Study Guide for Exam 1: Solutions

1. (a) First, recall[1] that the disjuction $A \vee B$ of two statements A and B is true if and only if at least one of the statements is true, and the conjunction $A \wedge B$ is true only when *both* A and B are true.

   Now, let R be a true statement, while P and Q are false. Then the disjunction $\underbrace{(P \wedge Q)}_{F} \vee \underbrace{R}_{T}$ is true, but the conjunction $\underbrace{P}_{F} \wedge \underbrace{(Q \vee R)}_{T}$ is false.

   Remark. Compare the given formulas with the ones given in exercise 1.14(c)-(f).

   (b) A quick reminder: the implication[2] $A \Rightarrow B$ is false only in the case when A is a true statement and B is false.

   Solution 1.

   Comparing the truth tables for both of the given statements, we conclude that they are equivalent:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

P	Q	R	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

   Solution 2.

   Since for any two statements A and B, the implication $A \Rightarrow B$ is equivalent to the disjunction $\sim A \vee B$, then

$$(P \wedge Q) \Rightarrow R = \sim (P \wedge Q) \vee R \overset{\text{De Morgan}}{=} (\sim P \vee \sim Q) \vee R.$$

   Similarly,

$$(P \Rightarrow R) \vee (Q \Rightarrow R) = (\sim P \vee R) \vee (\sim Q \vee R) = (\sim P \vee \sim Q) \vee R.$$

   Thus, the given statements are equivalent.

[1] Chapter 1, pp.3-4
[2] Chapter 1, p.5

2. (a) $\forall a \in \mathbb{R}\, \forall b \in \mathbb{R},\ P(a,b) \Rightarrow \left(P\left(a, \frac{a+b}{2}\right) \wedge P\left(\frac{a+b}{2}, b\right)\right)$.
   (b) $\forall x \in \mathbb{R},\ (P(0,x) \wedge P(x,1)) \Rightarrow P(x^2, x)$.
   (c) $\exists x \in \mathbb{R} \ni\ P(0,x) \wedge P(x,1) \wedge \sim P(x^2, x)$.

3. (a) Let A and B be subsets of $\mathbb{R}$. Then the given statement p is, basically, the implication $r \Rightarrow s$, where r is the statement

   the intersection $A \cap B$ is infinite,

   and the statement s reads

   both A and B are infinite.

   Recall[3] that the *contrapositive* of an implication $r \Rightarrow s$ is the implication $\sim s \Rightarrow \sim r$. In our case

   $\sim r$: *the intersection $A \cap B$ is finite,*
   $\sim s$: *at least one of the sets A and B is finite.*

   Hence, the contrapositive of p can be stated as

   if at least one of the sets A and B is finite, then $A \cap B$ is finite.

   (b) Let r and s be as above. By definition[4], the *converse* of an implication $r \Rightarrow s$ is the implication $s \Rightarrow r$. In our case it reads

   if both A and B are infinite sets, then the intersection $A \cap B$ is infinite.

   (c) The converse of p is false. Here is a counterexample: let A be the set of all even integers and B be the set of all odd integers. Then both A and B are infinite, but $A \cap B = \varnothing$ is a finite set.

   Remark. Make sure to know the difference between contrapositive, converse, inverse and negation of an implication $r \Rightarrow s$:

	Formula	Remarks
Contrapositive	$\sim s \Rightarrow \sim r$	It is logically equivalent to the original implication $r \Rightarrow s$.
Converse	$s \Rightarrow r$	Be careful. It is **not** equivalent to the original statement $r \Rightarrow s$.
Inverse	$\sim r \Rightarrow \sim s$	It is logically equivalent to the converse.
Negation	$r \wedge \sim s$	The negation of an implication is not an implication. *Exercise.* Let r be the statement "*you give a mouse a cookie*", and s denote the statement "*he will ask for a glass of milk*". Then the implication $r \Rightarrow s$ is a popular saying. How does the negation of this implication look like?

[3] Chapter 1, p.21
[4] Chapter 1, p.22