Quiz 4

Use induction to prove: if $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Solution.

Step 1. For $n = 1$, both sides of the given non-strict inequality are actually equal. That establishes the *basis of induction.*

Step 2. Now, suppose that the statement of the problem is true for a natural $n = k$. That is, we take for granted that for any $x > -1$, the inequality

$$(1 + x)^k \geq 1 + kx \tag{1}$$

holds. We would like to show that it holds for $n = k + 1$ as well.

Multiplying both sides of (1) by a positive number $1 + x$, we obtain

$$(1 + x)^{k+1} \geq (1 + kx)(1 + x). \tag{2}$$

Since $k > 0$, then

$$(1 + kx)(1 + x) = 1 + (k + 1)x + \underbrace{kx^2}_{\geq 0} \geq 1 + (k + 1)x.$$

Returning to (2), we get

$$(1 + x)^{k+1} \geq 1 + (k + 1)x.$$

This is precisely the desired inequality for $n = k + 1$. Thus, we have proven the *induction step.*