Math 3283W: solutions to skills problems due 11/6

(13.7(abc)) All three statements can be proven false using counterexamples that you studied on last week's skills homework.

(a) Let $S = \{\frac{1}{n} | n \in \mathbb{N}\}$. Every point of S is an isolated point, since given $\frac{1}{n} \in S$, there exists a neighborhood $N(\frac{1}{n}; \varepsilon)$ of $\frac{1}{n}$ that contains no point of S different from $\frac{1}{n}$ itself. (We could, for example, take $\varepsilon = \frac{1}{n} - \frac{1}{n+1}$.) Thus the set P of isolated points of S is just S. But $P = S$ is not closed, since 0 is a boundary point of S which nevertheless does not belong to S (every neighborhood of 0 contains not only 0 but, by the Archimedean property, some $\frac{1}{n}$, thus intersects both S and its complement); if S were closed, it would contain all of its boundary points.

(b) Let $S = \mathbb{N}$. Then (as you saw on the last skills homework) $int(S) = \emptyset$. But $\emptyset$ is closed, so its closure is itself; that is, $cl(int(S)) = cl(\emptyset) = \emptyset$, which is certainly not the same thing as S.

(c) Let $S = \mathbb{R} \setminus \mathbb{N}$. The closure of S is all of $\mathbb{R}$, since every natural number n is a boundary point of $\mathbb{R} \setminus \mathbb{N}$ and thus belongs to its closure. But $\mathbb{R}$ is open, so its interior is itself; that is, $int(cl(S)) = int(\mathbb{R}) = \mathbb{R}$, which is certainly not the same thing as S.

(13.10) Suppose that x is an isolated point of S, and let $N = (x - \varepsilon, x + \varepsilon)$ (for some $\varepsilon > 0$) be an arbitrary neighborhood of x. To show x is a boundary point of S, we must show two things: first, that N contains a point of S, and second, that N contains a point of $\mathbb{R} \setminus S$. The first is easy: part of the definition of "x is an isolated point of S" is that $x \in S$, so the neighborhood N contains the point x of S. The second is trickier. Since x is an isolated point of S, it is not an accumulation point of S, so (spelling out the negation of the definition of "accumulation point") we see that *there exists* some *deleted* neighborhood $N^*(x; \eta) = (x - \eta, x + \eta) \setminus \{x\}$ of x (for some $\eta > 0$) that contains no point of S. We need to use the fact that $(x - \eta, x + \eta)$ contains *no* point of S other than x itself to prove that our original neighborhood $N = (x - \varepsilon, x + \varepsilon)$ contains *at least one* point of $\mathbb{R} \setminus S$. The idea is that, regardless of whether ε or η is larger (we have no way of knowing which one), these are two open intervals with positive radius centered at the same point x, so they overlap; and any of the points in the (nonempty) intersection $N^*(x; \eta) \cap N$ is a point in the complement of S. If you want an explicit formula for such a "bad" point, let $\delta = \min\{\eta, \varepsilon\}$; then $x + \frac{\delta}{2}$ belongs both to N and to $\mathbb{R} \setminus S$.

(14.4) First, a preliminary result:

Lemma. If $A \subset B$ are sets of real numbers and B is bounded, then A is bounded.

Proof of lemma. To say B is bounded is to say there exist an upper bound m and a lower bound ℓ for B. Given any $x \in A$, we know $x \in B$ since A is contained in B. But then $x \leq m$ since m is an upper bound for B, and $\ell \leq x$ since ℓ is a lower bound for B. That is, $\ell \leq x \leq m$ for every $x \in A$, so ℓ (resp. m) is a lower (resp. upper) bound for A as well; that is, A is bounded, having both a lower bound and an upper bound.

Now we prove the claim of 14.4: the intersection of any collection of compact sets is compact. Let $\{C_i\}_{i \in I}$ be a family of compact sets. By the Heine-Borel theorem, we know that for subsets of $\mathbb{R}$, "compact" is equivalent to "closed and bounded". So each C_i is closed and bounded, and we will be done if we can prove that $\bigcap_{i \in I} C_i$ is closed and bounded as well. Call this intersection C. By Corollary 13.11*(a)*, we know that C is closed, since each C_i is closed and the intersection of any family of closed sets is closed. Now fix one of the sets C_i; call it C_{i_0}. (It doesn't matter *which* one we fix, just *that* we fix one.) Then it is certainly true that $C \subset C_{i_0}$; C is the intersection $\bigcap_{i \in I} C_i$, so if $x \in C$, we know $x \in C_i$ for all $i \in I$, and in particular, $x \in C_{i_0}$. But C_{i_0} is compact and hence bounded, so by the lemma above, C, being a subset of a bounded set, is bounded. Since C is both closed and bounded, the Heine-Borel theorem tells us C is compact.