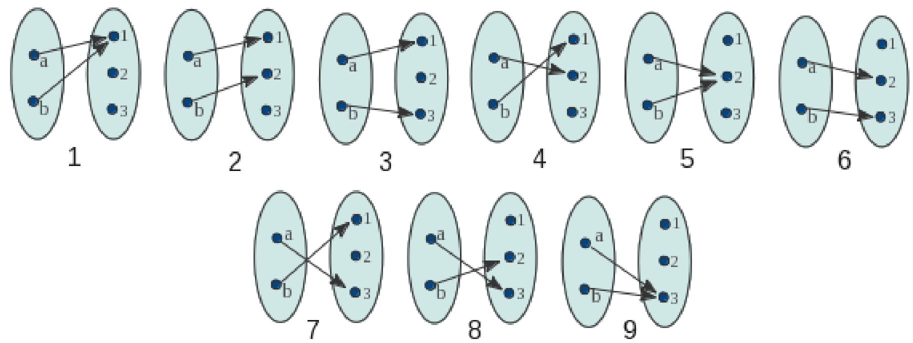
Skills problems 5: Solutions

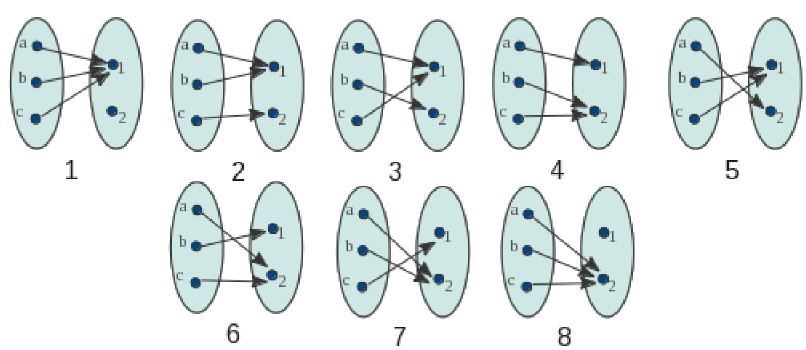
7.5 a) Let $A = \{\mathrm{a}, \mathrm{b}\}$ and $B = \{1, 2, 3\}$. A function $f : A \to B$ is determined by its values on the elements of the domain A. Hence, there are as many functions from A to B as there are ways to pick elements of B for $f(a)$ and $f(b)$. Namely, since $|B| = 3$, there are three possible values for $f(a)$ and three possible values for $f(b)$. Thus, there are [1] $3 \cdot 3 = 9$ possible functions from A to B.



There are no surjective functions from A to B as one can observe directly from the diagrams above: in each case, at least one element of the codomain B remains "unhit". More formally, notice that since, by definition, a function maps an element of its domain to a *single* element of the codomain, then there are at most as many elements in the range of a function as there are elements in its domain. In our case, $\mathrm{dom}(f) = A$ contains exactly two elements, while B has three. Hence, $|\mathrm{rng}(f)| \leq 2 < |B|$, and we conclude that the equality $B = \mathrm{rng}(f)$ cannot hold.

Six functions corrsponding to diagrams $2, 3, 4, 6, 7, 8$ drawn above are injective. Formally, one can count all injective functions from A to B as follows. First, we choose a value for $f(a)$ - it can be either one of the elements of set $B = \{1, 2, 3\}$. Next, we would like to set a value for $f(b)$. If we want f to be injective, then $f(b)$ cannot be equal to $f(a)$. We end up having only two possible values for $f(b)$. Therefore, there are $3 \cdot 2 = 6$ injective functions from A to B.

b) Let $A = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, $B = \{1, 2\}$. As in part a), the number of functions from A to B is equal to the number of ways to assign elements of B to $f(a)$, $f(b)$ and $f(c)$. Totally, there are $2 \cdot 2 \cdot 2 = 8$ ways to do that. The corresponding functions are schematically shown on the diagrams below.



[1] http://en.wikipedia.org/wiki/Rule_of_product

There are no injective functions from A to B. Indeed, since there are only two elements in set B, then at least two of the elements $f(a)$, $f(b)$, $f(c)$ have to be equal (pigenhole principle).

On the other hand, as one see from the diagrams, there are plenty of surjective functions. To count them, notice that there are only two *non-surjective* functions from A to B. Namely, one of them sends all elements of A to 1 and the other one maps everything to 2. Since totally there are eight functions from A to B, then exactly $8-2=6$ of them are surjective.

c) We just need to generalize the argument we used in parts a) and b). A function $f : A \to B$ is completely determined by the list of its values $f(a_1), f(a_2), \ldots, f(a_m)$, where $A = \{a_1, \ldots, a_m\}$. Since each $f(a_i)$ is an element of B, it may take one of $n = |B|$ possible values. Hence, there are $\underbrace{n \cdot n \ldots n}_{m} = n^m$ distinct ways to assign values to $f(a_1), f(a_2), \ldots, f(a_m)$. So there are n^m different functions from A to B.

