The following is a non-comprehensive list of solutions to the skills problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

33.3(a) Converges by the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{2n^3}\right| \to \frac{1}{2} < 1$$

33.3(e) Converges by the ratio test. First we compute the ratio.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)n(n-1)\cdots 1}{(n+1)^{n+1}} \cdot \frac{n^n}{n(n-1)\cdots 1}\right| \\ &= \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \frac{n^n}{(n+1)^n} \\ &= \left(\frac{n}{n+1}\right)^n\end{aligned}$$

Evaluating the limit of $\left(\frac{n}{n+1}\right)^n$ is hard – much harder than anything you'd see on the final! Probably the simplest way is to note that its reciprocal is

$$\left(\frac{n}{n+1}\right)^{-n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

and hopefully you recognize that last quantity; we learn in calculus that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Hence our ratio $\left|\frac{a_{n+1}}{a_n}\right|$ has a limit of $1/e$, which is less than one, so the series converges.

33.3(i) **Updated**: By a previous exercise, 17.15(a), we know the limit of $\sqrt{n+1}-\sqrt{n} = 0$, so the simple test for divergence doesn't help. But the series *does* diverge. Using algebra,

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n}}{1} &= \frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &> \frac{1}{\sqrt{n+1}+\sqrt{n+1}} \\ &= \frac{1}{2(n+1)^{1/2}}\end{aligned}$$

Hence $\sum \sqrt{n+1}-\sqrt{n} \geq \frac{1}{2}\sum\frac{1}{(n+1)^{1/2}}$. The latter is a divergent p-series (with $p = 1/2$) so the comparison test says our series diverges as well.

33.5(b) If you try the ratio test you'll find the series does not converge absolutely. In fact, the series does not converge at all, either conditionally or absolutely. The terms $\dfrac{(-2)^n}{n^2}$ do not converge to 0, because 2^n grows much more rapidly than n^2. Hence our simple test for divergence tells us that the series diverges.

33.5(c) This converges absolutely by the ratio test. Ask us if you need help with the details.

33.5(d) This series diverges. You can again check absolute converge with the ratio or root tests, but there's no need to use those tests or the alternating series test; the terms $(-5/2)^n$ do not converge to zero, as in (a).

33.6 One example is $a_n = (-1)^{n+1}\dfrac{1}{\sqrt{n}}$ and $b_n == (-1)^{n+1}\dfrac{1}{\sqrt{n}}$, so $\sum a_n$ and $\sum b_n$ are the same series and converge by the alternating series test. (You should double check this!) But $a_n b_n = \dfrac{1}{n}$, so $\sum a_n b_n$ diverges – it's the harmonic series.

33.4(a) We have $\sum a_n x^n$ with $a_n = n$. Using ratios,

$$R = \frac{1}{\lim \left|\dfrac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim \dfrac{n+1}{n}} = \lim \frac{n}{n+1} = 1.$$

Hence the radius of convergence is 1. At the endpoints $x = \pm 1$,

- $\sum n(1)^n = \sum n$ diverges to positive infinity.
- $\sum n(-1)^n$ also diverges; its sequence of partial sums,

$$s_1 = -1, \quad s_2 = -1 + 2 = 1, \quad , s_3 = -1 + 2 - 3 = -2, \quad s_4 = -1 + 2 - 3 + 4 = 2, \ldots$$

  oscillates wildly between positive and negative numbers.

Because the series diverges at both endpoints, the interval of convergence is $(-1, 1)$.

34.3(c) We have $\sum a_n x^n$ with $a_n = \dfrac{2^n}{n}$. Using ratios,

$$R = \frac{1}{\lim \left|\dfrac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim \dfrac{2^{n+1}}{n+1}\dfrac{n}{2^n}} = \lim \frac{(n+1)}{2n} = \frac{1}{2}.$$

Checking the endpoints $x = \pm 1/2$, we see:

- $\sum \dfrac{2^n}{n}\dfrac{1}{2^n} = \sum \dfrac{1}{n}$ diverges to positive infinity; it's the harmonic series.
- $\sum \dfrac{2^n}{n}\dfrac{(-1)^n}{2^n} = \sum \dfrac{(-1)^n}{n}$ converges by the alternating series test.

Thus the interval of convergence is $[-1/2, 1/2)$.

34.3(e) We have $\sum a_n x^n$ with $a_n = \dfrac{1}{n^n}$. Using roots,

$$\alpha = \lim \left|\frac{1}{n^n}\right|^{1/n} = \lim \frac{1}{n} = 0$$

which means the radius of convergence is $+\infty$ and the interval of convergence is all of $\mathbb{R}$.