Take-home writing problems 1: Solutions

1. a) If $S \subseteq T$, then $|S| \leq |T|$.

Proof. If $S = \varnothing$, then $|S| = 0$ and the statement follows. Otherwise, since any element of S belongs to T, then we can define a function $i : S \to T$ by setting $i(s) = s$ for all $s \in S$. Function i is clearly injective (indeed, an equality $i(s_1) = i(s_2)$ means exactly that $s_1 = s_2$). Hence, by definition 8.14, $|S| \leq |T|$. □

b) $|S| \leq |S|$

Proof. Since for any set S (including the case $S = \varnothing$) the containment $S \subseteq S$ holds, then by part a) we get $|S| \leq |S|$. □

c) If $|S| \leq |T|$ and $|T| \leq |U|$, then $|S| \leq |U|$.

Proof. The case when $S = \varnothing$ is easy. Indeed, in this case we have $|S| = 0 \leq |U|$, no matter what set U is.

Now, assume that $S \neq \varnothing$. Then, according to definition 8.14, the condition $|S| \leq |T|$ means exactly that there is an injective function $f : S \to T$. Moreover, since $|T| \leq |U|$, then there is an injection $g : T \to U$. We claim that the composition $g \circ f : S \to U$ is an injective function as well. Indeed, if for some $s_1, s_2 \in S$ the equality $g(f(s_1)) = g(f(s_2))$ holds, then due to injectivity of g, we must have $f(s_1) = f(s_2)$. Since f is injective, then $s_1 = s_2$, and our claim follows.

So we got an injective function from S to U. Thus $|S| \leq |U|$. □

d) If $m, n \in \mathbb{N}$ and $m \leq n$, then $|\{1, 2, \dots, m\}| \leq |\{1, 2, \dots, n\}|$.

Proof. Since $m \leq n$, then $\{1, 2, \dots, m\}$ is a subset of $\{1, 2, \dots, n\}$. Now, the desired statement follows from part a) proved above. □

2. Use Theorem 8.15 and the Schröder-Bernstein Theorem to provide a simpler proof than the one given in class that $[0, 1] \sim [0, 1)$.

Proof. According to the Schröder-Bernstein Theorem, proving that $[0, 1] \sim [0, 1)$ amounts to showing that $|[0, 1]| \leq |[0, 1)|$ and $|[0, 1)| \leq |[0, 1]|$.

First, let us prove that $|[0, 1]| \leq |[0, 1)|$. By definition 8.14, all we have to do is to construct an injective function $f : [0, 1] \to [0, 1)$. Fortunately, it is not hard to do. Such a function can be defined, for instance, by setting $f(x) = \frac{x}{2}$ for all $x \in [0, 1]$. Injectivity of f is pretty evident (if $f(x_1) = f(x_2)$, then $\frac{x_1}{2} = \frac{x_2}{2}$ and we conclude immediately that $x_1 = x_2$).

Now, we would like to show that $|[0, 1)| \leq |[0, 1]|$. In order to do that, just notice that $[0, 1) \subseteq [0, 1]$. Now, the desired statement follows directly from theorem 8.15(a). □

3. Use Theorem 8.15 and the Schröder-Bernstein Theorem to prove $(0, 1) \sim \mathbb{R}$.

Proof. The strategy we are going to use will be the same as in the previous exercise. Namely, we will show that $|(0,1)| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |(0,1)|$. The statement of the problem will follow then from the Schröder-Bernstein Theorem.

First, since the interval $(0,1)$ is a subset of $\mathbb{R}$, then the inequality $|(0,1)| \leq |\mathbb{R}|$ follows from theorem 8.15(a).

Now, we would like to show that $|\mathbb{R}| \leq |(0,1)|$. To do that, we will construct an injective function $f : \mathbb{R} \to (0,1)$. For instance, we can set $f(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$ for $x \in \mathbb{R}$. One can easily check that function f is injective - it follows from injectivity of arctangent. □