7.7 a) Function f is injective. Indeed, suppose that $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$. It means precisely that $n_1 + 3 = n_2 + 3$. Then, subtracting 3 from both sides, we obtain $n_1 = n_2$.
The function is not surjective: $1 \in \mathbb{N}$, but there is no $n \in \mathbb{N}$ such that $n + 3 = 1$.

b) Function f is injective, the proof is essentially the same as in part a). Namely, if $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{Z}$, then $n_1 - 5 = n_2 - 5$ and the equality $n_1 = n_2$ follows immediately.
Function f is surjective. Indeed, let $m \in \mathbb{Z}$. Then $f(m+5) = (m+5) - 5 = m$. Thus, any element of the codomain of f is in the range of f.

Since f is injective and surjective, it is bijective.

d) To check f for injectivity, suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in [1, \infty)$. By definition of f, it means that $x_1^3 - x_1 = x_2^3 - x_2$. We rewrite this equation as

$$x_1^3 - x_2^3 = x_1 - x_2 \Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2 \Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 - 1) = 0.$$

Notice now that since $x_1, x_2 \geq 1$, then $x_1^2 + x_1x_2 + x_2^2 - 1 \geq 2 > 0$. Thus, the above identity is satisfied only when $x_1 = x_2$. It implies that function f is injective.

Now, let $y \in [0, \infty)$ be an element of the codomain of function f. If we would like to prove that f is surjective, then we need to show that $f(x) = y$ for some $x \in [1, \infty)$. In other words, we need to show that the equation $x^3 - x = y$ has a solution in the interval $[1, \infty)$. To this end, recall the intermediate value theorem:

Theorem 1. *Let f be a function continuous on an interval $[a, b]$. If y is a number between (or equal to) $f(a)$ and $f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = y$.*

In our case, we take $a = 1$, $b = y + 1$ and we claim that $f(a) \leq y \leq f(b)$. Since $f(1) = 0$ and $y \geq 0$, the first inequality is automatic, and we just need to verify that $y \leq \underbrace{(y+1)^3 - (y+1)}_{f(b)}$. This follows from the following observation:

$$(y+1)^3 - (y+1) - y = y^3 + 3y^2 + y \geq 0.$$

Thus, the hypothesis of the intermediate value theorem is satisfied, and it follows now that the equation $x^3 - x = y$ has a solution in the interval $[1; y+1) \subset [1; \infty)$. We conclude that f is surjective.

7.9 a) Recall that in order to show that f is injective, one needs to prove that the implication $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ holds for all $x_1, x_2 \in \mathrm{dom}(f)$. But in the given case a proof of the converse implication [2] $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ is provided. This is not equivalent to the desired statement.

b) In the fourth sentence an assumption $x_1 = x_2$ is made, but this is the identity that we actually need to prove.

c) This is a valid proof of the contrapositive of the implication $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Since the contrapositive is logically equivalent to an original statement, everything is OK here.

d) This is a proof of the inverse of the desired implication $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. One cannot conclude from it that f is injective.

e) This is just a verification of injectivity of function f in a very special case, when $x_1 = 1$ and $x_2 = 2$. The general statement of the theorem does not follow from it.

f) Everything is correct.

7.10 a) $f(k) = \begin{cases} 1, & k \text{ is odd} \\ \frac{k}{2}, & k \text{ is even.} \end{cases}$

b) $f(k) = k + 1$.

c) $f(k) = 1$.

d) $f(k) = k$.

7.15 a) By definition of pre-image, $f^{-1}(f(C)) = \{x \in A | f(x) \in f(C)\}$. Now, if $x \in C$, then $f(x) \in f(C)$. Hence, $x \in f^{-1}(f(C))$. Therefore, $C \subseteq f^{-1}(f(C))$.

b) Let $x \in f^{-1}(D)$. By definition of pre-image, it means that $f(x) \in D$. Hence, $f(f^{-1}(D)) \subseteq D$.

d) First, we are going to show that $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$. Let $y \in f(C_1 \cup C_2)$. Then there exists $x \in C_1 \cup C_2$ such that $f(x) = y$. Since $x \in C_1 \cup C_2$, then $x \in C_1$ or $x \in C_2$. Hence, $f(x) \in f(C_1)$ or $f(x) \in f(C_2)$. That means exactly that $y \in f(C_1) \cup f(C_2)$, and the desired inclusion $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$ follows.

Now, we would like to show that the inclusion $f(C_1 \cup C_2) \supseteq f(C_1) \cup f(C_2)$ holds as well. Let $y \in f(C_1) \cup f(C_2)$. Then $y \in f(C_1)$ or $y \in f(C_2)$. Hence, we can find $x \in C_1$ or $x \in C_2$ such that $f(x) = y$. This is equivalent to saying that $y = f(x)$ for some $x \in C_1 \cup C_2$. Therefore, $y \in f(C_1 \cup C_2)$ and the inclusion follows.

[2] It is, in fact, just a part of the general definition of a function and has nothing to do with injectivity.