$. So $x \in A \setminus (B \cap C)$. Thus $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$, which shows that the two sets are equal.

6(a) Problem: Prove the statement "if $x+y$ is irrational, then x is irrational or y is irrational" using the contrapositive.

Proof The contrapositive is "if x is rational and y is rational, then $x+y$ is rational".

So assume x and y are rational. By definition, $x = \frac{m}{n}$ and $y = \frac{p}{q}$ for some integers m, n, p, q with $n \neq 0$, $q \neq 0$.

$$x + y = \frac{m}{n} + \frac{p}{q} = \frac{mq + pn}{nq} \quad (mq+pn,\ nq \text{ are integers, and } nq \neq 0),$$

so $x+y$ is a rational number as well, and we've proven the contrapositive.

6(b) Problem: Give a direct proof of the statement "if m and n are odd integers, then mn is an odd integer".

Proof An integer is odd if it is representable as $2k+1$ for some integer k.

Assume m, n are odd. Then $m = 2k+1$, $n = 2\ell + 1$ for some integers k and ℓ.

(WARNING: If you write "$2k+1$" for both, you are assuming $m = n$! In general, k and ℓ will be different.)

But then
$$\begin{aligned} m \cdot n = (2k+1)(2\ell+1) &= 4k\ell + 2k + 2\ell + 1 \\ &= 2\underbrace{(2k\ell + k + \ell)}_{\in \mathbb{Z}} + 1 \end{aligned}$$

is of the form $2(\text{integer})+1$, hence is odd.

7. (16 points) Consider the following relation on $\mathbb{R}$: xRy iff there exists a real number $r \neq 0$ such that $x = ry$. (In other words, xRy if you can multiply y by some nonzero real number to get x.)

(a) (12 points) Prove that R is an equivalence relation.

We must prove R is reflexive, symmetric, and transitive.

- Reflexive ($xRx\ \forall x \in \mathbb{R}$): Let $r=1$; then $x=rx \Rightarrow xRx$.
- Symmetric ($\forall x,y \in \mathbb{R}$, $xRy \Rightarrow yRx$)

  Let xRy, so $x=ry$ for $r\neq 0$. Then $y=\left(\frac{1}{r}\right)x$ - note that $r\neq 0 \Rightarrow \frac{1}{r}$ exists and is nonzero. Thus yRx.
- Transitive ($\forall x,y,z \in \mathbb{R}$, xRy and $yRz \Rightarrow xRz$).

  Let xRy and yRz, so $x=ry$ and $y=sz$, $r,s\neq 0$.
  Then $x=r(sz)=(rs)z$, $rs\neq 0$, so xRz.

(b) (4 points) Describe the equivalence class of $x = 0$ in this relation.

$E_0=\{y\in\mathbb{R} \mid 0Ry\}$

$=\{y \mid ry=0,\ \text{some } r\neq 0\}$

$=\{0\}$.

Common issues

- symmetric: need $y=\begin{pmatrix}\text{nonzero}\\ \text{constant}\end{pmatrix}x$, not $x=ry \Leftrightarrow ry=x$.
- transitive need different constants, $x=ry$, $y=sz$, not $x=ry$, $y=rz$ (or $ry=rz$, etc.)

8. (8 points) Give a counterexample to show the following equality is not true:

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D).$$

- Equivalence classes are subsets of the set $S=\mathbb{R}$, not $S\times S=\mathbb{R}\times\mathbb{R}$.

8) Give a counterexample to show the following equality is not true:

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D).$$

Solution. One can produce (infinitely) many such counterexamples. Perphaps, one of the simplest looks as follows. Let $A = B = \{\bigstar\}$ and $C = D = \{\spadesuit\}$. Then the Cartesian product $A \times B$ is the set $\{(\bigstar, \bigstar)\}$ and $C \times D = \{(\spadesuit, \spadesuit)\}$. Thus,

$$(A \times B) \cup (C \times D) = \{(\bigstar, \bigstar), (\spadesuit, \spadesuit)\}.$$

On the other hand, since $A \cup C = B \cup D = \{\bigstar, \spadesuit\}$, then

$$(A \cup C) \times (B \cup D) = \{(\bigstar, \bigstar), (\bigstar, \spadesuit), (\spadesuit, \bigstar), (\spadesuit, \spadesuit)\}.$$

So the set on the right-hand side of the given identity is strictly larger than the set on the left-hand side.