The following is a non-comprehensive list of solutions to the exam problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

(1) This was a collection of definitions and examples used in lecture, homework and quizzes.

(a) Any nonempty subset S of $\mathbb{R}$ which is bounded above has a supremum. (p121)

(b) $\mathbb{N}$ is unbounded above in $\mathbb{R}$. (p122)

(c) $x \in \mathbb{R}$ is an interior point of S if there is a neighborhood of x which is entirely contained in S; that is, $\exists \varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$.

(d) $\text{bd}\, \mathbb{N} = \mathbb{N}$.

(e) If $A_n = \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$, then $\bigcup_{n=1}^{\infty} A_n = (0, 2)$.

(f) $\mathbb{Q}$ is *dense* in $\mathbb{R}$ means for any real numbers x, y with $x < y$, there exists a rational number r such that $x < r < y$. (p125)

(2) This was a slight variant of an example done in lecture (which itself was presented as an example from a Spring 2009 midterm).

(a) The elements in the set are given by $\frac{n-2}{2} = 1 - \frac{2}{n}$, so

$$S = \{-1, 0, 1/3, 2/4, 3/5, 4/6, \ldots\}$$

We can see that S is bounded below by -1; you could prove this algebraically by starting with $\frac{n-2}{n} \geq -1$ and solving for n; the result is $n \geq 1$, which means it is true for every element in the set. Similarly, $n-2 < n$ for all n, which has the consequence that every fraction in the set must be less than 1, so S is bounded above.

(b) $m = \inf S = -1$ and $M = \sup S = 1$.

(c) To prove M is the supremum, we must verify two properties:

- M is an upper bound of S.
- No smaller number $M' < M$ can be an upper bound of S.

We already showed in (a) that $M = 1$ is an upper bound, so we can focus on the second. We need to show that, for any $M' < M$, there exists a number $s \in S$ such that $M' < s$. The standard way the book does this is to set $\varepsilon = M - M' > 0$ and solve $M - \varepsilon = 1 - \varepsilon < \frac{n-2}{n} = 1 - \frac{2}{n}$:

$$1 - \varepsilon < 1 - \frac{2}{n}$$
$$\frac{2}{n} < \varepsilon$$
$$\frac{2}{\varepsilon} < n$$

Since we can always be guaranteed of finding a n larger than $2/\varepsilon$, no matter how small ε is – use the Archemdean Property! – we have shown that there will be an element of the set larger than $M' = 1 - \varepsilon$. Hence M' is not an upper bound, and we have verified that $M = 1$ is in fact $\sup S$.

(3) (a) This was proven in class and was a writing problem.

To prove $A \cup B$ is open, we have to show any point in the union has a neighborhood which is contained in $A \cup B$. Let $x \in A \cup B$, which means either $x \in A$ or $x \in B$ (or conceivably both). If $x \in A$ then there exists a neighborhood N of x such that $N \subseteq A$, since A is open. But then that neighborhood is also in $A \cup B$:

$$x \in N \subseteq A \subseteq A \cup B.$$

Similarly, if x isn't in A but is in the open set B, then there is a neighborhood of x such that $N \subseteq B \subseteq A \cup B$.

(b) This is 11.7, a skills problem, with the base case provided for you.

Since the base case is provided, we move on to the induction step and assume $|x_1 + \cdots + x_k| \leq |x_1| + \cdots |x_k|$. Then

$$\begin{aligned} |x_1 + x_2 + \cdot + x_k + x_{k+1}| &\leq |(x_1 + x_2 + \cdot + x_k) + x_{k+1}| \\ &\leq |x_1 + x_2 + \cdots x_k| + |x_{k+1}| \text{ (base case)} \\ &\leq |x_1| + |x_2| + \cdots + |x_k| + |x_{k+1}| \text{ (by induction assumption)} \end{aligned}$$

(4) These problems were (or were similar to) examples in lecture and your first take-home problem.

(a) $|\emptyset| < |\{1, 2, 3, \ldots, 10\}| < |\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}| = |(0, 1)|$.

(b) The inclusion : $\mathbb{N} \to \mathbb{Z}$, $h(n) = n$ works for this problem. Inclusions were a major part of your first take-home program and are always injective. (Can you explain why?) It's not surjective because, for example, there is no natural number n for which $h(n) = 0$.

(c) The function $f(n) = n+1$ works. It is surjective because any $n \in \mathbb{N}$ is an output of f–namely, $n = f(n-1)$. Similarly, it is injective because if $f(n) = f(m)$ then $n + 1 = m + 1$ and $n = m$.