The following is a non-comprehensive list of solutions to the skills problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

12.3 (e) Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. For all $n \in \mathbb{N}$, $\frac{1}{n} \leq 1$, so 1 is an upper bound of S. Since 1 is actually in the set, it is the maximum and supremum of S.

(n) Let $S = \{r \in \mathbb{Q} : r^2 \leq 5\}$. The number $\sqrt{5} \in \mathbb{R}$ is an upper bound of S but is not actually in S, since $\sqrt{5}$ is irrational. It is the supremum of S, but S has no maximum value.

To prove $\sqrt{5} = \sup S$, let $\epsilon > 0$ and show that there exists an element of S in between $\sqrt{5} - \epsilon$ and ϵ. This follows from the density of the rationals within the reals:

$$\exists r \in \mathbb{Q} \text{ such that } \sqrt{5} - \epsilon < r < \sqrt{5}.$$

Furthermore, the fact that $r < \sqrt{5}$ means that $r^2 < 5$, so this r is in fact an element of our set. Hence $\sqrt{5} - \epsilon$ is not an upper bound of S for any $\epsilon > 0$, which proves that $\sqrt{5}$ is the supremum.

The fact that S has no maximum value also follows from the density of the rationals. Suppose there were a maximum value $m = \max S$. Then $m \in S$, since any maximum of a set must be in the set itself by definition, which means $m \neq \sqrt{5}$. But then there exists a rational number r such that $m < r < \sqrt{5}$. As above, r must be in S, contradicting the claim that m is the maximum.

12.4 (e) 0 is a lower bound of $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$, and in fact $0 = \inf S$. In this case S has no minimum value, since the infimum is not actually in the set.

To prove $0 = \inf S$, let $\epsilon > 0$ and show that there is an element of S in between 0 and $0+\epsilon$. (Rephrasing, this means no number greater than 0 can be a lower bound, which is another way of saying that 0 is the *greatest* lower bound.) But that follows quickly from the Archimedean Property or, more accurately, the Theorem which immediately follows it. By Theorem 12.10(c), there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \in S$ satisfies

$$0 < \frac{1}{n} < \epsilon.$$

(n) The set $S = \{r \in \mathbb{Q} : r^2 \leq 5\}$ has no minimum value, but its infimum is $-\sqrt{5}$. The proofs of these facts are essentially the same as 12.3(n), replacing $\sqrt{5}$ and $\sqrt{5} - \epsilon$ with $-\sqrt{5}$ and $-\sqrt{5} + \epsilon$, reversing the inequalities, etc. Ask us if you need help with these questions.

13.3 (a) The set has no interior; any neighborhood centered at a number $1/n$ will necessarily include numbers which are not in the set itself – irrational numbers, for example – so none of the neighborhoods will ever be contained in the original set.

(b) $[0,3] \cup (3,5) = [0,5) = \{x : 0 \le x < 5\}$. The interior of this set is $(0,5)$, because for any $x \in (0,5)$ you can always find a neighborhood $N(x;\epsilon)$ which stays entirely within the set. (For example, let $\epsilon = \min\{|5-x|, |x-0|\}$.)

(c) This set has no interior for essentially the same reason as (a) — any neighborhood $N(r;\epsilon)$ centered at a point r in the set will contain irrational numbers which are not in the original set.

13.5 (a) Every point in the original set is a boundary point, since every neighborhood $N\left(\frac{1}{n};\epsilon\right)$ will contain a point in the set ($1/n$ itself) and irrational numbers which are not in the set. However, 0 is also a boundary point, since every $N(0;\epsilon)$ will contain elements of the set–we can find n so that $0 \le \frac{1}{n} < \epsilon$ by Theorem 12.10(c)–and numbers which are not in the set. Hence the boundary is

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$$

(b) The boundary of $[0,3] \cup (3,5) = [0,5)$ is $\{0,5\}$. Similar examples have been done in lecture, but ask us in email, Moodle or office hours if you need help sorting this out.

(c) Every number in this set is a boundary point, since every $N(r;\epsilon)$ will contain numbers in the set (like r itself) and numbers not in the set (such as irrationals). However, the *irrational* numbers between 0 and $\sqrt{2}$ are also boundary points. For example, $1/\sqrt{2} = \frac{\sqrt{2}}{2}$ satisfies

$$0 < \frac{\sqrt{2}}{2} < \sqrt{2}$$

and is a boundary point of the set: each $N(\sqrt{2}/2;\epsilon)$ will contain rational numbers in the original set, as well as irrational numbers (like itself) which are not.

Hence the boundary is the set of *all real numbers* between 0 and $\sqrt{5}$, i.e. the set $[0,\sqrt{5}]$.