The following is a non-comprehensive list of solutions to the exam problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

(1) (a) $\displaystyle\sum_{n=3}^{\infty} \frac{2}{n(n-1)} = \sum_{n=3}^{\infty} \left[\frac{2}{n-1} - \frac{2}{n}\right].$

This is a telescoping series. If you write out the first few partial sums you'll find $s_n = 1 - 2/n$. Since $s_n \to 1$, the series converges to 1.

(b) $\displaystyle\sum_{n=0}^{\infty} \left[(-3)^{n+1}\left(\frac{1}{2}\right)^n\right] = \sum_{n=0}^{\infty}(-3)\left(-\frac{3}{2}\right)^2.$

This is a geometric series $\sum ar^n$ with $a = -3$ and $r = -3/2$. It diverges since $|r| > 1$. We know

$$1 + r + \cdots r^n = \frac{1 - r^{n+1}}{1 - r}$$

so the partial sums of this series are

$$a + ar + \cdots ar^n = \frac{a(1 - r^{n+1})}{1 - r} = \frac{-3(1 - (-3/2)^{n+1})}{1 + 3/2}$$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ The first few partial sums are

$$\begin{aligned} s_1 &= -1 \\ s_2 &= -1 + 1 = 0 \\ s_3 &= -1 + 1 - 1 = -1 \\ s_4 &= -1 + 1 - 1 + 1 = 0 \end{aligned}$$

and, in general,

$$s_n = \begin{cases} 0, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

which does not converge. Hence this series diverges.

(2) (a) Using algebra and the limit laws (Theorem 17.1),

$$\begin{aligned} \lim \frac{2n+1}{n+2} &= \lim \frac{2 + 1/n}{1 + 2/n} \\ &= \frac{\lim(2 + 1/n)}{\lim(1 + 2/n)} \\ &= \frac{2 + \lim 1/n}{1 + \lim 2/n} = \frac{2}{1} = 2. \end{aligned}$$

(b) A few different answers are possible. $n > 3/\varepsilon$ or $n > 3/\varepsilon - 2$ were the most common. Most people did a chain of equalities and inequalities similar to:

$$|s_n - s| = \left|\frac{2n+1}{n+2} - 2\right| = \left|\frac{2n+1-2n-4}{n+2}\right| = \left|\frac{-3}{n+2}\right| = \frac{3}{n+2} < \frac{3}{n} < \varepsilon,$$

and $3/n < \varepsilon$ yields $3/\varepsilon < n$.

(c) Give any $\varepsilon > 0$, choose $N = 3/\varepsilon$ (or whatever you found above). Then for any $n > N$, we have

$$\left|\frac{2n+1}{n+2} - 2\right| = \left|\frac{2n+1-2n-4}{n+2}\right| = \left|\frac{-3}{n+2}\right| = \frac{3}{n+2} < \frac{3}{n} < \varepsilon.$$

Hence $s_n \to 2$.

(3) (a) It's probably easiest to prove this by induction. The base case is given: $a_1 = 1 \leq 3$. Now assume $a_k \leq 3$, in which case

$$a_{k+1} = \sqrt{2a_k + 3} \leq \sqrt{2(3)+3} = \sqrt{9} = 3$$

as desired, where we have made use of the induction assumption $a_k \leq 3$ to get the inequality in the line above.

(b) This is also an inductive proof. The base case is $a_2 = \sqrt{2+3} = \sqrt{5} \geq 1 = a_1$. Now assume $a_k \geq a_{k-1}$, in which case

$$a_{k+1} = \sqrt{2a_k + 3} \geq \sqrt{2a_{k-1} + 3} = a_k.$$

Hence $a_n$ is increasing as desired.

(c) The sequence $a_n$ is increasing by (b). It is bounded below by 1 since $a_1 = 1$ and the sequence increases after that. It is bounded above by (a). Hence the Monotone Convergence Theorem says the sequence converges. To find the limit we use our "standard" technique for these problems,

$$\begin{aligned} \lim a_n &= \lim a_{n+1} \\ \lim a_n &= \lim \sqrt{2a_n + 3} \\ a &= \sqrt{2a+3} \\ a^2 &= 2a + 3 \\ a^2 - 2a - 3 &= 0 \\ (a-3)(a+1) &= 0 \end{aligned}$$

And we conclude $a = \lim a_n = 3$. (The solution $a = -1$ is an extraneous one introduced when we squared both sides.)

(4) (a) $s_n \to +\infty$ iff $\forall M \in \mathbb{R} \exists N$ such that $n > N \Rightarrow s_n > M$. (Alternatively, in words: $s_n$ diverges to infinity off for every real number $M$ there exists an $N$ such that $s_n > M$ for any $n > N$.)

(b) One example: $s_n = (-1)^n$, $t_n = (-1)^n$. Then both sequences diverge (they oscillate between $-1$ and 1) but their product is $(s_n t_n) = \left((-1)^{2n}\right) = (1, 1, 1, 1, 1, 1, 1, \ldots)$ which is constantly equal to 1 and therefore converges to 1.

(c) $s_n$ is Cauchy iff for all $\varepsilon > 0$ there exists $N$ such that $n, m > N \Rightarrow |s_n - s_m| < \varepsilon$.

(d) This is impossible: immediately after we defined Cauchy sequence in class, we proved a theorem that sequences are Cauchy if and only if they converge.

(e) This is not bounded above; it's the sequence of partial sums of the harmonic series which diverges to $+\infty$ – in other words, it gets larger than any possible number, and can therefore never be bounded brave.